A note on graph reconstruction

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ABSTRACT. Suppose G and G' are graphs on the same vertex set V such that for each $x \in V$ there is an isomorphism θ_x of $G - x$ to $G' - x$. We prove in this paper that if there is a vertex $x \in V$ and an automorphism σ of $G - x$ such that θ_x agrees with σ on all except for at most three vertices of $V - x$, then G is isomorphic to G'. As a corollary we prove that if a graph G has a vertex which is contained in at most three bad pairs, then G is reconstructible. Here a pair of vertices x, y of a graph G is called a bad pair if there exist $u, v \in V(G)$ such that $\{u, v\} \neq \{x, y\}$ and $G - \{x, y\}$ is isomorphic to $G - \{u, v\}$.

All graphs discussed here are finite simple graphs. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. If A is a subset of $V(G)$, we use $G|A$ and $G - A$ to denote the subgraphs of G induced by A and $V(G) - A$ respectively. When $A = \{x\}$ is a singleton, we use $G - x$ instead of $G - \{x\}$. For two subsets X, Y of $V(G)$, we use $e_G(X, Y)$ to denote the number of edges joining a vertex of X to a vertex of Y. For brevity, we write $e_G(x, X)$ for $e_G(\{x\}, X)$ and $e_G(X)$ for $e_G(X, X)$. The degree of x in G is denoted by $d_G(x)$.

We shall use some notations defined in [4]. Two graphs G and H are hypomorphic if there exists a bijection $f: V(G) \to V(H)$ such that $G - x$ is isomorphic to $H - f(x)$ for each vertex x of G. Such a mapping f is called a *hypomorphism* of G to H. Obviously each isomorphism is a hypomorphism. The converse is not true. However the well-known reconstruction conjecture (cf. [3]) asserts that the existence of a hypomorphism of G to H implies the existence of an isomorphism of G to H. Let $f: V(G) \to V(H)$ be a

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hypomorphism; and for each $x \in V(G)$, let $p_x\colon G - x \to H - f(x)$ be an isomorphism. Define $\theta_x = f^{-1}p_x$, where mappings are composed from right to left. The mapping θ_x is a permutation of $V(G) - x$, and does not act on x. We call θ_x a *partial permutation* of $V(G)$.

Note that $H - f(x) = f\theta_x(G-x)$,so that $H = \cup_x(H - f(x)) = f(\cup_x \theta_x(G-x))$. The graph $G' = \cup_x \theta_x(G-x)$ is called a *hypomorph* of G. A hypomorph G' of G is actually a graph on the same vertex set $V = V(G)$ which is hypomorphic to G with identity being a hypomorphism. We note that G may have many hypomorphs, derived from different hypomorphisms. In particular G is a hypomorph of itself. If all hypomorphs of G are isomorphic then we say G is *reconstructible*.

W.L. Kocay [4] studied some basic properties of these partial permutations θ_x as well as the partial automorphisms $\theta_{xy} = \theta_x^{-1}\theta_y$. Some sufficient conditions in terms of these mappings are given in [4] so that G is isomorphic to its hypomorph G'. It was shown in [4] that if there exist distinct vertices $x, y \in V(G)$ such that $\theta_x \in \mathrm{Aut}(G-x)$ and $\theta_y \in \mathrm{Aut}(G-y)$ then G is isomorphic to G'.

In this note, we show that if there is a vertex $x \in V(G)$ such that θ_x is very "close" to an automorphism of $G - x$, then G is isomorphic to G'. To be precise, we will prove the following:

Theorem 1. *Suppose G' is the hypomorph of G defined as above. If for a vertex x of G, there exists an automorphism $\sigma \in Aut(G-x)$ of $G-x$ which agrees with θ_x on all except for at most three vertices of $G-x$, then G is isomorphic to G'.*

We call an unordered pair of vertices $\{x, y\}$ of a graph G a *bad pair* if there exist $u, v \in V(G)$ such that $\{u, v\} \neq \{x, y\}$ and $G - \{x, y\}$ is isomorphic to $G - \{u, v\}$. It was proved in [2] that if a graph G has a vertex x which is contained in no bad pairs then G is reconstructible. As a consequence of Theorem 1, we obtain the following result:

Corollary 1. *If G has a vertex which is contained in at most three bad pairs, then all hypomorphs of G are isomorphic to G and hence G is reconstrcutible.*

Proof: Suppose that G has a vertex x which is contained in at most three bad pairs. We need to show that any hypomorph G' of G is isomorphic to G.

