Hamiltonian Graphs with Large Neighborhood Unions

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ABSTRACT. One of the fundamental results concerning cycles in graphs is due to Ore: If G is a graph of order $n \geq 3$ such that $d(x) + d(y) \geq n$ for every pair of nonadjacent vertices $x, y \in V(G)$, then G is hamiltonian. We generalize this result using neighborhood unions of k independent vertices for any fixed integer $k \geq 1$. That is, for $A \subseteq V(G)$, let $N(A) = \cup_{a \in A} N(a)$, where $N(a) = \{b : ab \in E(G)\}$ is the neighborhood of a. In particular we show: In a $4(k-1)$-connected graph G of order $n \geq 3$, if $|N(S)| + |N(T)| \geq n$ for every two disjoint independent vertex sets S and T of k vertices, then G is hamiltonian. A similar result for hamiltonian connected graphs is obtained too.

1 Introduction

Only finite simple graphs will be considered. In general $G = (V, E)$ will denote a graph with vertex set V and edge set E. Terminology will in general follow that used in the text of Bondy and Murty [1]. Given a graph G, a hamiltonian path of the graph is a path that contains every vertex of G. Similarly, a hamiltonian cycle of G is a cycle that contains every vertex of G. A graph G is *hamiltonian* if it contains a hamiltonian cycle. A graph

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is *hamiltonian connected* if there is a hamiltonian path between any two distinct vertices. In this paper, $N(x)$ denotes the open neighborhood of a vertex x, which we generalize to a subset A of vertices. Let A and B be two vertex subsets of G. We define

$$N(A) = \cup_{a\in A}N(a) \quad \text{and} \quad N_B(A) = N(A) \cap B.$$

One of the oldest results giving sufficient conditions for a graph to be hamiltonian was given by Dirac.

Theorem 1 (Dirac [7]) *If G is a graph of order $n \geq 3$ such that the minimum degree $\delta(G) \geq n/2$, then G is hamiltonian.*

Since Dirac published this theorem, the approach for developing sufficient conditions for a graph to be hamiltonian usually involved generalized degrees of a graph. Ore relaxed the condition in Dirac's theorem and obtained the following.

Theorem 2 (Ore [15]) *If G is a graph of order $n \geq 3$ such that $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices u, $v \in V$, then G is hamiltonian.*

Recently, several papers have explored the effect of various neighborhood union conditions for hamiltonian graphs, beginning with the next result.

Theorem 3 (Faudree, Gould, Jacobson, Schelp [9]) *Let G be a 2-connected graph of order n. If for every pair of distinct nonadjacent vertices u and v*

$$|N(u) \cup N(v)| \geq (2n-1)/3,$$

then G is hamiltonian.

The above three results are generalized in [3].

Theorem 4 (Chen [3]) *Let G be a 2-connected graph of order n. If*

$$d(u) + d(v) + 2|N(u) \cup N(v)| \geq 2n - 1$$

for every pair of nonadjacent vertices u and v, then G is hamiltonian.

Later there are many stronger results are obtained in [4], [5], [10], [17]. The graph $K_2 + 3K_p$ illustrates that $(2n-1)/3$ in Theorem 3 is, in some sense, best possible. However, the following three theorems show that the $(2n-1)/3$ can be lowered considerably under some circumstances.

Theorem 5 (Faudree, Gould, Jacobson, and Lesniak, [8]) *If G is a 2-connected graph of sufficiently large order n such that $|N(u) \cup N(v)| \geq n/2$ for all distinct u and $v \in V(G)$, then G is hamiltonian.*

Theorem 6 (Jackson [14]) *Let G be a 3-connected graph of order n. If $|N(u) \cup N(v)| \geq (n+1)/2$ for any pair of nonadjacent vertices, then G is hamiltonian.*

Theorem 7 (Broersma, Van Den Heuvel, and Veldman [2]) *Let G be a 3-connected graph of order n. If $|N(u) \cup N(v)| \geq n/2$ for every pair of nonadjacent vertices u and v, then G is either hamiltonian or the Petersen graph.*

Fraisse extended Theorem 3 from two nonadjacent vertices to larger independent sets of vertices.

