The Irregularity of a Graph

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ABSTRACT. The *imbalance* of edge $(x, y) = |\deg(x) - \deg(y)|$. The sum of all edge imbalances in a graph is called its *irregularity*. We determine the maximum irregularity of various classes of graphs. For example, the irregularity of an arbitrary graph with n vertices is less than $\frac{4n^3}{27}$, and this bound is tight.

Introduction

Suppose G is a graph with n vertices and E edges. The *imbalance* of the edge $e = (x, y)$, denoted by $\text{imb}_G(e)$, is defined to be $|\deg(x) - \deg(y)|$. The *irregularity* of G, denoted by $\text{irr}(G)$, is defined by

$$\text{irr}(G) = \sum_e \text{imb}(e) = \sum_{(x,y)} |\deg(x) - \deg(y)|.$$

Clearly G is a regular graph if and only if its irregularity is 0. The idea of the imbalance of an edge appeared implicitly in [3] where it was related to Ramsey problems with repeat degrees. This was generalized in [6]. Specifically, if $r = r(G, H)$ denotes the classic Ramsey number, then Chen, Erdos, Rousseau, and Schelp show that in any 2-coloring of the edges of K_n, there exists either a red G or a blue H in which the maximum imbalance is $\leq r - 2$ provided $n \geq 4(r-1)(r-2)$. There have been other attempts to measure how irregular a graph is, but heretofore this has not been captured by a single parameter [1,2,4,5,7]. The focus of this paper is on graphs with large irregularity.

Elementary Properties

Proposition 1. *For any edge e in a graph G, imb$(e) \leq n - 2$.*

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Proposition 2. *If T is a tree, then irr$(T) \leq (n-1)(n-2)$.*

Equality is achieved by the star $K_{1,n-1}$. This graph also maximizes the irregularity of a graph with a fixed number of edges.

Proposition 3. *For any graph G, irr$(G) = O(nE) = O(n^3)$.*

Our next result shows what happens to the irregularity when an edge is added to G. If u is a vertex in G it is convenient to define

$$\deg^{>}(u) = |\{x\colon (x,u) \text{ in } E(G) \text{ and } \deg(u) > \deg(x)\}|,$$
$$\deg^{=}(u) = |\{x\colon (x,u) \text{ in } E(G) \text{ and } \deg(u) = \deg(x)\}|,$$

and

$$\deg^{<}(u) = |\{x\colon (x,u) \text{ in } E(G) \text{ and } \deg(u) < \deg(x)\}|.$$

Clearly $\deg(u) = \deg^{>}(u) + \deg^{=}(u) + \deg^{<}(u)$.

Lemma 4. (Edge Addition Lemma) *If (u,v) is not in $E(G)$, set $G' = G+(u,v)$. If $\deg(u) \geq \deg(v)$, then irr$(G') =$ irr$(G) + 2[\deg^{>}(u) + \deg^{=}(u) - \deg^{<}(v)]$.*

Proof: The only edges whose imbalances change between G and G' are those incident with u and/or v. The imbalance of (u,v) will be $[(\deg(u)+1) - (\deg(v)+1)]$. Each edge of G that is incident with u will have its imbalance changed by 1. Which direction that change occurs in depends on which end vertex had the greater degree in G. The total contribution at u will be $[\deg^{>}(u) + \deg^{=}(u) - \deg^{<}(u)]$, while the total contribution at v will be $[\deg^{>}(v) + \deg^{=}(v) - \deg^{<}(v)]$. Thus

$$\begin{aligned} \text{irr}(G') &= \text{irr}(G) + (\deg(u)+1) - (\deg(v)+1) \\ &\quad + [\deg^{>}(u) + \deg^{=}(u) - \deg^{<}(u)] \\ &\quad + [\deg^{>}(v) + \deg^{=}(v) - \deg^{<}(v)] \\ &= \text{irr}(G) + 2[\deg^{>}(u) + \deg^{=}(u) - \deg^{<}(v)]. \end{aligned}$$

Corollary 5. *For any G, irr(G) is even.*

Lemma 6. (Edge Deletion Lemma) *If (u,v) is in $E(G)$, set $G'' = G - (u,v)$. If $\deg(u) > \deg(v)$, then irr$(G'') =$ irr$(G) + 2[\deg^{<}(v) + \deg^{=}(v) - \deg^{>}(u)]$. If $\deg(u) = \deg(v)$, then irr$(G'') =$ irr$(G) + [\deg(u) + \deg^{=}(u) - 1 - \deg^{>}(u)] + [\deg^{<}(v) + \deg^{=}(v) - 1 - \deg^{>}(v)]$.*

Bipartite Graphs

If G is bipartite, then G must be a subgraph of $K_{t,n-t}$ for some t. We assume that $1 \leq t \leq \frac{n}{2}$. In such a graph the maximum degree is $\leq n-t$.

