SIMPLICIAL GRAPHS AND RELATIONSHIPS TO DIFFERENT GRAPH INVARIANTS

BERT RANDERATH AND LUTZ VOLKMANN

ABSTRACT. In this paper we show that simplicial graphs, in which every vertex belongs to exactly one simplex, characterize graphs satisfying equality in some graph invariants concerning independence, clique covering, domination or distance.

1. Terminology and Introduction

In this paper we consider finite, undirected and simple graphs with vertex set $V(G)$. For $A \subseteq V(G)$ let $G[A]$ be the subgraph induced by A in G. Moreover, $N(x, G)$ denotes the set of vertices adjacent to the vertex x and more generally, $N(X, G) = \cup_{x \in X} N(x, G)$ for $X \subseteq V(G)$. We write $N[x, G]$ instead of $N(x, G) \cup \{x\}$. A maximal complete subgraph of a graph is a clique. A vertex v of a graph G is simplicial, if every two vertices of $N(v, G)$ are adjacent in G. Equivalently, a simplicial vertex is a vertex that appears in exactly one clique. A clique of a graph G containing at least one simplicial vertex is called a simplex of G. A graph G is simplicial if every vertex of G is either simplicial or adjacent to a simplicial vertex of G. A clique covering of G is a collection of cliques, such that the union of their vertex sets is $V(G)$. The clique covering number of G denoted by $\theta(G)$ is the minimum number of cliques, such that their union is a clique covering of G. An independent set I of vertices of a graph G is a set of pairwise nonadjacent vertices. Let $i(G)$ and $\alpha(G)$ denote the minimum and maximum number of vertices, such that their union is a maximal independent set of G. Since every clique of a graph G contains at most one vertex of an independent set of G, we have

$$\alpha(G) \leq \theta(G).$$

A set D of vertices of a graph G is called a dominating set of G, if every vertex $v \in V(G) - D$ is adjacent to at least one vertex of D. Let $\gamma(G)$ and $\Gamma(G)$ denote the minimum and maximum number of vertices, such that their union is a minimal domination set of G. In case of equality $i(G) =$

$\alpha(G)$ ($\gamma(G) = \Gamma(G)$) a graph G is said to be well covered (well dominated). Since every maximal independent set of G is a minimal dominating set of G, these graph invariants are related by the following inequalities:

$$\gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G)$$

The distance between two vertices u and v in a graph G is denoted by $d_G(u,v)$. The kth power of a graph G is the graph G^k with the same vertex set $V(G)$ and an edge is joining distinct vertices $u, v \in V(G)$ whenever $d_G(u,v) \leq k$. If we consider the last four graph invariants in G^k, distance related generalized graph invariants of G appear ($i(G^k) = i_k(G)$, $\alpha(G^k) = \alpha_k(G)$ the k-packing number, $\gamma(G^k) = \gamma_k(G)$ the k-covering number, $\Gamma(G^k) = \Gamma_k(G)$). They were first introduced by Meir and Moon [8]; further studies are given in the papers of Chang, Nemhauser [1] and Topp, Volkmann [11]. Since $N[x, G]$ induces a complete subgraph in G^2 for every vertex x of a graph G, we have:

$$\theta(G^2) \leq \gamma(G).$$

Simplicial graphs were introduced by Cheston, Hare and Laskar [2] . These graphs appear often in studies about well covered graphs as one can see in the survey paper of Plummer [9] . In particular, the well covered simplicial graphs were characterized by Prisner, Topp and Vestergaard [10] . The class of simplicial graphs for which every vertex belongs to exactly one simplex seems to be a useful tool for characterizing graphs satisfying equality in some graph invariants. In this paper we characterize graphs G with $\alpha_2(G) = \theta(G)$, $\alpha_2(G) = \alpha(G)$ and those satisfying $\theta(G^2) = \theta(G)$. Topp and Volkmann [11] characterized trees T with k-packing number $\alpha_k(T)$ equal to the k-covering number $\gamma_k(T)$. Hattingh and Henning [6] extended this result for block graphs - graphs without noncomplete subgraphs induced by vertices of a cycle. Furthermore, we present a generalization of these results with emphasis on chordal graphs.

