On the bondage number of block graphs

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Abstract

Let $\gamma(G)$ be the domination number of a graph G. The bondage number $b(G)$ of a nonempty graph G is the minimum cardinality among all sets of edges X for which $\gamma(G-X) > \gamma(G)$.
In this paper we show that $b(G) \leq \Delta(G)$ for any block graph G, and we characterize all block graphs with $b(G) = \Delta(G)$.

1 Introduction

Let $G = (V(G), E(G))$ be a finite, undirected graph with neither loops nor multiple edges. For $u \in V(G)$ we denote by $N(u)$ the open neighborhood of u. More general we define $N(U) = \bigcup_{u \in U} N(u)$ for a set $U \subseteq V(G)$ and $N[U] = N(U) \cup U$.
A set D of vertices in G is a dominating set if $N[D] = V(G)$. A dominating set of minimum cardinality in G is called a minimum dominating set (MDS), and its cardinality is termed the domination number of G, denoted by $\gamma(G)$.
The bondage number $b(G)$ of a nonempty graph is the minimum cardinality among all sets of edges X for which $\gamma(G-X) > \gamma(G)$ holds. Brigham, Chinn, and Dutton [2] defined a vertex v to be critical if $\gamma(G-v) < \gamma(G)$. A vertex v of a graph G is called a cut vertex of G if $G-v$ has more components than G. A connected graph without cut vertices is called a block. A block of a graph G is a subgraph of G which is itself a block and which is maximal with respect to that property. A block H of G is called an end block of G if H has at most one cut vertex of G. If a block has at least 3 vertices, we call this block a large block. A graph G is called a block graph if each block of G is complete. For graph theory not presented here we follow [4].
In 1990, Fink, Jacobson, Kinch and Roberts [3] introduced the bondage

number, and they proved $b(T) \leq 2$ for every tree T. Two years later Hartnell and Rall [5] characterized all trees with bondage number 2. Further results on the bondage number were published in the articles of Hartnell and Rall [6] and Teschner [7-9]. Results on the bondage number of cactus graphs can be found in Teschner and Volkmann [10].
In the sequel, we will need the following known results.

Proposition 1.1 *[1] If there is a vertex $u \in V(G)$ with $\gamma(G-u) \geq \gamma(G)$ (that means, u is not critical), then $b(G) \leq deg\ u \leq \Delta(G)$.*

Proposition 1.2 *[2] If G has a nonisolated vertex v such that $N(v)$ is complete, then the neighbors of v are not critical.*

Corollary 1.3 *If G has a large end block E, then $b(G) \leq |V(E)| - 1 < \Delta(G)$.*

Proposition 1.4 *[9] Let G be a nonempty graph and $u,\ v \in V(G)$. Then $b(G) \leq \min\ \{deg\ u + \ deg\ v - 1\ ;\ d(u,v) \leq 2\}$.*

Proof: If u is adjacent to v, then the result is due to [3]. In the case $d(u,v) = 2$ let w be a vertex adjacent to u and v. Now we remove all edges of G which are incident to u and v, except of the two edges uw and vw. In the resulting graph G', the vertex w is adjacent to the end vertices u and v. Obviously $b(G') = 1$, and hence our hypothesis is valid.•

2 The general upper bound

Theorem 2.1 *If G is a nontrivial connected block graph, then $b(G) \leq \Delta(G)$.*

Proof: If $G = K_n$, then $b(G) = \lceil \frac{n}{2} \rceil \leq n - 1 = \Delta(G)$ is immediate. If G has at least two blocks, let E be an end block of G with the cut vertex v, and let $u \in V(E)$ with $u \neq v$. Then $N(u)$ is complete, and by Proposition 1.2 the vertex v is not critical. Hence Proposition 1.1 yields $b(G) \leq \Delta(G)$.
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That the upper bound of Theorem 2.1 is best possible may be seen by the next result.
Let G be a graph with the vertex set $V(G) := \{v_1, \ldots, v_n\}$. Then the corona $G \circ K_1$ of G and K_1 is the graph with the vertex set $\{v_1, \ldots, v_n\} \cup \{w_1, \ldots, w_n\}$ and the edge set $E(G) \cup \{v_i w_i\ ;\ 1 \leq i \leq n\}$.