Theorem 8 (Fraisse [11]) *Let G be a k-connected graph of order n. If for every independent vertex set S of cardinality k,*

$$|N(S)| > \frac{k(n-1)}{k+1},$$

then G is hamiltonian.

The graph $K_k + (k+1)K_p$ illustrates that the above result is, in some sense, best possible. Notice that the connectivity of $K_k + (k+1)K_p$ is k. It is natural to ask under what circumstances the bound $\frac{k(n-1)}{k+1}$ can be lower to $n/2$. In particular, the following question is asked.

Question 1 *For any positive integer k, does there exist a positive integer $f(k)$ such that: for any $f(k)$-connected graph G of order $n \geq 3$, if $|N(S)| \geq n/2$ for every independent set S of k vertices, then G is hamiltonian?*

Notice that the complete bipartite graphs $K_{p,p+1}$ show that bound $n/2$ is best possible if the answer of the question is a positive one. The graph $K_{2k-2} + (2k-1)K_p$ with $p \geq 1$ illustrates that $f(k) \geq 2k-1$ if such a $f(k)$ exists. The following theorem affirms that such a $f(k)$ exists and it is bounded above by $4(k-1)$ for every positive integer k.

Theorem 9 *Let k be a positive integer and G be a $4(k-1)$-connected graph of order $n \geq 3$. If $|N(S)| \geq n/2$ for every independent set of k vertices, then G is hamiltonian.*

In fact we will prove the following stronger result which generalizes Ore's theorem on hamiltonian cycles.

Theorem 10 *Let k be a positive integer and G be a $4(k-1)$-connected graph of order $n \geq 3$. If*

$$|N(S)| + |N(T)| \geq n$$

for every two disjoint independent sets S and T of k vertices, then G is Hamiltonian.

When $k = 1$, the above result generalizes Theorem 2. When $k = 2$, the Petersen graph is an example showing that 4-connected is necessary in some sense. We will also prove two similar results on hamiltonian paths and hamiltonian connected graphs respectively listed below.

Theorem 11 *Let k be a positive integer and G be a $(4k-5)$-connected graph of order n. If*

$$|N(S)| + |N(T)| \geq n - 1$$

for every two disjoint independent sets S and T of k vertices, then G contains a hamiltonian path.

Theorem 12 *Let k be a positive integer and G be a $(4k-3)$-connected graph of order n. If*

$$|N(S)| + |N(T)| \geq n + 1$$

for every two disjoint independent sets S and T of k vertices, then G is hamiltonian connected.

Clearly, Theorem 11 follows directly from Theorem 10. When $k = 1$, Theorem 12 generalizes a result of Ore [16] that if $d(x) + d(y) \geq n + 1$ for each pair of nonadjacent vertices x and y, then G is hamiltonian connected. For the case $k = 2$, Gould and Yu recently proved the following result which is independent to above result.

Theorem 13 (Gould and Yu [13]) *Let G be a 3-connected graph of order n, if*

$$|N(S)| + |N(T)| \geq n + 1$$

for each pair of distinct vertex sets of two vertices, then G is hamiltonian connected.

2 Insertible Vertices On Maximal Cycles

Let G be a graph. We assume that all cycles and paths of G are given with a fixed orientation. For a path (or a cycle) X of G, we let X^- denote X with the reverse orientation. For u, $v \in V(X)$, let $X[u, v]$ denote the subpath of X from u to v. We also use $X[u, v]$ to denote $V(X[u, v])$ when no confusion can arise. Set $X(u,v] = X[u,v] - \{u\}$ and define $X[u,v)$ and $X(u,v)$ similarly. For $S \subseteq V(X)$, let S^+ (S^-) denote the set of successors (predecessors) of vertices of S in X and let $G(S)$ denote the subgraph of G induced by the vertices of S. A uv-path of G is a path of G connecting u and v with the fixed orientation from u to v. Let B be a connected subgraph of G and let u and v be two vertices of B. Then uBv will denote an arbitrary uv-path in B. A maximal cycle C of G is a cycle such that no other cycle in G contains all of vertices of C as a proper subset of vertices.