For each $x \in V$, let $\theta_x\colon V - x \to V - x$ be an isomorphism of $G - x$ to $G' - x$. We now show that if $\theta_x(y) \neq y$ for some vertex $y \in V(G-x)$, then $\{x, y\}$ is a bad pair of G. Indeed, let $z = \theta_x(y)$ and let $w = \theta_z^{-1}(x)$, where $\theta_z\colon V - z \to V - z$ is an isomorphism of $G - z$ to $G' - z$. Then $G - \{x, y\}$ is isomorphic to $G' - \{x, z\}$, which is isomorphic to $G - \{z, w\}$. Since $z \neq x$, and $z \neq y$, $\{x, y\} \neq \{z, w\}$, and hence $\{x, y\}$ is a bad pair of G.

Because x is contained in at most three bad pair of G, we have that $\theta_x(y) = y$ for at least $|V(G)| - 4$ vertices y of $G - x$, i.e., the identity, which is an automorphism of $G - x$, agrees with θ_x on all except for at most three vertices of $G - x$. Therefore G is isomorphic to G' by Theorem 1.

We now proceed to prove Theorem 1. The following lemma is an easy consequence of the fact that $d_G(v) = d_{G'}(v)$ for all $v \in V$ (cf. [1]).

Lemma 1. *Suppose $\sigma \in Aut(G - x)$ is an automorphism of $G - x$. If there is a subset $A \subset V - \{x\}$ such that $\sigma(A) = \theta_x(A)$, then let $\sigma(A) = B$, we have $e_G(x, B) = e_{G'}(x, B)$.*

Proof: Since $\sigma \in \text{Aut}(G - x)$ and $\sigma(A) = B$, we obtain $e_G(B) = e_G(A) = e_{G'}(\theta_x(A)) = e_{G'}(B)$. Similarly $e_G(B, (V - x) - B) = e_{G'}(B, (V - x) - B)$.

It is clear that $e_G(B, V) = e_G(B) + e_G(B, (V - x) - B) + e_G(x, B)$, and $e_{G'}(B, V) = e_{G'}(B) + e_{G'}(B, (V - x) - B) + e_{G'}(x, B)$. Also we have $e_G(B, V) = \sum_{v \in B} d_G(v) - e_G(B) = \sum_{v \in B} d_{G'}(v) - e_{G'}(B) = e_{G'}(B, V)$. Therefore $e_G(x, B) = e_{G'}(x, B)$. □

Corollary 2. *If $v \in V(G - x)$ and $\sigma(v) = u = \theta_x(v)$, while $\sigma \in Aut(G - x)$, then $(x, u) \in E(G)$ if and only if $(x, u) \in E(G')$.*

Proof of Theorem 1: Let $x \in V$ be a vertex of G and let $\sigma \in \text{Aut}(G - x)$ be an automorphism of $G - x$ which agrees with θ_x on all except for at most three vertices. We shall prove that G is isomorphic to G'.

If $\sigma(v) = \theta_x(v)$ for all $v \in V(G - x)$ then G and G' are identical. Indeed, for any edge $(u, v) \in E(G)$ which does not contain x as an end point, we have $(\sigma^{-1}(u), \sigma^{-1}(v)) \in E(G)$. Therefore $(\theta_x(\sigma^{-1}(u)), \theta_x(\sigma^{-1}(v))) \in E(G')$. But $\theta_x = \sigma$, so $(u, v) \in E(G')$. For an edge $(x, u) \in E(G)$ which does contain x as an end point, we have $(x, u) \in E(G')$ by Corollary 2. Thus G is isomorphic to G'.

Next we consider the case that there are exactly two vertices, say v_1, v_2, of $G - x$, such that $\sigma(v_i) \neq \theta_x(v_i)$ $(i - 1, 2)$. Let $u_i = \sigma(v_i)$ for $i = 1, 2$. Then we must have $\theta_x(v_1) = u_2$ and $\theta_x(v_2) = u_1$. By Lemma 1, we have $(x, v) \in E(G)$ if and only if $(x, v) \in E(G')$ for all vertices $v \in V - x$ not equal to u_1 or u_2; and $e_G(x, \{u_1, u_2\}) = e_{G'}(x, \{u_1, u_2\})$. If $e_G(x, \{u_1, u_2\}) = 2$ or 0, then it is easy to show (similar to the argument in the previous paragraph) that the mapping $g \colon V \to V$ defined by $g = \theta_x \sigma^{-1}$ on $V - x$ and $g(x) = x$ is an isomorphism of G to G'.

