E-cordial Graphs

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Abstract
A graph $G = (V, E)$ is called E-cordial if it is possible to label the edges with the numbers from the set $N = \{0, 1\}$ and the induced vertex labels $f(v)$ are computed by $f(v) = \sum_{\forall u} f(u, v)$ (mod 2), where $v \in V$ and $\{u, v\} \in E$ so that the conditions $| v_f(0) - v_f(1) | \leq 1$ and $| e_f(0) - e_f(1) | \leq 1$ are satisfied, where $v_f(i)$ and $e_f(i), i = 0, 1$ denote the number of vertices and edges labeled with 0's and 1's, respectively. The graph G is called E-cordial if it admits an E-cordial labelling. In this paper we investigate E-cordiality of several families of graphs such as complete bipartite graphs, complete graphs, wheels, etc.

1 Introduction

Cordial graphs were first introduced by I. Cahit [4],[5] in 1987, as a weaker version of graceful and harmonious graphs and was based on { 0,1 }-binary labelling of vertices. Other types of cordial graphs were considered in [8]-[11]. On the otherhand edge-graceful labelling of graphs was introduced by Lo in 1985 [12]. Let $G(V, E)$ be a simple graph with $| V |= p$ and $| E |= q$. Then G is said to be *edge-graceful* if there exists a bijection $f : E \rightarrow \{1, 2, ..., q\}$ such that the induced mapping $f^+ : V \leftarrow \{0, 1, 2, ..., p-1\}$ which is defined by $f^+(v) = \sum_{\forall (u,v) \in E} f(u, v) \pmod p$ is also bijection. Lee conjectured that every tree with odd number of vertices is edge-graceful [6],[7]. The graph labellings introduced in this paper may be considered as a weaker version of edge-graceful labellings but have considerable differences than the cordial graphs. Terms not defined in the paper can be found in [1]-[3].
In this paper, we have studied cordiality on binary edge labelling of graphs. We have observed that any graph fails to be E-cordial when the number of

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vertices of the graph $| V | \equiv 2 \pmod 4$.

An **E-cordial** labelling of a graph can be defined as follows:

Definition 1.1 *Let f be a binary edge labelling of graph $G = \{V, E\}$, i.e. $f : E(G) \to \{0,1\}$, and the induced vertex labelling is given as $f(v) = \sum_{\forall u} f(u,v) \pmod 2$, where $v \in V$ and $\{u,v\} \in E$.*
f is called an E-cordial labelling of G, if the following conditions are satisfied:
1)$| e_f(0) - e_f(1) | \leq 1$,
2)$| v_f(0) - v_f(1) | \leq 1$;
where $e_f(0)$, $e_f(1)$ denote the number of edges, and $v_f(0)$, $v_f(1)$ denote the number of vertices labelled with 0's and 1's respectively.
The graph G is called E-cordial if it admits an E-cordial labelling.

Before going any further it would be useful to give some basic theorems related to the E-cordiality of graphs.

Lemma 1.1 *If a labelling f of any graph satisfies $| e_f(0) - e_f(1) | \leq 1$, then $v_f(1) \equiv 0 (\mathrm{mod} 2)$.*

Proof: Since the edges labelled with 1, change the parity of two vertices at a time, when we label the edges of a graph such that $| e_f(0) - e_f(1) | \leq 1$, we necessarily end up with an even number of vertices labelled with 1, regardless of the number of edges and vertices of the graph. □

Theorem 1.1 *Necessary condition for a graph G, to admit an E-cordial labelling is that $n \not\equiv 2 \pmod 4$, where n denotes the number of vertices of G.*

Proof: For a graph with $n \equiv 2 \pmod 4$ vertices, to admit an E-cordial labelling, we must have $v_f(0) = v_f(1) = \frac{n}{2}$. However, $\frac{n}{2} \equiv 1 \pmod 2$ is an odd number and this contradicts Lemma 1.1. □

Corollary 1.1 *If G is a graph with $n \equiv 1 \pmod 4$ vertices, and f is an E-cordial labelling of G, then $v_f(0) = v_f(1) + 1$*

Proof: It is possible to write n as the sum of two adjacent numbers. i.e. Let $a = \frac{n-1}{2} \equiv 0 \pmod 2$ and $b = a + 1 = \frac{n+1}{2} \equiv 1 \pmod 2$ then $n = a + b$. So it is clearly seen in the light of Lemma 1.1 that $v_f(1) = a$ and $v_f(0) = b$, so $v_f(0) = v_f(1) + 1$. □