Let G be a given nonhamiltonian graph of order n and C be a maximal cycle of G with an orientation. Let H be a connected component of $G - V(C)$ and let $\{v_1, v_2, \cdots, v_h\}$ be h vertices in $N_C(H)$ such that $x_i v_i \in E(G)$ where $x_i \in H$ for $1 \leq i \leq h$. We also assume that $v_1, v_2, \cdots, v_h$ are labeled in the order along the orientation of C, that is $v_i \in C(v_{i-1}, v_{i+1})$. The vertices $v_1, v_2, \cdots, v_h$ divides the cycle C into h segments,

$$Q_i = C(v_i, v_{i+1}] = w_{i1} w_{i2} \cdots w_{iq_i} v_{i+1} \quad \text{for} \quad 1 \leq i \leq h,$$

where the subscripts of v_{i+1} are taken modulo h.

Motivated by the algorithm used by C. Q. Zhang in [18], we define the insertible vertices as follows. A vertex $w_i \in Q_i$ is called an insertible vertex if there are a pair of consecutive vertices w and $w^+ \in C - Q_i$ such that $w_i w$, $w_i w^+ \in E(G)$. If w is an insertible vertex, we define that $I(w_i)$ be the vertex in $C - Q_i \cup \{v_i\}$ such that $w_i I(w_i)$, $w_i(I(w_i))^+ \in E(G)$ and $|C[w, I(w_i)]|$ is as large as possible.

Suppose that $w_{i1}, w_{i2}, \cdots, w_{i\alpha}$ are insertible vertices. Let β_1 be the largest integer in $[1, \alpha]$ such that $I(w_{i1}) = I(w_{i\beta_1})$, and β_2 be the largest integer in $[\beta_1 + 1, \alpha]$ such that $I(w_{i\beta_1+1}) = I(w_{i\beta_2})$, $\cdots$, $\beta_t = \alpha$. Then we insert the segment $C[w_{i1}, w_{i\beta_1}]$ between $I(w_{i1})$ and $(I(w_{i\beta_1}))^+$, the segment $C[w_{i\beta_1+1}, w_{i\beta_2}]$ between $I(w_{i\beta_1+1})$ and $(I(w_{i\beta_1+1}))^+$, $\cdots$, the segment $C[w_{i\beta_{t-1}+1}, w_{i\beta_t}]$ between $w_{i\beta_{t-1}+1}$ and $(w_{i\beta_{t-1}+1})^+$. Since we will use such insertion very often, we call such insertion a *segment insertion* and denote the insertion by $SI[C[w_{i1}, w_{i\alpha}]]$.

Lemma 1 *For each Q_i there is a non-insertible vertex in $Q_i - \{v_{i+1}\}$.*

Proof: We assume, to the contrary, that all vertices in $Q_i - \{v_{i+1}\}$ are insertible. Using the segment insertion $SI[w_{i1}, w_{iq_i}]$, we obtain a $v_{i+1}v_i$-path $P[v_{i+1}, v_i]$ with $V(P[v_{i+1}, v_i]) = V(C)$. The existence of the cycle $v_i x_i H x_{i+1} v_{i+1} P[v_{i+1}, v_i]$ contradicts the maximality of C. □

For each $1 \leq i \leq k$, let t_i be the smallest integer such that w_{it_i} is not an insertible vertex and let $S_i = \{w_{i1}, w_{i2}, \cdots, w_{it_i}\}$. Notice that from Lemma 1, $S_i \cap N_C(H) = \phi$. Further, we have