Theorem 2.2 *Let $G = K_n \circ K_1$, then $b(G) = \Delta(G) = n$.*

Proof: Assume, there exists a set $X := \{x_1, \ldots, x_{n-1}\} \subseteq E(G)$ such that $\gamma(G - X) > \gamma(G) = n$.
Without loss of generality let $\{x_1, \ldots, x_s\}$ be the end edges of X (where $0 \leq s \leq n-1$). Furthermore let $\{w_1, \ldots, w_s\} \subseteq V(G)$ be the end vertices incident to $\{x_1, \ldots, x_s\}$, and let $\{w_{s+1}, \ldots, w_n\}$ be the remaining end vertices. Finally let $\{v_1, \ldots, v_n\} \subseteq V(G)$ be the neighbors of the vertices $\{w_1, \ldots, w_n\}$. Then we will show that $D := \{w_1, \ldots, w_s, v_{s+1}, \ldots, v_n\}$ dominates $G - X$.
Obviously D dominates all the vertices w_i in $G - X$. If $s \geq 1$, assume that one of the vertices v_j (where $j \leq s$) is not dominated by D in $G - X$, say v_1. That implies that (in $G - X$) v_1 only is adjacent to vertices v_j (where $j \leq s$, $j \neq 1$). Now we count the removed edges: s end edges plus at least $n - 1 - (s - 1)$ edges of the K_n, which is a contradiction to $|X| = n - 1$. Hence D dominates $G - X$ such that $\gamma(G - X) \leq n$, a contradiction to the main assumption.•

Theorem 2.3 *Let G be a block graph with exactly one large block. Then $b(G) = \Delta(G)$ if and only if*
1) $G = K_3$ or
2) $G = K_n \circ K_1$ (where $n \geq 3$)

Proof: Let Ω be the set of end vertices of G. If $v \in V(G)$ then we denote by $c(v)$ the distance from v to the unique large block of G.
Case 1 : If $G = K_n$, then $b(G) = \lceil \frac{n}{2} \rceil$. $\Delta(G) = n - 1$ equals $\lceil \frac{n}{2} \rceil$ if and only if $n = 2$ or $n = 3$. Since G must have a large block, the case $G = K_3$ remains. *Case 2 :* Let $\Omega \neq \emptyset$ and $c(v) = 1$ for all $v \in \Omega$.
If $G = K_n \circ K_1$, then $b(G) = \Delta(G)$ follows from Theorem 2.2. If there exists a vertex adjacent to two end vertices, we obtain $b(G) = 1$ from Proposition 1.4. In the remaining case there exists a vertex $w \in V(K_n)$ with $N(w) = V(K_n)$. Obviously w is not critical in G. Hence Proposition 1.1 yields $b(G) \leq deg\ w = n - 1 < \Delta(G)$.
Case 3 : There exists an $u \in \Omega$ with $c(u) \geq 2$. Now choose a vertex $v \in \Omega$ with $c(v) \geq c(w)$ for all $w \in \Omega$. Let s be the unique neighbor of v. Then there are two possibilities: $deg\ s = 2$ or s is adjacent to at least two end vertices. In each case we obtain $b(G) \leq 2$ according to Proposition 1.4.•

3 The coalescence

The coalescence of two disjoint graphs G and H, denoted by $G \cdot H$, is obtained by identifying a vertex v_1 of G and a vertex v_2 of H. Thus the identified vertex v becomes a cut vertex of $G \cdot H$.
If v_1 is critical in G and v_2 is critical in H, the coalescence is called simple. If

a property of G and H is also valid for the graph $G \cdot H$ obtained by a simple coalescence, then the property is called hereditary. In this connection we will call the graphs G and H original graphs.
Some of the following results have been shown in [10].