2. Preliminary Results

The first proposition yields a nice property of well covered graphs with equal independence number α and clique covering number θ.

Proposition 2.1. *If G is a well covered graph with $\alpha(G) = \theta(G)$, then in every minimum clique covering $\mathcal{C}$ all cliques are pairwise vertex disjoint.*

Proof: Let $\mathcal{C} = \{C_1, C_2, \ldots, C_q\}$ be a minimum clique covering of G. If $q = 1$, then the result is valid. Now suppose that $q > 1$ and assume to the contrary that there are two cliques of $\mathcal{C}$, say C_{q-1} and C_q, containing a common vertex v. In addition, let I be a maximal independent set of G

containing the vertex v. Since G is well covered and $\alpha(G) = \theta(G) = q$, it follows that $|I| = q$. Furthermore, $|I \cap V(C_i)| \leq 1$ for $i = 1, 2, \ldots, q-2$ and $|I \cap (V(C_{q-1}) \cup V(C_q))| = |\{v\}| = 1$. Hence we obtain the contradiction $|I| \leq q - 1$, which completes the proof . ∎

The following property of simplicial graphs was found by Cheston, Hare and Laskar [2].

Proposition 2.2. [2] *If G is a simplicial graph and $S_1, S_2, \ldots S_q$ are the simplexes of G, then $V(G) = \cup_{i=1}^{q} V(S_i)$ and $q = \theta(G)$.*

The last result of this section is due to Chang and Nemhauser [1] .

Proposition 2.3. [1] *If G is chordal and S is a clique of G^k, then the subgraph $G[V(S)]$ induced by the vertex set of S is connected and $d_G(x, y) = d_{G[V(S)]}(x, y)$ for all $x, y \in S$.*

3. Main Results

It is possible to combine the inequalities of Section 1 in the following way:

$$\alpha_2(G) \leq \theta(G^2) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \theta(G) \qquad (*)$$

The next theorem gives a characterization of graphs G with $\alpha_2(G) = \theta(G)$, $\alpha_2(G) = \alpha(G)$ and those satisfying $\theta(G^2) = \theta(G)$.

Theorem 3.1. *For a graph G the following statements are equivalent:*

(i) Every vertex of G belongs to exactly one simplex of G;
(ii) G satisfies $\alpha_2(G) = \alpha(G)$;
(iii) G satisfies $\theta(G^2) = \theta(G)$;
(iv) G satisfies $\alpha_2(G) = \theta(G^2) = \gamma(G) = i(G) = \alpha(G) = \theta(G)$.

Proof: Obviously the implications $(iv) \rightarrow (iii)$ and $(iv) \rightarrow (ii)$ are valid. For the proof of the implication $(i) \rightarrow (iv)$ let G be a graph such that every vertex of G belongs to exactly one simplex of G, and let S^* be a vertex set of G containing exactly one simplicial vertex of every simplex of G. It is obvious that S^* is independent. According to Proposition 2.2 we have $\theta(G) = |S^*|$. Since all simplexes $S_1, S_2, \ldots, S_q$ of G are pairwise vertex disjoint we have $V(G) = \dot\cup_{i=1}^{q} V(S_i) = \dot\cup_{s \in S^*} N[s, G]$. Thus, for all distinct vertices $x, y \in S^*$ we deduce $N[x, G] \cap N[y, G] = \emptyset$ and $d_G(x, y) > 2$. Therefore, S^* is an independent set of G^2 and so $\theta(G) = |S^*| \leq \alpha_2(G)$. Together with $(*)$ we obtain $\alpha_2(G) = \theta(G^2) = \gamma(G) = i(G) = \alpha(G) = \theta(G)$.
For the proof of the implication $(ii) \rightarrow (i)$ let G be a graph with $\alpha_2(G) = \alpha(G)$ and let I be an independent set of G^2 with $|I| = \alpha_2(G)$. Thus, two distinct vertices of I are neither neighbors nor have a common