Lemma 2 *Let $1 \leq i \neq j \leq h$ be two distinct integers. Then for any $1 \leq s_i \leq t_i$ and $1 \leq s_j \leq t_j$, the following two properties hold.*

I There does not exist a path $R[w_{is_i}, w_{js_j}]$ *such that*

$$R[w_{is_i}, w_{js_j}] \cap C = \{w_{is_i}, w_{js_j}\}.$$

II For every $w \in C[w_{is_i}^+, w_{js_j}]$, *if* $ww_{is_i} \in E(G)$, *then* $w^- w_{js_j} \notin E(G)$. *Similarly, for any* $w \in C[w_{js_j}, w_{is_i}]$, *if* $ww_{js_j} \in E(G)$, *then* $w^- w_{is_i} \notin E(G)$.

Proof: We prove this lemma by induction on $s_i + s_j$. For $s_i = s_j = 1$, Clearly $w_{is_i} = v_i^+$ and $w_{js_j} = v_j^+$. By standard argument on maximal cycles in a nonhamiltonian graph, we see that both **I** and **II** hold. Assume that both **I** and **II** are true for any pair of $r_i + r_j < s_i + s_j$ with $1 \leq r_i \leq s_i$ and $1 \leq r_j \leq s_j$.

From the induction hypothesis on **I**, there is no edges between the vertex sets $\{w_{i1}, w_{i2}, \cdots, w_{is_i}^-\}$ and $\{w_{j1}, w_{j2}, \cdots, w_{js_j}\}$. Thus no vertices in $\{w_{i1}, w_{i2}, \cdots, w_{is_i}^-\}$ are inserted between any pair of vertices in $\{w_{j1}, w_{j2}, \cdots, w_{js_j}\}$ when we use the segment insertion $SI[C[w_{i1}, w_{is_i}]]$. Using the segment insertion $SI[C[w_{i1}, w_{is_i}^-]]$, we obtained a path $P[w_{is_i}, v_i]$ such that $V(P[w_{is_i}, v_i]) = V(C)$. From the induction hypothesis on **II**, $I(w_{jr_j}) \neq I(w_{ir_i})$ for any $1 \leq r_i \leq s_i - 1$ and $1 \leq r_j \leq s_j - 1$. Then $I(w_{ir_j})^+$ is also the immediate successor of $I(w_{ir_j})$ on the path $P[w_{is_i}, v_i]$. Using the segment insertion $SI[C[w_{j1}, w_{js_j}^-]]$, we can insert $w_{j1}, w_{j2}, \cdots, w_{js_j}^-$ in $P[w_{is_i}, v_j]$ or $P[w_{js_j}, v_i]$ to obtained two vertex disjoint paths $Q[w_{is_i}, v_j]$ and $Q[w_{js_j}, v_i]$ such that $V(Q[w_{is_i}, v_j]) \cup V(Q[w_{js_j}, v_i]) = V(C)$ and for any pair of two consecutive vertices w and w^- of $C - C[w_{i1}, w_{is_i}^-] \cup C[w_{j1}, w_{js_j}^-]$, one and only one the following three properties holds.

- w and w^- are two consecutive vertices on one of the paths $Q[w_{is_i}, v_j]$ and $Q[w_{js_j}, v_i]$,
- there is a segment $C[w_{ir_i}, w_{ir_i^*}]$ inserted between w and w^- with $N(w_{ir_i}) \cap N(w_{ir_i^*}) \supseteq \{w, w^-\}$,
- there is a segment $C[w_{jr_j}, w_{jr_j^*}]$ inserted between w and w^- with $N(w_{jr_j}) \cap N(w_{jr_j^*}) \supseteq \{w, w^-\}$.

