On a Class of Graphic Matrices*

R.P. Swaminathan
Department of Computer Science
University of Cincinnati
Cincinnati, OH 45221–0008
email: swamy@jupiter.csm.uc.edu.

Dedicated to Rajasree

ABSTRACT. A $\{0,1\}$-matrix M is *tree graphic* if there exists a tree T such that the edges of T are indexed on the rows of M and the columns are the incidence vectors of the edge sets of paths of T. Analogously, M is *ditree graphic* if there exists a ditree T such that the directed edges of T are indexed on the rows of M and the columns are the incidence vectors of the directed-edge sets of dipaths of T. In this paper, a simple proof of an excluded-minor characterization of the class of tree-graphic matrices that are ditree-graphic is given. Then, using the same proof technique, a characterization of a "special" class of tree-graphic matrices (which are contained in the class of consecutive 1's matrices) is stated and proved.

1 Introduction

A standard graph theory reference is Bondy and Murty [1]. Throughout, if G is a graph, then $V(G)$ denotes its vertex set and $E(G)$ denotes its edge set. Moreover, for convenience trees and cycles are equated with their edge sets.

A $\{0,1\}$-matrix M is *tree graphic* if there exists a tree T such that the edges of T are indexed on the rows of M and the columns are the incidence vectors of the edge sets of paths of T. If such a tree T exists, then T is a *tree realization* for M. (Note that not every path of T need correspond to a column of M.) A M-*path* of T is a path in T, the incidence vector of the edge set of which is a column of M.

*This work has been supported in part by National Sciences Foundation Grant CCR–93–09470.

Next, by replacing edges with directed edges in the above definition, the class of ditree (ie., directed tree) graphic matrices is defined as follows. A $\{0,1\}$-matrix M is *ditree graphic* if there exists a ditree T such that the directed edges of T are indexed on the rows of M and the columns are the incidence vectors of the directed-edge sets of dipaths of T. If such a ditree T exists, then T is a *ditree realization* for M.

Tree-graphic and ditree-graphic matrices have a wide variety of applications. For instance, the tree-graphic class arises from topological analysis of electrical networks and identifying network structures in linear programming problems, and the ditree-tree graphic class arises from satisfiability in propositional logic, information-storage-retrieval, and network-reliability problems. Moreover, testing whether a given $\{0,1\}$-matrix is tree (ditree) graphic and if so, constructing a tree (ditree) realization can be done in "almost-linear" (in the number of non-zero entries) time. See Bixby and Wagner [2], and Swaminathan and Wagner [5], [6], [7] for more details.

A $\{0,1\}$-matrix M is a *consecutive 1's matrix*, abbreviated *C1M*, if for some permutation of its rows and columns, the 1's in every column are arranged consecutively. Observe that if M is a C1M, then M is tree (ditree) graphic and some tree (ditree) realization of M is a path (dipath). In [8], Tucker gave an excluded-minor characterization of C1Ms. In this paper, first, a simple proof of an excluded-minor characterization of the class of tree-graphic matrices that are ditree-graphic is given. Then, using the same proof technique, Tucker's characterization [8] when restricted to a "special" class of tree-graphic matrices that are C1Ms is stated and a short proof is given.

2 Ditree–Graphic Matrices

A *minor* of a $\{0,1\}$-matrix is a submatrix obtained by deleting subsets (possibly empty) of its row and column sets. A *wheel-matrix*, denoted W_n, for $n \geq 3$, is a $n \times n$ $\{0,1\}$-matrix having exactly two non-zero entries in every row and column such that no two rows or columns are identical. Observe that the wheel-matrix W_n, for odd $n \geq 3$, is tree graphic and is not ditree graphic. Furthermore, it is interesting to note that for every $n \geq 3$, W_n is not a C1M and that Tucker's characterization [8] of C1Ms using a set of five excluded minors contains W_n.