neighbor, i.e. $N[x, G] \cap N[y, G] = \emptyset$ for all distinct $x, y \in I$. Since I is a maximum independent set, I is also a dominating set of G. According to the last two arguments $V(G) = \dot\bigcup_{x \in I} N[x, G]$ holds. Now it suffices to show that each $w \in I$ is a simplicial vertex. Suppose to the contrary that there exists a nonsimplicial vertex $v \in I$. Then v has at least two nonadjacent neighbors x and y. Hence $(I - \{v\}) \cup \{x, y\}$ is an independent set of G, contradicting that I is a maximum independent set. This contradiction completes the proof of this implication.
For the proof of the implication $(iii) \to (i)$ let G be a graph $\theta(G^2) = \theta(G)$. Because of the inequalities $(*)$, the graph G is well covered and $\alpha(G) = \theta(G) = q$. From Proposition 2.1 follows that in every minimum clique covering $\mathcal{C} = \{C_1, C_2, \ldots, C_q\}$ all cliques are pairwise vertex disjoint, i.e. $V(G) = \dot\bigcup_{i=1}^{q} V(C_i)$. Now it suffices to show that each clique $C \in \mathcal{C}$ is a simplex of G. If $q = 1$, then G is a complete graph and the result is valid. Assume $q \geq 2$ and suppose to the contrary that there is a clique of $\mathcal{C}$, say C_q, which is not a simplex of G, i.e. every vertex $x \in C_q$ is adjacent to a vertex $y \notin C_q$. For every pair of vertices x, y of the induced subgraph $H_i := G[V(C_i) \cup \{N(V(C_i), G) \cap V(C_q)\}]$ for $i = 1, 2, \ldots, q-1$ of G we see $d_G(x, y) \leq 2$ and furthermore $\bigcup_{i=1}^{q-1} V(H_i) = V(G)$. Now let H_i^* be a clique of G^2 containing the vertices of H_i for $i = 1, 2, \ldots, q-1$. Then $\mathcal{H} = \{H_1^*, H_2^*, \ldots, H_{q-1}^*\}$ is a clique covering of G^2. This yields the contradiction $q - 1 \geq \theta(G^2) = q$ and completes the proof. ∎

A graph G is chordal, if G contains no induced cycle of length at least four. Prisner, Topp and Vestergaard [10] characterized the well covered simplicial graphs and the well covered chordal graphs, as follows.

Theorem 3.2. [10] *For a simplicial {chordal} graph G the following statements are equivalent:*

- *(i) G is well covered, i.e. $i(G) = \alpha(G)$;*
- *(ii) G is well dominated, i.e. $\gamma(G) = \Gamma(G)$;*
- *(iii) G satisfies $\gamma(G) = \alpha(G)$;*
- *(iv) G satisfies $i(G) = \Gamma(G)$;*
- *(v) Every vertex of G belongs to exactly one simplex of G.*

The next theorem has been proved recently by Dean, Zito [3] and Scheinerman [see 3]. For convenience, we call a graph G without cycle of length four as induced subgraph C_4-free.

Theorem 3.3. [3] *Let G be a C_4-free graph satisfying $\alpha(G) = \theta(G)$. Then the following statements are equivalent:*

- *(i) G is well covered, i.e. $i(G) = \alpha(G)$;*
- *(ii) Every vertex of G belongs to exactly one simplex of G.*

This result generalizes the equivalence of (i) and (v) of Theorem 3.2 for chordal graphs, since every chordal graph G is C_4-free and fulfils $\alpha(G) = \theta(G)$, proved by A. Hajnal and J.Suranyi [5]. Theorems 3.1, 3.2 or 3.3 applied on the kth power of a graph G yield in a simple way distance related generalizations. We focus our interest now on the generalized Theorems 3.2 and 3.3.