The irregularity will be maximized by having as many edges as possible have the maximum possible imbalance. To this end choose t to be its minimum possible value given n and E. So if we fix r to be $r = \min\{j : j(n-j) \geq E\}$, it is straightforward to obtain

$$r = \left\lceil \frac{n - \sqrt{n^2 - 4E}}{2} \right\rceil .$$

We now construct $G_1 = G_1(n, E)$, a bipartite graph with n vertices, E edges, and large irregularity. Create r red vertices, say $x_1, x_2, \ldots, x_r$. Of these all but x_r will be adjacent to $y_1, y_2, \ldots, y_{n-r}$, and thus have degree $n - r$. The vertex x_r will be adjacent to $y_1, y_2, \ldots, y_s$ where $s = E - (r - 1)(n - r)$. Figure 1 shows $G_1(10, 18)$:

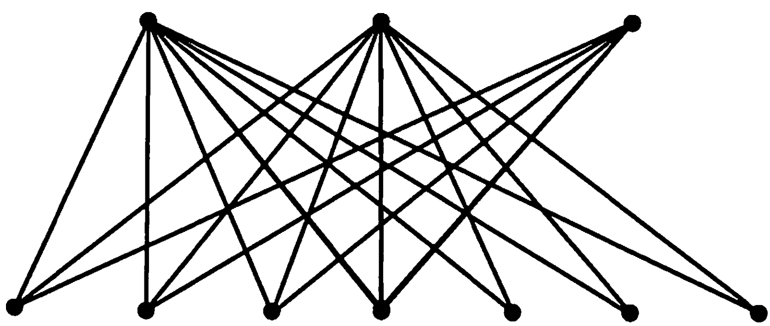

Figure 1

Proposition 7. *irr*$(G_1(n, E)) = (r-1)(n-r)^2 + s^2 - sr^2 - (r(n-r) - E)(r-1)^2$.

It is instructive to plot irr$(G_1(n, E))$ for fixed n and all possible values of E. Figure 2 exhibits such a discrete plot with $n = 20$. Each jump down represents an increase in r. Evidently the maximum occurs when the bipartite graph is complete. The smooth curve represents the function that extends irr to the interval $[0, \frac{n^2}{4}]$. A straightforward, symbolic computation shows that the maximum of this function is $\frac{n^3}{6\sqrt{3}}$ which occurs when $E = \frac{n^2}{6}$.

Concentrating for the moment on complete bipartite graphs it is immediate that irr$(K_{t,n-t}) = t(n-t)(n-2t)$ provided that $t \leq \frac{n}{2}$. This will be maximized when t is either the floor or ceiling of $\frac{n}{2}(1 - \frac{1}{\sqrt{3}})$. For those t, irr$(K_{t,n-t}) \approx \frac{n^3}{6\sqrt{3}}$. Thus we have:

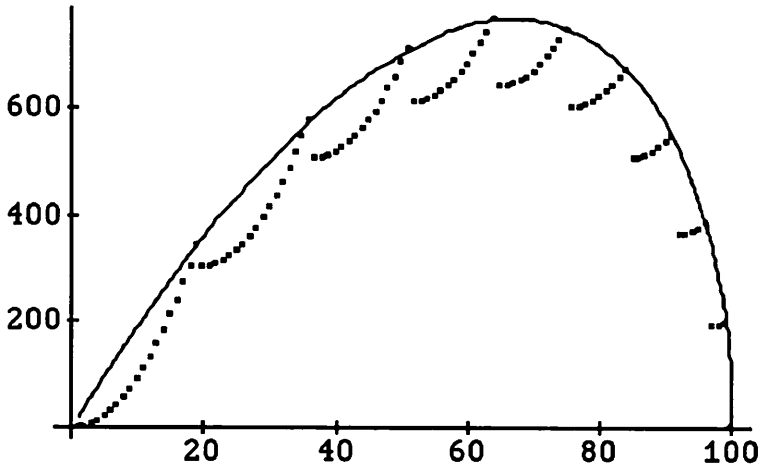

Figure 2

Proposition 8. *If G is bipartite, then* $irr(G) \le \frac{n^3}{6\sqrt{3}}$.