Theorem 1 below is the first main result of the paper. It was proved by Bland and Ko [3], and independently by Swaminathan and Wagner in an unpublished report [5] with extensions to matroids and totally unimodular matrices. The alternate proof given here is short and simple.

Theorem 1. *A tree-graphic matrix M is ditree graphic if and only if M does not have the wheel-matrix W_n, for any odd $n \geq 3$, as a minor.*

Proof: One half of the theorem is easy. In particular, suppose that a tree-graphic matrix M is ditree graphic. It is easily verified that if M is ditree graphic, then so is every minor of M, and that no wheel-matrix W_n, for odd $n \geq 3$, is ditree graphic. Thus, M does not have W_n for any odd n, as a minor.

Now consider the other half of the theorem. That is, suppose a given tree-graphic matrix M has no wheel-matrix W_n, for odd $n \geq 3$, as a minor. Let T be a tree realization of M. For every vertex v of T, define a graph $T(v)$ as follows. The vertex set of $T(v)$ is the set of vertices of T that are adjacent to v by an edge of T; two vertices of $T(v)$, say u_1 and u_2, are adjacent in $T(v)$ if there exists a M-path in T containing the edges u_1v and u_2v.

First, consider the case when for some vertex v of T, the graph $T(v)$ is non-bipartite. Then, $T(v)$ has an odd cycle (odd number of edges) C. Let $u_1, \dots, u_k$ be the vertex set of C, and let e_i be the edge of T that joins u_i and v. Without loss of generality, assume u_i and u_{i+1} are adjacent in C, with subscripts taken modulo k. By the definition of $T(v)$, for each i, there exists a M-path which contains e_i and e_{i+1} (mod k). Now a wheel-matrix W_n for some odd n, is obtained from M by deleting the set of rows that corresponds to the edges in $T - \{e_1, \dots, e_k\}$, a contradiction.

Next, consider the case when for each vertex v of T, the graph $T(v)$ is bipartite. Then, for each v of T, the vertices of T adjacent to v are partitioned into two sets. This induces a partition of the edges of T incident to v such that any two such edges that are in a M-path are in different members of the partition.

Choose a vertex v of T, and assign directions to the edges of T incident to v so that all of the edges of one member of the partition are directed into v and all of the edges of the other member of the partition are directed out of v. Next choose an edge $e \in T$ that is incident to v, and let u be the other end of e. Now assign directions to the edges of T incident to u in an analogous manner with the restriction that u and v induce a consistent direction on e. Continuing this procedure for each vertex of T yields a direction on all of the edges of T.

The assignment of directions to the edges of T constructed above proves T is a ditree realization of M as follows. Consider a M-path P of T. Let e_1 and e_2 be adjacent edges of T in P, and let v be the common end of e_1 and e_2. Thus, e_1 and e_2 are in different members of the partition associated with v. Therefore, by the above construction, one of e_1 and e_2 is directed into v and the other is directed out of v. It follows that P is a dipath, and so T is a ditree realization of M. That is, M is ditree graphic, as required. □

A *wheel-graph*, denoted W_n, for $n \geq 3$, is a graph with vertex set $\{v_0, v_1, \ldots, v_n\}$ and edge set $\{e_1, \ldots e_n, f_1, \ldots f_n\}$ such that $e_i = v_0 v_i$ and $f_i = v_i v_{i+1}$ (mod n). A *minor* of a connected graph G is a subgraph obtained by deleting a subset and contracting a disjoint subset of $E(G)$.

Let G be a connected graph and T be a spanning tree of G. Then, the pair (G, T) is a *gt-pair*. Every edge in $E(G) - T$ induces a unique cycle in G, and is called a *fundamental cycle* of (G, T). If M is a tree-graphic matrix and T is a tree realization of M, then T can be extended to a graph G by adding a unique edge between the two ends of every M-path of T. Clearly, the resulting pair (G, T) is a gt-pair, and is referred to as a *gt-realization* of M. Observe that every row or column of M corresponds to a unique edge in G. Therefore, for any $n \geq 3$, if the wheel-matrix W_n is a minor of M obtained by deleting the set of rows I and columns J, then the wheel-graph W_n is a minor of G obtained by contracting the set of edges of T corresponding to I and deleting the set of edges of $E(G) - T$ corresponding to J.