Proposition 3.1 *[10] Let $G \cdot H$ be a coalescence where at least one of the identified vertices v_1 and v_2 is critical in its original graph, e.g. a simple coalescence. Then $\gamma(G \cdot H) = \gamma(G) + \gamma(H) - 1$.*

Proposition 3.2 *[10] The vertex property of being critical is hereditary.*

Proposition 3.3 *[10] The vertex property of being not critical is hereditary.*

Proposition 3.4 *[10] Let $G \cdot H$ be any coalescence of G and H such that $\gamma(G \cdot H) = \gamma(G) + \gamma(H) - 1$. Then $b(G \cdot H) \leq \min\{b(G), b(H)\}$.*

Lemma 3.5 *The property that any critical vertex of a graph remains critical, even after removing t arbitrary edges (where the domination number remains unchanged), is hereditary.*

Proof: Let $v_1 \in V(G)$ and $v_2 \in V(H)$ be the identified critical vertices which become the vertex v in $G \cdot H$ (a simple coalescence).
Assume there are edges $X := \{x_1, \ldots, x_t\} \subseteq E(G \cdot H)$, so that the vertex w which is critical in $G \cdot H$ is not critical anymore in $G \cdot H - X$, i.e.

$$\gamma(G \cdot H - X - w) \geq \gamma(G \cdot H - X) = \gamma(G \cdot H) > \gamma(G \cdot H - w) .$$

By Proposition 3.3 w is critical in its original graph, say G, too, since w is critical in $G \cdot H$. Without loss of generality let $X_1 := \{x_1, \ldots, x_s\} \subseteq E(G)$ and $X_2 := \{x_{s+1}, \ldots, x_t\} \subseteq E(H)$ (where $0 \leq s \leq t$). By the hypothesis we have $\gamma(G - X_1 - w) < \gamma(G - X_1) = \gamma(G)$ as well as $\gamma(H - X_2 - v_2) < \gamma(H - X_2) = \gamma(H)$. Since

$$\gamma(G \cdot H - X - w) \leq \gamma(G - X_1 - w) + \gamma(H - X_2 - v_2) \leq$$

$$\gamma(G) - 1 + \gamma(H) - 1 < \gamma(G \cdot H) = \gamma(G \cdot H - X) ,$$

we obtain a contradiction and w is critical in $G \cdot H - X$ as well.•

Theorem 3.6 *Let G, H be block graphs with $b(G) = \Delta(G) = \Delta(H) = b(H)$ where the property of Lemma 3.5 is valid for $t = \Delta(G) - 2 = \Delta(H) - 2$. And let $G \cdot H$ be a simple coalescence of G and H such that $\Delta(G \cdot H) = \Delta(G) = \Delta(H)$. Then $b(G \cdot H) = \Delta(G \cdot H)$.*

Proof: Let $v_1 \in V(G)$ and $v_2 \in V(H)$ be the identified critical vertices which become the vertex v in $G \cdot H$. Proposition 3.4 shows that $b(G \cdot H) \leq \Delta(G) = \Delta(G \cdot H)$. Let $\Delta := \Delta(G \cdot H)$.
Assume that $b(G \cdot H) \leq \Delta - 1$. Let $X := \{x_1, \ldots, x_{\Delta-1}\} \subseteq E(G \cdot H)$ be edges, so that $\gamma(G \cdot H - X) > \gamma(G \cdot H)$.
<u>*Case 1*:</u> All the edges of X belong to the same original graph, say G.
Since $b(G) = \Delta(G)$, we have $\gamma(G - X) = \gamma(G)$. Let D be a $MDS(G - X)$. Then $|D| = \gamma(G)$, and let $D' \in MDS(H - v_2)$. The set $\tilde{D} := D \cup D'$ dominates $G \cdot H - X$, and $\tilde{D}$ has the cardinality $\gamma(G) + \gamma(H) - 1 = \gamma(G \cdot H)$, a contradiction.
<u>*Case 2*:</u> The edges $X_1 := \{x_1, \ldots, x_k\}$ belong to G, and the edges $X_2 := \{x_{k+1}, \ldots, x_{\Delta-1}\}$ belong to H (where $1 \leq k \leq \Delta - 2$).
By the hypothesis the vertex v_1 remains critical in $G - X_1$, since $b(G) = \Delta(G)$ and thus $\gamma(G) = \gamma(G - X_1)$ (the property of Lemma 3.5). Hence there is a $D \in MDS(G - X_1)$ with $v_1 \in D$. Analogously v_2 remains critical in $H - X_2$, since $b(H) = \Delta(H)$ and thus $\gamma(H) = \gamma(H - X_2)$ (the same property again). Hence there is a $D' \in MDS(H - X_2)$ with $v_2 \in D'$. We have $|D| = \gamma(G)$ and $|D'| = \gamma(H)$. Now $\tilde{D} := D \cup D'$ dominates $G \cdot H - X$, and $\tilde{D}$ has the cardinality $\gamma(G) + \gamma(H) - 1 = \gamma(G \cdot H)$, a contradiction again. Hence $b(G \cdot H) = \Delta(G \cdot H)$.•