Thus we assume that $e(x, \{u_1, u_2\}) = 1$. Without loss of generality we assume that $(x, u_1) \in E(G)$. If $(x, u_2) \in E(G')$ then again the mapping g defined in the previous paragraph is an isomorphism of G to G'. Thus we assume that $(x, u_1) \in E(G')$. We claim that in this case G is identical to G', i.e., the identity is an isomorphism of G to G'.

Otherwise there are vertices $a, b \in V$ such that $(a, b) \in E(G)$ and $(a, b) \notin E(G')$. It is easy to verify (similar to the proof of the case $\sigma = \theta_x$) that

G and G' are identical on $V - \{u_1, u_2\}$, and they are also identical on $\{x, u_1, u_2\}$. Thus we may assume that $a \in \{u_1, u_2\}$ and $b \in V - \{x, u_1, u_2\}$.

Without loss of generality, we can assume that $a = u_1$. (If $a = u_2$ then we have the equivalent of $a = u_1$ in G' and we may interchange the roles of G and G'). Since $G - b$ is isomorphic to $G' - b$, these two graphs have the same degree sequence. However for all vertices $v \neq u_1, u_2$, we have $d_{G-b}(v) = d_{G'-b}(v)$ (as $d_G(v) = d_{G'}(v)$ and $(b, v) \in E(G)$ if and only if $(b, v) \in E(G')$). Therefore we must have $\{d_{G-b}(u_1), d_{G-b}(u_2)\} = \{d_{G'-b}(u_1), d_{G'-b}(u_2)\}$.

Since $(u_1, b) \in E(G)$, we have $(\sigma^{-1}(u_1), \sigma^{-1}(b)) \in E(G)$ and hence $(\theta_x\sigma^{-1}(u_1), \theta_x\sigma^{-1}(b)) \in E(G')$, i.e, $(u_2, b) \in E(G')$. Similarly, $(u_1, b) \notin E(G')$ implies $(u_2, b) \notin E(G)$. Therefore $d_G(u_1) = d_{G-b}(u_1) + 1$, $d_G(u_2) = d_{G-b}(u_2)$, $d_{G'}(u_1) = d_{G'-b}(u_1)$, and $d_{G'}(u_2) = d_{G'-b}(u_2) + 1$.

As $d_G(u_1) = d_{G'}(u_1)$, $d_G(u_2) = d_{G'}(u_2)$, we conclude that $d_G(u_1) = d_G(u_2) = d_{G'}(u_1) = d_{G'}(u_2)$, which implies that $d_{G-x}(u_1) \neq d_{G'-x}(u_2)$. This is a contradiction, as $\theta_x\sigma^{-1}$ is an isomorphism of $G - x$ to $G' - x$ which sends u_1 to u_2. Therefore G and G' are identical.

Finally we consider the case that there are three vertices of $G - x$, say v_1, v_2, v_3, such that $\sigma(v_i) \neq \theta_x(v_i)$ $(i = 1, 2, 3)$. Without loss of generality, we may assume that $\theta_x(v_1) = \sigma(v_2) = u_2, \theta_x(v_2) = \sigma(v_3) = v_3$ and $\theta_x(v_3) = \sigma(v_1) = u_1$.

By Lemma 1, $(x, v) \in E(G)$ if and only if $(x, v) \in E(G')$ for all vertices v of $G - x$ not equal to u_1, u_2 or u_3; and $e_G(x, \{u_1, u_2, u_3\}) = e_{G'}(x, \{u_1, u_2, u_3\})$. If $e_G(x, \{u_1, u_2, u_3\}) = 3$ or 0, then again it is easy to verify that the mapping $g\colon V \to V$ defined as $g = \theta_x\sigma^{-1}$ on $V - x$ and $g(x) = x$ is an isomorphism of G to G'.

We now consider the case that $e_G(x, \{u_1, u_2, u_3\}) = 1$. The case $e_G(x, \{u_1, u_2, u_3\}) = 2$ will follow easily by considering the complement graphs.