To prove property **I**, we assume, to the contrary, there is a path $R[w_{is_i}, w_{js_j}]$ such that $R[w_{is_i}, w_{js_j}] \cap V(C) = \{w_{is_i}, w_{js_j}\}$. By Lemma 1, $R[w_{is_i}, w_{js_j}] \cap V(H) = \phi$. Then the existence of the cycle

$$Q[w_{is_i}, v_j]v_j x_j H x_i v_i Q^-[v_i, w_{js_j}]R[w_{js_j}, w_{is_i}]$$

contradicts the maximality of C.

To prove property **II**, we assume, again to the contrary, there are two consecutive vertices w and w^- in $C[v_{is_i}^+, v_{js_j}]$ such that $ww_{is_i} \in E(G)$ and

$w^- w_{js_j} \in E(G)$. Because of $ww_{is_i} \in E(G)$, by our induction hypothesis, we have $w^- w_{jr_j} \notin E(G)$ for every $1 \leq r_j \leq s_j - 1$. Hence no vertices in $C[w_{j1}, w_{js_j}^-]$ are inserted between w and w^-. In the same manner, we can show that no vertices in $C[w_{i1}, w_{is_i}^-]$ are inserted between w and w^-. Thus w and w^- are two consecutive vertices on the path $Q[w_{is_i}, v_j]$. The existence of the cycle

$$Q[w_{is_i}, w^-]w^- w_{js_j} Q[w_{js_j}, v_i]v_i x_i H x_j v_j Q^-[v_j, w]ww_{is_i}$$

contradicts the maximality of C. □

Let $w_i = w_{it_i}$ for $1 \leq i \leq h$ and let $W = \{w_1, w_2, \cdots, w_h\}$. If $h \geq k$, we let $W_i = \{w_i, w_{i-1}, \cdots, w_{i-k+1}\}$ for $1 \leq i \leq h$, where the subscripts are taken modulo h.

Lemma 3 *If $h \geq max\{4(k-1), 2\}$, then there is an i_0 such that*

$$|N(W_{i_0})| + |N(W_{i_0+k})| \leq |V(G)| - 1.$$

Proof: It is readily seen that $h \geq max\{4(k-1), 2\} \geq 2k$. To show the above inequality, we only need to show that

$$\sum_{i=1}^{h}(|N(W_i)| + |N(W_{i+k})|) < h|V(G)|,$$

which is equivalent to show that

$$\sum_{i+1}^{h}|N(W_i)| < \frac{h}{2}|V(G)|.$$

We define an additive weight function τ on $V(G)$ such that

$$\tau(v) = |\{i \ : \ x \in N(W_i)\}| \quad \text{and} \quad \tau(A) = \sum_{a \in V(A)} \tau(a)$$

for every $v \in V(G)$ and every subgraph A of G. Clearly, $\tau(v) = 0$ if and only if $v \notin N(W)$. Notice that

$$\sum_{v \in V(G)} \tau(v) = \sum_{i=1}^{h} |N(W_i)|.$$

We will show that $\sum_{v \in V(G)} \tau(v) < \frac{h}{2}|V(G)|$.

Since $N(W) \cap V(H) = \phi$, $\tau(v) = 0$ for each $v \in V(H)$. Thus

$$\tau(H) = 0 \tag{1}$$

For each $v \in V(G - C \cup H)$, by **I** of Lemma 2, $|N(v) \cap W| \leq 1$. Then $\tau(v) \leq k$. Thus

$$\tau(G - C \cup H) \leq k|V(G - C \cup H)|. \tag{2}$$

We will show that

$$\tau(C) \leq \frac{h}{2}|V(C)|. \tag{3}$$

Notice that, from (1), (2), and (3), we have

$$\begin{aligned} \tau(G) &= \tau(C) + \tau(H) + \tau(G - C \cup H) \\ &\leq \frac{h}{2}|V(C)| + k|V(G - C \cup H)| \\ &\leq \frac{h}{2}(|V(G)| - |V(H)|) < \frac{h}{2}|V(G)| \end{aligned}$$

since $h \geq 2k$. Thus we only need to prove (3) to complete the proof.