Lemma 3.7 *For a block graph $G = K_n \circ K_1$ the property of Lemma 3.5 is valid for $t \leq \Delta(G) - 2 = n - 2$.*

Proof: Let $G = K_n \circ K_1$, then by Theorem 2.2 we have $b(G) = \Delta(G) = n$. Exactly the end vertices are the critical vertices of G. Assume that the end vertex w is not critical in the graph $G - X$ (where $X := \{x_1, \ldots, x_{n-2}\} \subseteq E(G)$). Then $\gamma(G - X - w) \geq \gamma(G - X)$, and w can not be isolated in $G - X$ because isolated vertices always are critical. Then Proposition 1.1 yields $b(G - X) \leq deg_{G-X} w = 1$, and hence $b(G) \leq |X| + 1 = n - 1 < \Delta(G)$, a contradiction. •

Corollary 3.8 *Any block graph G obtained by successive simple coalescences of the graphs $K_n \circ K_1$ (where n is arbitrary, but fixed) such that the maximum degree never changes, has the property $b(G) = \Delta(G) = n$.*

4 Decomposition

Theorem 4.1 *Let $G = (K_n \circ K_1) \cdot H$ be a block graph with $b(G) = \Delta(G)$, and let v be the cut vertex belonging to $K_n \circ K_1$ and H (obtained from identifying an end vertex $v_1 \in V(K_n \circ K_1)$ and a vertex $v_2 \in V(H)$ by an arbitrary coalescence).*
Then $b(H) = \Delta(H) = \Delta(G) = n$ and v_2 is critical in H.

Proof: Since the end vertex v_1 is critical in $K_n \circ K_1$ we have $\gamma(G) = \gamma(K_n \circ K_1) + \gamma(H) - 1$ by Proposition 3.1. Then Proposition 3.4 yields $\Delta(G) = b(G) \leq \min\{b(K_n \circ K_1), b(H)\} = \min\{n, b(H)\}$. $\Delta(G) \geq n$ is obvious because of the structure of G, hence $\Delta(G) = n$.
Assume that $\Delta(H) < n$. Since H is a block graph we would have $b(H) \leq \Delta(H) < n$ by Theorem 2.1, and by Proposition 3.4 $\Delta(G) = b(G) \leq b(H) < n$, a contradiction. If $b(H) < n$, we would obtain the same contradiction. Hence $b(H) = \Delta(H) = n$.
Now assume that v_2 is not critical in H, i.e. $\gamma(H - v_2) \geq \gamma(H)$.
Let $w \in V(K_n \circ K_1)$ be the unique neighbor of v_1 in $K_n \circ K_1$, and let $\{w_1, \ldots, w_{n-1}\} \subseteq V(K_n \circ K_1)$ be the remaining neighbors of w. Finally, let $\{x_1, \ldots, x_{n-1}\} \subseteq V(K_n \circ K_1)$ be the end edges incident to $\{w_1, \ldots, w_{n-1}\}$. Then