Without loss of generality, we assume that $(x, u_1) \in E(G)$. If $(x, u_2) \in E(G')$ then again the mapping g defined above is an isomorphism of G to G'. We now assume that $(x, u_2) \notin E(G')$. Thus we have either $(x, u_1) \in E(G')$ or $(x, u_3) \in E(G')$

Case 1: Suppose $(x, u_1) \in E(G')$. We shall show that in this case the mapping g which sends u_2 to u_3, sends u_3 to u_2, and fixes every other vertices of V is an isomorphism of g to G'. Suppose $d_G(u_1) = k + 1$. then it is easy to see that $d_{G-x}(u_2) = d_G(u_2) = d_{G-x}(u_3) = d_G(u_3) = k$. Let $S = V - \{u_1, u_2, u_3\}$. For any vertex $u \in S$, we have $\{d_{G-u}(u_1), d_{G-u}(u_2), d_{G-u}(u_3)\} = \{d_{G'-u}(u_1), d_{G'-u}(u_2), d_{G'-u}(u_3)\}$, because $G - u$ is isomorphic to $G' - u$ (hence these two graphs have the same degree sequence), and $d_{G-u}(v) = d_{G'-u}(v)$ for all vertices $v \neq u_1, u_2, u_3$ (as $d_G(v) = d_{G'}(v)$ and $(u, v) \in E(G)$ if and only if $(v, u) \in E(G')$). This implies that for any vertex $u \in S$, we have $(u_1, u) \in E(G)$ if and only if $(u_1, u) \in E(G')$. Indeed if $(u_1, u) \notin E(G)$ and $(u_1, u) \in E(G')$, then $d_{G-u}(u_1) = k + 1$ and $d_{G'-u}(u_i) \leq k$ for all $i =$

1, 2, 3; and hence $\{d_{G-u}(u_1), d_{G-u}(u_2), d_{G-u}(u_3)\} \neq \{d_{G'-u}(u_1), d_{G'-u}(u_2), d_{G'-u}(u_3)\}$. Similar contradiction can be derived if $(u_1, u) \in E(G)$ and $(u_1, u) \notin E(G')$.

Recall that $\theta_x\sigma^{-1}(u_3) = u_1$ and $\theta_x\sigma^{-1}(u_1) = u_2$, we conclude that for any $u \in S - \{x\}$, $(u, u_3) \in E(G)$ if and only if $(u, u_1) \in E(G')$ if and only if $(u, u_1) \in E(G)$ if and only if $(u, u_2) \in E(G')$. Furthermore $(x, u_3) \notin E(G)$ and $(x, u_2) \notin E(G')$. Thus for all $u \in S$, we have $(u, u_3) \in E(G)$ if and only if $(u, u_2) \in E(G')$. To prove that the mapping g defined above is an isomorphism of G to G', it remains to show that $(u_i, u_j) \in E(G)$ if and only if $(g(u_i), g(u_j)) \in E(G')$ for $i, j \in \{1, 2, 3\}$.

First observe that because $G - u_1$ is isomorphic to $G' - u_1$, and $d_{G-u_1}(u) = d_{G'-u_1}(u)$ for any $u \in S$, we have $\{d_{G-u_1}(u_2), d_{G'-u_1}(u_3)\} = \{d_{G'-u_1}(u_2), d_{G'-u_1}(u_3)\}$.

We now consider two subcases:

Case 1(a): Suppose that $(u_1, u_2) \in E(G)$. Then $(u_2, u_3) \in E(G')$, as $\theta_x\sigma^{-1}(u_1) = u_2$ and $\theta_x\sigma^{-1}(u_2) = u_3$. Since $d_{G-u_1}(u_2) = k - 1 \in \{d_{G'-u_1}(u_2), d_{G'-u_1}(u_3)\}$, we must have $(u_1, u_2) \in E(G')$ or $(u_1, u_3) \in E(G')$.

If $(u_1, u_2) \in E(G')$, then $(u_1, u_3) \in E(G)$, and hence $\{d_{G-u_1}(u_2), d_{G-u_1}(u_3)\} = \{k-1, k-1\}$. In order that $\{d_{G'-u_1}(u_2), d_{G'-u_1}(u_3)\} = \{k-1, k-1\}$, we must have $(u_1, u_3) \in E(G')$. This implies that $(u_2, u_3) \in E(G)$, and hence $\{u_1, u_2, u_3\}$ induces a complete graph in both graphs G and G'.