Corollary 3.1. *Let G be a graph with $\alpha_k(G) = \theta(G^k)$ and furthermore let G^k be C_4-free. Then the following statements are equivalent:*

(i) *G satisfies $i_k(G) = \alpha_k(G)$;*
(ii) *G satisfies $\gamma_k(G) = \Gamma_k(G)$;*
(iii) *G satisfies $\gamma_k(G) = \alpha_k(G)$;*
(iv) *G satisfies $i_k(G) = \Gamma_k(G)$;*
(v) *Every vertex of G belongs to exactly one simplex of G^k.*

Now we have to explain the statement: *S is a simplex of G^k*. There exists a simplicial vertex s_0 of G^k contained in $V(S)$. So we see that every vertex of G having distance at most k from s_0 is also contained in $V(S)$. Since S is a clique of G^k we deduce $d_G(x,y) \leq k$ for every pair of distinct vertices $x, y \in V(S)$. Nevertheless, the distance in a graph G between two vertices x, y of S can increase in the subgraph $G[V(S)]$ of G induced by the vertices of S, i.e. $d_G(x,y) \leq d_{G[V(S)]}(x,y)$. In particular, the greatest distance in $G[V(S)]$ between two vertices of S -the diameter of $G[V(S)]$- can be greater than k. We call a simplex S of G^k with the additional property that the diameter of the subgraph $G[V(S)]$ of G is at most k a *k^*-simplex* of G. In the special case of chordal graphs Proposition 2.3 yields the missing additional property, such that for chordal graphs G every simplex of G^k is also a k^*-simplex of G. So, if we apply Corollary 3.1 to chordal graphs, then statement (v) changes a little bit.

Corollary 3.2. *Let G be a chordal graph with $\alpha_k(G) = \theta(G^k)$ and furthermore let G^k be C_4-free. Then the following statements are equivalent:*

(i) *G satisfies $i_k(G) = \alpha_k(G)$;*
(ii) *G satisfies $\gamma_k(G) = \Gamma_k(G)$;*
(iii) *G satisfies $\gamma_k(G) = \alpha_k(G)$;*
(iv) *G satisfies $i_k(G) = \Gamma_k(G)$;*
(v) *Every vertex of G belongs to exactly one k^*-simplex of G, i.e. $V(G) = \dot{\cup}_{i=1}^{q} V(S_i)$, where $S_1, \ldots, S_q$ are the k^*-simplexes of G.*

There is a question arising from the last corollary: which chordal graphs fulfil the assumptions? The class of chordal graphs G with likewise chordal kth power G^k fulfil the assumptions, since G^k as chordal graph is C_4-free and by reason of the equality of independence number α and clique covering number θ for chordal graphs, proved by A. Hajnal and J.Suranyi [5].

This class of graphs has been examined by Chang, Nemhauser [1] , Duchet [4] and Laskar, Shier [7]. Jamison (Corollary 6.9 in [4]) showed that the class of block graphs is also included in this subclass of chordal graphs. Thus Corollary 3.2 generalizes the following theorem which is due to Hattingh and Henning [6]. In particular, the statement (v) of Corollary 3.2 is equivalent to statement (ii) of Theorem 3.4 for connected block graphs.

Theorem 3.4. [6] *Let G be a connected block graph. Then the following statements are equivalent:*

- *(i)* $\gamma_k(G) = \alpha_k(G) = n$;
- *(ii)* *One of the following statements holds:*
  - *(a)* *G has diameter at most k and $n = 1$.*
  - *(b)* *There exists a decomposition of G into n subgraphs $G_1, G_2, ..., G_n$ in such a way that:*
    - *(1)* *G_i is a connected block graph of diameter k ($i = 1, 2, ..., n$),*
    - *(2)* *for each $i \in \{1, 2, ..., n\}$ there exists $u_i \in V(G_i) - V(G_0)$ such that $d_G(u_i, V(G_0)) = k$, where G_0 is the subgraph of G generated by the edges which do not belong to any of the subgraphs $G_1, G_2, \ldots, G_n$;*
- *(iii)* *G is a well-k-dominated, i.e. $\gamma_k(G) = \Gamma_k(G)$.*