A gt-pair (G, T) is *orientable* if for some assignment of directions to the edges of G, every fundamental cycle of (G, T) becomes a dicycle. Since the proof of Theorem 1 implies that M is ditree graphic (given that M is tree graphic) if and only if any gt-realization of M is orientable, it follows that any gt-realization (G, T) of M is orientable if and only if G has no wheel-graph W_n, for odd $n \geq 3$ as a minor, obtained by contracting a subset of T and deleting a subset of $E(G) - T$.

Let $G = (V, E)$ be a connected graph and let F be a non-empty subset of edges. The subgraph of G induced by F is denoted by $G[F]$. Let $\{E_1, E_2\}$ be a partition of E. For $k > 0$, the partition $\{E_1, E_2\}$ is a *k-separation* of G if $|E_1| \geq k \leq |E_2|$, and $|V(G[E_1]) \cap V(G[E_2])| = k$. For a positive integer n, the graph G is *n-connected* if it has no k-separation for $k < n$.

Corollary 2 below is an extension of Theorem 1. It can also be viewed as a characterization of 2-connected series-parallel graphs (graphs obtained from two parallel edges on two vertices by subdividing edges with a new vertex and adding parallel edges, repeatedly). See Purdy and Swaminathan [4] for a proof and related characterizations of series-parallel graphs.

Corollary 1. *For every spanning tree T of a 2-connected graph G, the gt-pair (G, T) is orientable if and only if G does not have W_3 as a minor.* □

3 Consecutive 1's Matrices

An *arrow* is a tree of four edges with one degree-3 vertex, one degree-2 vertex and three degree-1 vertices. Consider the gt-pair $(W_4, \hat{T}_4)$ where W_4 is wheel on four vertices and $\hat{T}_4$ is an arrow, and define $\hat{W}_4$ as a tree-graphic matrix whose gt-realization is $(W_4, \hat{T}_4)$. Observe that $\hat{W}_4$ is a non-C1M.

A tree-graphic matrix M is 3-*connected* if the graph G of any gt-realization

(G, T) of M is 3-connected. In this case, using a theorem of Whitney [9], it can be shown that (G, T) is unique for M. The details are omitted.

Theorem 3 below is the second and final result of the paper. It is precisely Tucker's excluded-minor characterization [8] of C1Ms when restricted to 3-connected tree-graphic matrices. See Tucker [8] for more details.

Theorem 2. *A 3-connected tree-graphic matrix M is a C1M if and only if M does not have $\hat{W}_4$ or W_n, for any $n \geq 3$, as a minor.*

Proof: One half of the theorem is easy. Namely, if M is a C1M, then since every minor of a C1M is also a C1M and the matrices $\hat{W}_4$ and W_n, for any $n \geq 3$, are not C1Ms, it follows that M does not have $\hat{W}_4$ and W_n as a minor.

Now consider the other half of the theorem. Assume that M does not have $\hat{W}_4$ and W_n, for any $n \geq 3$, as a minor. Suppose M is not a C1M. Let (G, T) be a gt-realization of M. If the degree of every vertex in T is at most two, then M is a C1M. Therefore, T has at least one vertex whose degree is at least three and G is not a triangle (cycle of three edges). For every vertex v of G, define $T(v)$ as in the proof of Theorem 1. Clearly, $T(v)$ has no loops. For some v of G, if $T(v)$ has a cycle having three or more edges, then as shown in the proof of Theorem 1, every such cycle induces a wheel-matrix W_n, for some $n \geq 3$, as a minor of M, a contradiction. Thus, for every vertex v of G, either $T(v)$ has no cycles or every cycle of $T(v)$ has exactly two edges.