$$\gamma(G) = \gamma(G - \bigcup_{i=1}^{n-1} x_i) \geq \gamma(H - v_2) + \gamma(K_n \circ K_1) \geq \gamma(H) + \gamma(K_n \circ K_1) > \gamma(G),$$

a contradiction. Thus v_2 must be critical in H, which completes the proof.•

5 Characterization

Theorem 5.1 *Let G be a connected block graph without large blocks. Then $b(G) = \Delta(G)$ if and only if*
1) $G = K_2$ or
2) $G = P_n$ with $n \equiv 1 (mod\ 3)$

Proof: By the hypothesis G is a tree. Then from [3] we know that $b(G) \leq 2$. If $b(G) = \Delta(G) = 1$, then $G = K_2$ is obvious. If $\Delta(G) = 2$, then G is a path. From [3] we know that $b(P_n) = 2$ if and only if $n \equiv 1 (mod\ 3)$. Hence the proof is complete.•

Theorem 5.2 *Let G be a connected block graph with at least one large block.*
Then $b(G) = \Delta(G) = m \geq 3$ if and only if G is obtained by successive simple coalescences of the graphs $K_m \circ K_1$ (where $m \geq 3$ is arbitrary, but fixed) such that the maximum degree never changes.

Proof: Let n be the number of large blocks of G. If $b(G) = \Delta(G)$, then we will prove the theorem by induction on n. For $n = 1$, it follows from Theorem 2.3 that we only obtain a block graph G with $b(G) = \Delta(G) \geq 3$, if $G = K_m \circ K_1$ for arbitrary $m \geq 3$. For $n \geq 2$ we assume that the assertion is valid for $n - 1$. Since $b(G) = \Delta(G) \geq 3$, we first observe that no vertex is adjacent to two end vertices and that no vertex of degree 2 is

adjacent to an end vertex or to a neighbor of an end vertex. Furthermore, if there is a large end block in G, then Corollary 1.3 yields $b(G) < \Delta(G)$, a contradiction.
Hence each end block is isomorphic to the K_2. Now let $v_0, v_1, \ldots, v_t$ be a path between two end vertices of G such that the largest possible number of blocks is 'visited'. If B_0 is the end block containing v_0, then there exists exactly one block B_1 with $V(B_0) \cap V(B_1) = \{v_1\}$. Necessarily B_1 is a complete subgraph K_s (where $s \geq 3$), and Proposition 1.2 yields that v_1 is not critical in G. By Proposition 1.1 $\Delta(G) = b(G) \leq deg\ v_1 \leq \Delta(G)$, thus $s = deg\ v_1 = \Delta(G) = m$. Now assume that $deg\ w < deg\ v_1$ for some vertex $w \in V(B_1)$, $w \neq v_1$. Since B_1 is a complete subgraph of G, $deg\ w = m-1$, and Proposition 1.4 yields $b(G) \leq deg\ v_0 + deg\ w - 1 = m - 1 < \Delta(G)$, a contradiction. Since $deg\ w \leq \Delta(G) = deg\ v_1$ is obvious, we have $deg\ w = m = \Delta(G)$ for each vertex $w \in V(B_1)$.
Hence G has the form $G = (K_m \circ K_1) \cdot H$, where the coalescence is obtained by identifying the end vertex v_3 of $K_m \circ K_1$ and a vertex u of H. Therefore Theorem 4.1 yields $b(H) = \Delta(H) = \Delta(G) = m$ and u critical in H, so that the coalescence $(K_m \circ K_1) \cdot H$ is simple as required. Since H is a connected block graph with $n - 1 \geq 1$ large blocks, we deduce from the induction hypothesis that H has the desired form.
For the opposite direction we have $b(K_m \circ K_1) = \Delta(K_m \circ K_1) = m$ for $m \geq 3$ from Theorem 2.3. Now Proposition 3.2 and Corollary 3.8 yield $b(G) = \Delta(G) = m.\bullet$

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