Corollary 1.2 *If G is a graph with $n \equiv 3 \pmod 4$ vertices, and f is an E-cordial labelling of G, then $v_f(1) = v_f(0) + 1$.*

Proof: It is possible to write n as the sum of two adjacent numbers. i.e. Let $a = \frac{n-1}{2} \equiv 0 \pmod 2$ and $b = a + 1 = \frac{n+1}{2} \equiv 1 \pmod 2$ then $n = a + b$. So it is clearly seen in the light of Lemma 1.1 that $v_f(0) = a$ and $v_f(1) = b$, so $v_f(1) = v_f(0) + 1$. □

2 Trees

A tree is a connected graph that contains no subgraphs isomorphic to a cycle and there exists exactly one path between any two vertices. Let v denote the number of vertices and e the number of edges of a tree, then $v = e + 1$.

Theorem 2.1 *Every tree is E-cordial if and only if* $n \not\equiv 2 \pmod 4$.

Proof: Necessity follows from Theorem 1.1. For sufficiency we use induction on n.
Assume that f is an E-cordial labelling of an $(n-1)$-vertex tree T_{n-1}. Then there are three possible cases for T_{n-1}, these are:
1) $n - 1 \equiv 0 \pmod 4$
2) $n - 1 \equiv 1 \pmod 4$
3) $n - 1 \equiv 3 \pmod 4$
Now assume that u is a vertex of T_{n-1} while w is not. Add a new edge $\{u, w\}$ to T_{n-1}, thus obtaining a new tree with n vertices, namely T_n, $(w \in T_n)$.
Case 1: $n - 1 \equiv 0 \pmod 4$ implies that $v_f(0) = v_f(1)$ and $| e_f(0) - e_f(1) | = 1$.
Adding a new vertex results in T_n with $n \equiv 1 \pmod 4$. In order for T_n to have an E-cordial labelling f', it is required that $e_{f'}(0) = e_{f'}(1)$ and $v_{f'}(0) = v_{f'}(1) + 1$ (by Corollary 1.3).
1.a) Assume $f(u) = 0$ and $e_f(0) = e_f(1) + 1$. Let $f'(u, w) = 1$, then $f'(u) = 1$, $f'(w) = 1$. So we have $e_{f'}(0) = e_{f'}(1)$ and $v_{f'}(1) = v_{f'}(0) + 3$. Thus f' is not an E-cordial labelling of T_n in this case.

1.b) Assume $f(u) = 0$ and $e_f(1) = e_f(0) + 1$. Let $f'(u, w) = 0$, then $f'(w) = 0$, $f'(u) = 0$. So we have $e_{f'}(0) = e_{f'}(1)$ and $v_{f'}(0) = v_{f'}(1) + 1$. Thus f' is an E-cordial labelling of T_n.

1.c) Assume $f(u) = 1$ and $e_f(0) = e_f(1) + 1$. Let $f'(u, w) = 1$, then $f'(u) = 0$, $f'(w) = 1$. So we have $e_{f'}(0) = e_{f'}(1)$ and $v_{f'}(0) = v_{f'}(1) + 1$. Thus f' is an E-cordial labelling of T_n.

1.d) Assume $f(u) = 1$ and $e_f(1) = e_f(0) + 1$. Let $f'(u, w) = 0$, then $f'(u) = 1$, $f'(w) = 0$. So we have $e_{f'}(0) = e_{f'}(1)$ and $v_{f'}(0) = v_{f'}(1) + 1$. Thus f' is an E-cordial labelling of T_n.

Observation: If f is an E-cordial labelling of T_{n-1} where $n - 1 \equiv 0 \pmod 4$ with $e_f(0) = e_f(1) + 1$, in order for f' to be an E-cordial labelling of T_n, $T_n = T_{n-1} \cup \{u, w\}$, label of vertex u in T_{n-1} must be $f(u) = 1$

(proved in *case 1.a*).

Case 2: $n-1 \equiv 1 \pmod 4$ implies that $v_f(0) = v_f(1)+1$ (Corollary 1.3) and $e_f(0) = e_f(1)$, as obtained in *Case 1*. Adding a new vertex results in T_n with $n \equiv 2 \pmod 4$ and it is not possible to have an E-cordial labelling of T_n (Theorem 1.1), as it will be shown below:

2.*a*) Assume $f(u) = 0$ and let $f'(u,w) = 0$, then $f'(u) = 0$, $f'(w) = 0$ and $e_{f'}(0) = e_{f'}(1)+1$, $v_{f'}(0) = v_{f'}(1)+2$.