Since G is 3-connected, every edge of $T(v)$ is in a cycle. This is seen as follows. Suppose $T(v)$ has an edge u_1u_2 that is not in any cycle. By the definition of $T(v)$, v is adjacent to u_1 and u_2, and all the three vertices v, u_1, u_2 are in a fundamental cycle C of (G, T). Let u_3u_4 denote the unique edge in $C - T$. Then, since every cycle of $T(v)$ has exactly two edges and G is not a triangle, G has a 2-separation $\{E_1, E_2\}$ such that $|E_1| \geq 2 \leq |E_2|$, and $V(G[E_1]) \cap V(G[E_2])$ is either $\{v, u_3\}$ or $\{v, u_4\}$, a contradiction. Thus, every edge of $T(v)$ is in a cycle.

Pick a degree-3 vertex of T and call it y. Since every edge of $T(y)$ is in a cycle and every cycle of $T(y)$ has exactly two edges, it follows that there are three edges p, q, r incident to y such that each of the pairs p and q, and q and r is in at least two distinct fundamental cycles of (G, T), and p and r are not in any fundamental cycle of (G, T). Moreover, since G is not a triangle, if all those edges in $E(G) - T$ each of which induce an unique fundamental cycle of (G, T) containing the edges p and q (or q and r) are incident to the same vertex z (say) in G, then G has a 2-separation $\{E_1, E_2\}$ such that $|E_1| \geq 2 \leq |E_2|$ and $V(G[E_1]) \cap V(G[E_2])$ is $\{y, z\}$. This contradicts the hypothesis that G is 3-connected. Therefore, there are at least two edges in $E(G) - T$ having distinct end vertices such that each of them induce an unique fundamental cycle of (G, T) containing p and q (or

q and r). Let $a, b \in E(G) - T$ (respectively, $c, d \in E(G) - T$) denote two such edges, and let C_a and C_b (respectively, C_c and C_d) denote the unique fundamental cycles containing both p and q (respectively, q and r). But for the edges in $C_a \cup C_b \cup C_c \cup C_d$, delete all the edge of G not in T and contract all the edges of G in T and call the resulting gt-pair (G', T'). It is easy to verify that T' has $\hat{T}_4$ (arrow) as a minor (with y as the degree-3 vertex) obtained by contracting some of the edges in T'. Therefore, (G', T') has $(W_4, \hat{T}_4)$ has a minor. Since (G', T') is a minor of (G, T), it follows that (G, T) has $(W_4, \hat{T}_4)$ as a minor. This in turn implies that the matrix M has $\hat{W}_4$ as a minor, a contradiction to the assumption that M does not have $\hat{W}_4$ as a minor. □

Acknowledgements. I would like to thank Don Wagner for several discussions on graphic matrices, and András Frank for his ideas concerning the proof of Theorem 1 given in this paper.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, North-Holland, New York, 1976.

[2] R. E. Bixby and D. K. Wagner, An almost–linear time algorithm for graph realization, *Mathematics of Operations Research* **13** (1988), 99–123.

[3] R. G. Bland and C. W. Ko, Characterizations of Camion trees and depth–first search trees by excluded configurations, (1990), submitted for publication.

[4] C.N. Purdy and R. Swaminathan, On a characterization of series–parallel graphs, *Ars Combinatoria*, (1994), to appear.

[5] R. Swaminathan and D. K. Wagner, A forbidden–minor characterization of orientable graph–tree pairs, (1989), unpublished.

[6] R. Swaminathan and D. K. Wagner, The arborescence–realization problem, *Discrete Applied Mathematics*, (1991), to appear.

[7] R. Swaminathan and D. K. Wagner, On the consecutive–retrieval problem, *SIAM Journal on Computing* **23**(2) (1994), 398–414.

[8] A. C. Tucker, A structure theorem for the consecutive 1's property, *Journal of Combinatorial Theory* **12** (1972), 153–162.

[9] H. Whitney, 2-isomorphic graphs, *American Journal of Mathematics* **55** (1933), 245–254.