Since W is an independent vertex set, it is readily seen that the cycle C is a disjoint union of intervals $T = C[a, c]$ with $a, c^+ \notin N(W)$ and $C(a, c] \subseteq N(W)$. Notice that $C(a, c] = \phi$ if $a = c$. Such intervals are called $W-segments$. To show that $\tau(C) \leq \frac{h}{2}|V(C)|$, we only need to prove that $\tau(T) \leq \frac{h}{2}|V(T)|$ for each W-segment T.

Let T be a W-segment. From **I** of Lemma 2, there is i such that $T \subseteq C[w_i, w_{i+1})$. Without loss of generality, we assume that $T \subseteq C[w_h, w_1)$. By **II** of Lemma 2,

$$N(w_h),\ N(w_{h-1}),\ \cdots,\ N(w_2),\ N(w_1)$$

form consecutive closed subintervals of T (possibly some of them are empty) which can only have their endvertices in common. Further, we assume that

$$T = \{a, b_1, b_2, \cdots, b_s,\ c_1,\ c_2,\ \cdots,\ c_t,\ d_1,\ \cdots,\ d_r\}$$

such that

- $a \notin N(W)$,
- $\{b_1, \cdots, b_s\} = \phi$ if $|N(w_h) \cap T| \leq 1$,
- c_1 is the last vertex of T, along the orientation of C, adjacent to w_h if $|N(w_h) \cap T| \geq 2$,
- $\{d_1, \cdots, d_r\} = \phi$ if $|N(w_1) \cap T| \leq 1$,
- c_t is the first vertex of T, along the orientation of C, adjacent to w_1 if $|N(w_1) \cap T| \geq 2$,

Notice that $\{d_1, \cdots, d_r\} \subseteq C[v_1, w_1^-]$ since w_1 is a non-insertible vertex. Clearly, $|V(T)| = s + t + r + 1$ and

$$\begin{aligned}\sum_{i=1}^{s} \tau(b_i) &= ks \leq \frac{hs}{2}, \\ \sum_{i=1}^{r} \tau(d_i) &= kr \leq \frac{hr}{2}.\end{aligned}$$

Since $w_1, w_2, \cdots, w_h$ are not insertible vertices and $T \subsetneq C[w_h, w_1)$, it follows that $|N(w_i) \cap C[c_1, c_t]| \leq 1$ for each $1 \leq i \leq h$. For $1 \leq i \leq t$, let

$$\alpha_i = \max\{j \ : \ c_i w_j \in E(G)\} \quad \text{and} \quad \beta_i = \min\{j \ : \ c_i w_j \in E(G)\}.$$

Then we have

$$h \geq \alpha_1 \geq \beta_1 > \alpha_2 \geq \beta_2 > \cdots > \alpha_t \geq \beta_t.$$

By the definition of $W_1, W_2, \cdots, W_h$, we have $c_i \in N(W_j)$ only if $j \in \{\alpha_i + k - 1, \alpha_i + k - 2, \cdots, \beta_i\}$. Hence $\tau(c_i) \leq \alpha_i - \beta_i + k$. Thus

$$\sum_{i=1}^{t} \tau(c_i) \leq \sum_{i=1}^{t} (\alpha_i - \beta_i) + tk \leq \sum_{i=1}^{t} (\alpha_i - \beta_i + 1) - t(k-1) \leq h + t(k-1).$$

If $t \geq 2$, then $\frac{2t}{t-1} \leq 4$. Then we have

$$\sum_{i=1}^{t} \tau(c_i) \leq h + t(k-1) \leq h + \frac{t-1}{2}\frac{2t}{t-1}(k-1) \leq h + \frac{t-1}{2}h \leq \frac{h}{2}(t+1).$$