If $(u_1, u_2) \notin E(G')$ then $(u_1, u_3) \in E(G')$. This implies that $(u_2, u_3) \in E(G)$ and $(u_1, u_3) \notin E(G)$, as $\theta_x\sigma^{-1}$ is an isomorphism of G to G'.

In any case the restriction of g to $\{u_1, u_2, u_3\}$ is an isomorphism.

Case 1(b): Suppose that $(u_1, u_2) \notin E(G)$. Then $(u_2, u_3) \notin E(G')$. If $(u_1, u_3) \in E(G)$ then $(u_1, u_2) \in E(G')$. This implies that $(u_1, u_3) \notin E(G')$ for otherwise we would have $\{d_{G-u_1}(u_2), d_{G-u_1}(u_3)\} = \{k, k-1\}$ and $\{d_{G'-u_1}(u_2), d_{G'-u_1}(u_3)\} = \{k-1, k-1\}$. Since $\theta_x\sigma^{-1}(u_2) = u_3$ and $\theta_x\sigma^{-1}(u_3) = u_1$, we know that $(u_2, u_3) \notin E(G)$. Thus the restriction of g to $\{u_1, u_2, u_3\}$ is an isomorphism.

If $(u_1, u_3) \notin E(G)$ the $(u_1, u_2) \notin E(G')$. This implies that $(u_1, u_3) \notin E(G')$, for otherwise we would nave $\{d_{G-u_1}(u_2), d_{G-u_1}(u_3)\} = \{k, k\}$ and $\{d_{G'-u_1}(u_2), d_{G'-u_1}(u_3)\} = \{k, k-1\}$. Thus again we have $(u_2, u_3) \notin E(G)$, and therefore the restriction of g to $\{u_1, u_2, u_3\}$ is an isomorphism.

Case 2: Suppose that $(x, u_3) \in E(G')$. We shall show that in this case the mapping g which sends u_1 to u_3, sends u_3 to u_1, and fixes every other vertex of V, is an isomorphism of G to G'. The proof is very similar to that of Case 1 and we omit some details.

Let $k = d_{G-x}(u_1)$. Then $d_G(u_1) = d_{G'}(u_1) = d_G(u_3) = d_{G'}(u_3) = k + 1$ and $d_G(u_2) = d_{G'}(u_2) = k$.

Similar to the argument in the proof of Case 1, we can show that for any vertex $u \in S = V - \{u_1, u_2, u_3\}$, we have $\{d_{G-u}(u_1), d_{G-u}(u_2), d_{G-u}(u_3)\} = \{d_{G'-u}(u_1), d_{G'-u}(u_2), d_{G'-u}(u_3)\}$. This implies that for any vertex $u \in S$, we have $(u_2, u) \in E(G)$ if and only if $(u_2, u) \in E(G')$ (cf. the proof of Case 1).

It remains to show that g restricted to $\{u_1, u_2, u_3\}$ is an isomorphism. Similarly $\{d_{G-u_2}(u_1), d_{G-u_2}(u_3)\} = \{d_{G'-u_2}(u_1), d_{G'-u_2}(u_3)\}$, because $G - u_2$ is isomorphic to $G' - u_2$, and $d_{G-u_2}(u) = d_{G'-u_2}(u)$ for any $u \in S$. Again we consider two subcases:

Case 2(a): Suppose that $(u_2, u_3) \in E(G)$. Then $(u_1, u_3) \in E(G')$. This implies that $(u_1, u_2) \in E(G')$ for otherwise $\{d_{G-u_2}(u_1), d_{G-u_2}(u_3)\} \neq \{d_{G'-u_2}(u_1), d_{G'-u_2}(u_3)\}$. This then implies that $(u_1, u_3) \in E(G)$, and hence the restriction of g to $\{u_1, u_2, u_3\}$ is an isomorphism.

Case 2(b): Suppose that $(u_2, u_3) \notin E(G)$. Then $(u_1, u_3) \notin E(G')$. Similarly by using the condition that $\{d_{G-u_2}(u_1), d_{G-u_2}(u_3)\} = d_{G'-u_2}(u_1), d_{G'-u_2}(u_3)\}$, we can show that g restricted to $\{u_1, u_2, u_3\}$ is an isomorphism. This completes the proof of Theorem 1.

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