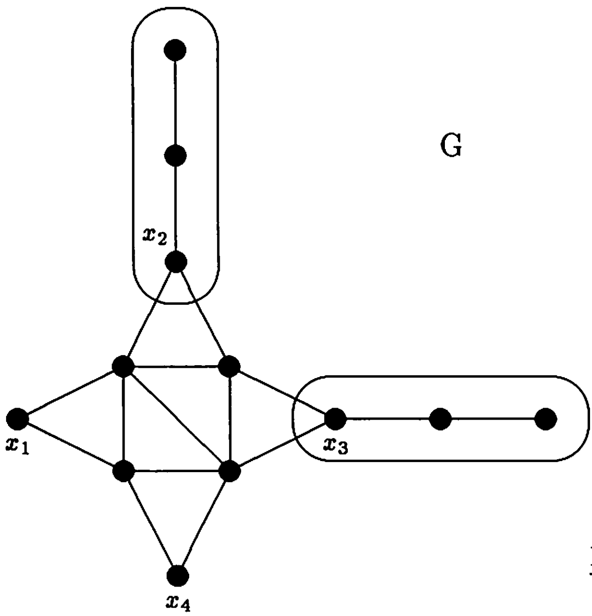

Figure 1:

The condition that G^k is C_4-free is necessary in Corollary 3.2, as we can see by the example in Figure 1: a chordal graph G with $i_2(G) = \alpha_2(G) = 3$ having a cycle $x_1x_2x_3x_4x_1$ of length four as induced subgraph of G^2, furthermore G^2 is not simplicial and so G is not 2^*-simplicial decomposable.

Remark 3.1. *With some effort it is possible to show that the statements (ii),(iii) and (v) of Corollary 3.2 remain equivalent, if we omit the condition that G^k is C_4-free. We do not know, whether the other condition $\alpha_k(G) = \theta(G^k)$ of Corollary 3.2 is necessary. We hope that it is possible to close this gap in order to improve the equivalences of (ii),(iii) and (v) of Corollary 3.2 for all chordal graphs.*

References

1. G.J. Chang and G.L. Nemhauser, *The k-domination and k-stability problems on sunfree chordal graphs,* SIAM J. Algebraic Discrete Methods **5** (1984), 332-345.
2. G.H. Cheston, E.O. Hare and R.C. Laskar, *Simplicial graphs,* Congr. Numer. **67** (1988), 105-113.
3. N. Dean and J. Zito, *Well-covered graphs and extendability,* Discrete Math. **126** (1994), 67-80.
4. P. Duchet, *Classical perfect graphs: An introduction with emphasis on triangulated and intervall graphs,* Ann. Discrete Math. **21** (1984), 67-96.
5. A. Hajnal und J. Suranyi, *Über die Auflösung von Graphen in vollständige Teilgraphen,* Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **1** (1958), 113-121.
6. J.H. Hattingh and M.A. Henning, *A characterization of block graphs that are well-k-dominated,* J. Comb. Math. Comb. Comp. **13** (1993), 33-38.
7. R. Laskar and D. Shier, *On powers and centers of chordal graphs,* Discrete Appl. Math. **6** (1983), 139-147.
8. A. Meir and J.W. Moon, *Relations between packing and covering number of a tree,* Pacific J. Math. **61** (1975), 225-233.
9. M.D. Plummer, *Well-covered graphs: a survey,* Quaestiones Math. **16** (1993), 253-287
10. E. Prisner, J. Topp and P.D. Vestergaard, *Well covered simplicial, chordal and circular arc graphs,* J. Graph Theory, to appear.
11. J. Topp and L. Volkmann, *On packing and covering numbers of graphs,* Discrete Math. **96** (1991), 229-238.

LEHRSTUHL II FÜR MATHEMATIK, RWTH AACHEN, 52056 AACHEN, GERMANY
E-mail address: rander@math2.rwth-aachen.de, volkm@math2.rwth-aachen.de