Furthermore, there exists a complete bipartite graph whose irregularity is arbitrarily close to this bound.

The preceding result relied heavily upon the assumption that G is bipartite. As we shall see there exist graphs whose irregularity is substantially larger than that of the complete bipartite graphs. Although we have been unable to find a triangle free graph with larger irregularity, all we can prove is:

Proposition 9. *If G is triangle free, then* $irr(G) < \frac{n^3}{9}$.

Proof: Suppose $\deg(v) = \delta$, the minimum degree in G. Let $G' = G - \{v\}$. Let $N(v)$ denote the set of v's neighbors. We can obtain the irregularity of G from that of G' by adding the imbalance of each edge incident with v and accounting for changes in the imbalance at the edges that are incident with the neighbors of v.

$$\mathrm{irr}(G) \le \mathrm{irr}(G') + \sum_{x \in N(v)} \left(|\deg(x) - \deg(v)| + \sum_{v \ne z \in N(x)} 1 \right).$$

Since G is triangle free, x and v cannot have a common neighbor. Thus $\deg(x) \le n-1-(\delta-1) = n-\delta$ and $|\deg(x)-\deg(v)| \le n-2\delta$. Consequently

$$\mathrm{irr}(G) \le \mathrm{irr}(G') + \delta(n-2\delta) + \delta(n-\delta-1) = \mathrm{irr}(G') + \delta(2n-3\delta-1).$$

If we inductively assume that $\mathrm{irr}(G') < c(n-1)^3$ for some constant c, then $\mathrm{irr}(G) < cn^3$ provided that $\delta(2n-3\delta-1) < c(3n^2-3n+1)$. Now

$\delta(2n - 3\delta - 1)$ will be maximized when $\delta = \frac{2n-1}{6}$. It is straightforward to check that $c = \frac{1}{9}$ is the smallest value that will support the induction.

The General Case

Let $H_{r,n}$ denote $K_r \vee I_{n-r}$ where I_m denotes an independent set with m vertices and $G_1 \vee G_2$ denotes the join, i.e. vertex disjoint copies of G_1 and G_2 together with every possible edge joining a vertex of G_1 and a vertex of G_2. Figure 3 shows $H_{3,7}$.

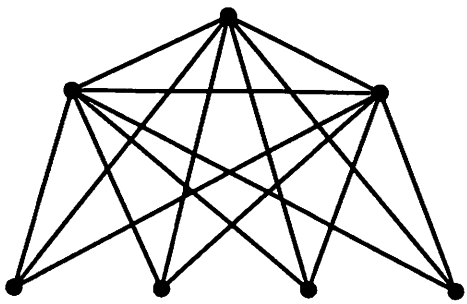

Figure 3

Proposition 10. *There exist integers r and n such that irr$(H_{r,n})$ is arbitrarily close to $\frac{4n^3}{27}$.*

Proof: By construction the only edges that contribute any imbalance are the join edges. There are $r(n-r)$ of these and each has imbalance $n-1-r$. Thus $\mathrm{irr}(H_{r,n}) = r(n-r)(n-r-1)$. It is straightforward to determine that for fixed n, $\mathrm{irr}(H_{r,n})$ will be maximized when r is either the floor or the ceiling of $\frac{2n-1-\sqrt{n^2-n+1}}{3}$. Plugging this value into $\mathrm{irr}(H_{r,n})$ for large n asymptotically yields $\frac{4n^3}{27}$.

In the graph $H_{r,n}$ let $x_2, x_2, \ldots, x_r$ denote the vertices of the clique and $y_2, y_2, \ldots, y_{n-r}$ denote the vertices of the independent set. Since $\deg(y_j) = \deg^<(y_i) = r$, the edge addition lemma guarantees that adding an edge to $H_{r,n}$ would decrease the irregularity. If $r \geq n-1$, then $H_{r,n} = K_n$. If $r < n-1$, then $\deg(x_i) > \deg(y_j)$. Since $\deg^<(y_j) = r$ and $\deg^>(x_i) = n-r$, the edge deletion lemma guarantees that deleting the edge (x_i, y_j) from $H_{r,n}$ would decrease the irregularity whenever $r < \frac{n}{2}$. Similarly, since $\deg^<(x_k) = 0$, $\deg^=(x_k) = r-1$, and $\deg^>(x_k) = n-r$, the edge deletion lemma guarantees that deleting the edge (x_i, x_j) would add $2(r-2)-2(n-r)$ to the irregularity of $H_{r,n}$. This will be negative precisely if $r < \frac{n}{2}+1$. This inspires the idea of a critical graph. A graph G is said to be *critical* if whenever G' is obtained from G by exactly one edge addition or deletion, then $\mathrm{irr}(G') \le \mathrm{irr}(G)$. We have just shown:

Proposition 11. *If $r \le \frac{n}{2}$, then $H_{r,n}$ is critical.*

Theorem 12. *If G is critical, then $G \cong H_{r,n}$ for some r.*

Proof: Suppose G is critical and $\deg(x) = \Delta(G)$, the maximum degree of G. If x is not adjacent to y, then $G' = G + (x, y)$ will have irregularity $\mathrm{irr}(G') = \mathrm{irr}(G) + 2[deg^{>}(x) + \deg^{=}(x) - \deg^{<}(y)]$. Since $\deg^{<}(y) < \Delta(G)$, $\mathrm{irr}(G') > \mathrm{irr}(G)$. Thus we may assume that $\Delta(G) = n - 1$. Now assume that the vertices of G have been labelled so that $\deg(x_1) = \cdots = \deg(x_r) = \Delta(G)$ and $\Delta(G) > \deg(u) \ge \deg(t)$ for every t in $V(G)$ that is not one of the x_i's. If $\deg(u) = r$, then $G \cong H_{r,n}$. Thus we may assume there exist v and z not among the x_i's such that (u, v) is in $E(G)$ but (u, z) is not. If no such v has $\deg(v) < \deg(u)$, then $\deg^{>}(u) = 0$. This, together with the Edge Deletion Lemma, implies that G is not critical. Thus we may assume that $\deg(u) > \deg(v)$. By the Edge Deletion Lemma applied to the edge (u, v), since G is critical

$$\deg^{>}(u) \ge \deg^{<}(v) + \deg^{=}(v) \ge r + 1.$$

By the Edge Addition Lemma applied to u and z,

$$\deg^{<}(z) \ge \deg^{>}(u) + \deg^{=}(u) \ge r + 1.$$

Thus there exists w not one of the x_i's such (w, z) in $E(G)$ and $\deg(w) > \deg(z)$. Once again we use edge deletion applied to (w, z) to see that

$$\deg^{>}(w) \ge \deg^{<}(z) + \deg^{=}(z) \ge \deg^{>}(u) + \deg^{=}(u).$$

If $\deg(w) > \deg(u)$, this contradicts how u was chosen. Alternatively, $\deg^{>}(w) = \deg^{>}(u) + \deg^{=}(u)$ and $\deg^{=}(w) = 0$. Consequently w is not adjacent to u. Adding the edge (u, w) to G increases the irregularity by $\deg^{>}(w) - \deg^{<}(u) = r + 1 - r > 0$, contradicting the assumption that G is critical.

Corollary 13. *For any graph G, $\mathrm{irr}(G) < \frac{4n^3}{27}$.*

Proof: Clearly a graph of maximum irregularity must be critical. From Theorem 12 such a graph is $H_{r,n}$ for some r. Thus

$$\begin{aligned} \mathrm{irr}(G) &\le \max\{\mathrm{irr}(H_{r,n})\} \\ &\le \max\{r(n - r)(n - r - 1)\} \\ &< \max\{r(n - r)^2\} = \frac{4n^3}{27}. \end{aligned}$$

References

[1] Y. Alavi, A. Boals, G. Chartrand, P. Erdös, and O. Oellermann, k-path irregular graphs, *Congressus Numerantium* **65** (1988), 201–210.

[2] Y. Alavi, G. Chartrand, F.R.K. Chung, P. Erdös, R.L. Graham, and O.R. Oellermann, Highly irregular graphs, *J. Graph Theory* **11** (1987), 235–249.

[3] M. Albertson and D. Berman, Ramsey graphs without repeated degrees, *Congressus Numerantium* **83** (1991), 91–96.

[4] G. Chartrand, K. Holbert, O. Oellermann, and H. Swart, F-degrees in graphs, *Ars Combinatoria* **24** (1987), 133–148.

[5] G. Chartrand, P. Erdös, and O. Oellermann, How to define an irregular graph, The *College Mathematics Journal* **19** (1988), 36–42.

[6] G. Chen, P. Erdös, C. Rousseau, and R. Schelp, Ramsey problems involving degrees in edge-colored complete graphs of vertices belonging to monochromatic subraphs, *European J. Combinatorics* **14** (1993), 183–189.

[7] D. Jackson and R. Entringer, Totally segregated graphs, *Congressus Numerantium* **55** (1986), 159–165.