If $t = 1$, $\tau(c_1) \leq h = \frac{h}{2}(t+1)$. Hence, in any case, we have

$$\sum_{i=1}^{t} \tau(c_i) \leq \frac{h}{2}(t+1).$$

Thus

$$\begin{aligned}\tau(T) &= \sum_{i=1}^{s} \tau(b_i) + \sum_{j=1}^{t} \tau(c_j) + \sum_{\ell=1}^{r} \tau(d_\ell) \\ &\leq \frac{h}{2}(t+1) + \frac{h}{2}s + \frac{h}{2}r = \frac{h}{2}(s+t+r+1) = \frac{h}{2}|V(T)|,\end{aligned}$$

which completes our proof of Lemma 3. $\square$

3 Proofs of Theorems 10 and 12

3.1 Proof of Theorem 10

Let G be a $4(k-1)$-connected nonhamiltonian graph of order $n \geq 3$ such that

$$|N(X)| + |N(Y)| \geq n$$

for each pair of two disjoint vertex sets X and Y with $|X| = |Y| = k$.

If $k = 1$, the condition $d(x) + d(y) \geq n$, for any pair of nonadjacent vertices, implies that G is a 2-connected graph. If $k \geq 2$, $4(k-1) \geq 2k \geq 2$. Hence G is a 2-connected graph.

Let C be a maximal cycle of G and H be a connected component of $G - C$. Clearly, $|N_C(H)| \geq \max\{2, 4(k-1)\}$. Hence by Lemma 3, there are two disjoint vertex sets X_0 and Y_0 with $|X_0| = |Y_0| = k$ such that $|N(X_0)| + |N(Y_0)| \leq |V(G)| - 1$, a contradiction. □

3.2 Proof of Theorem 12

Let G be a $(4k-3)$-connected graph of order n such that $|N(X)|+|N(Y)| \geq n+1$ for every pair of two disjoint vertex sets X and Y of k vertices. If $k = 1$, the condition $d(x)+d(y) \geq n+1$ for any pair of nonadjacent vertices implies that G is 3-connected. If $k \geq 2$, $4k-3 \geq 2k+1 \geq 3$. Hence G is a 3-connected graph.

Suppose that, to the contrary, G is not hamiltonian connected. There exist a pair of vertices x_0 and y_0 such that there is no x_0y_0-path containing all vertices of G. Let G^* be a new graph obtained from G by adding a new vertex u_0 and two new edges u_0x_0 and u_0y_0. Clearly, G^* is not hamiltonian. Note that the neighborhood of v in G and the neighborhood in G^* are same for every $v \in V(G) - \{x_0, y_0\}$. We use $N(v)$ to denote both the neighborhood of v in G and the neighborhood of v in G^* . In the same manner, we use $N(A)$ for each $A \subseteq V(G) - \{x_0, y_0\}$. Let C be a longest cycle containing u_0 in G^*. Clearly, C is a maximal cycle in G^*. Let H be a connected component of $G^* - C$ and $h = \max\{4(k-1), 2\}$. Since G is an $h+1$-connected graph, $|N_C(H)| \geq h+1$. Let $\{v_1, v_2, \cdots, v_h, v_{h+1}\} \subseteq N_C(H)$ such that $u_0 \in C(v_{h+1}, v_1]$. Let w_i be the first non-insertible vertex in $C(v_i, v_{i+1}]$ for $1 \leq i \leq h+1$. Clearly, $w_i \notin \{u_0, x_0, y_0\}$ for every $1 \leq i \leq h$. Denote

$$\begin{aligned} W &= \{w_1, w_2, \cdots, w_h\} \quad \text{and} \\ W_i &= \{w_i, w_{i-1}, \cdots, w_{i-k+1}\} \quad \text{for} \quad 1 \leq i \leq h, \end{aligned}$$

where the subscripts are taken modulo h. By Lemma 3 there is an $1 \leq i_0 \leq h$ such that

$$|N(W_{i_0})| + |N(W_{i_0-k})| \leq |V(G^*)| - 1 = n,$$

a contradiction. □

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