2.*b*)Assume $f(u) = 0$ and let $f'(u,w) = 1$, then $f'(u) = 1$, $f'(w) = 1$ and $e_{f'}(1) = e_{f'}(0)+1$, $v_{f'}(1) = v_{f'}(0)+2$.

2.*c*)Assume $f(u) = 1$ and let $f'(u,w) = 0$, then $f'(u) = 1$, $f'(w) = 0$ and $e_{f'}(0) = e_{f'}(1)+1$, $v_{f'}(0) = v_{f'}(1)+2$.

2.*d*)Assume $f(u) = 1$ and let $f'(u,w) = 1$, then $f'(u) = 0$, $f'(w) = 1$ and $e_{f'}(1) = e_{f'}(0)+1$, $v_{f'}(0) = v_{f'}(1)+2$.
So it is not possible to have an E-cordial labelling of T_n when $n \equiv 2 \pmod 4$.

Case 3: When $n-1 \equiv 2 \pmod 4$, f is not an E-cordial labelling of T_{n-1}, but we have $|e_f(0)-e_f(1)| = 1$, $|v_f(0)-v_f(1)| = 2$, as obtained in *Case 2* above. Adding a new vertex results in T_n, $n \equiv 3 \pmod 4$. In order for T_n to admit an E-cordial labelling we must have $e_{f'}(0) = e_{f'}(1)$ and $v_{f'}(1) = v_{f'}(0)+1$, since by Corollary 1.2 $v_{f'}(1) = v_{f'}(0)+1$ is not possible.

3.*a*) Assume $f(u) = 0$ and $v_f(0) = v_f(1)+2$.
(*i*) If $e_f(0) = e_f(1)+1$, let $f'(u,w) = 1$.
Then $f'(u) = 1$, $f'(w) = 1$ and $e_{f'}(0) = e_{f'}(1)$, $v_{f'}(1) = v_{f'}(0)+1$.
(*ii*) If $e_f(1) = e_f(0)+1$, let $f'(u,w) = 0$.
Then $f'(u) = 0$, $f'(w) = 0$ and $e_{f'}(0) = e_{f'}(1)$, $v_{f'}(0) = v_{f'}(1)+3$.

3.*b*) Assume $f(u) = 0$ and $v_f(1) = v_f(0)+2$. According to the outcome of *Case 2.b* we have $e_f(1) = e_f(0)+1$. Let $f'(u,w) = 0$, then $f'(u) = 0$, $f'(w) = 0$ and $e_{f'}(0) = e_{f'}(1)$,$v_{f'}(1) = v_{f'}(0)+1$.

3.*c*) Assume $f(u) = 1$ and $v_f(0) = v_f(1)+2$
(*i*) If $e_f(0) = e_f(1)+1$, let $f'(u,w) = 1$.
Then $f'(u) = 0$,$f'(w) = 1$ and $e_{f'}(0) = e_{f'}(1)$, $v_{f'}(0) = v_{f'}(1)+3$.
(*ii*) If $e_f(1) = e_f(0)+1$, let $f'(u,w) = 0$.
Then $f'(u) = 1$,$f'(w) = 0$ and $e_{f'}(0) = e_{f'}(1)$,$v_{f'}(1) = v_{f'}(0)+1$.

3.d) Assume $f(u) = 1$ and $v_f(1) = v_f(0) + 2$. from *Case 2.b* $e_f(1) = e_f(0) + 1$, and let $f'(u,w) = 0$. Then $f'(u) = 1$,$f'(w) = 0$ and $e_{f'}(0) = e_{f'}(1)$, $v_{f'}(1) = v_{f'}(0) + 1$.

Case 4: $n - 1 \equiv 3 \pmod 4$ implies that $v_f(1) = v_f(0) + 1$ and $e_f(0) = e + f(1)$, by Corollary 1.2 and as we have obtained in *Case 3*. Adding a new vertex to T_{n-1} results in T_n with $n \equiv 0 \pmod 4$.

4.a) Assume $f(u) = 0$ and let $f'(u,w) = 0$. Then $f'(u) = 0$, $f'(w) = 0$ and $e_{f'}(0) = e_{f'}(1) + 1$, $v_{f'}(0) = v_{f'}(1)$.

4.b) Assume $f(u) = 0$ and let $f'(u,w) = 1$. Then $f'(u) = 1$, $f'(w) = 1$ and $e_{f'}(1) = e_{f'}(0) + 1$, $v_{f'}(1) = v_{f'}(0) + 4$.

4.c) Assume $f(u) = 1$ and let $f'(u,w) = 0$. Then $f'(u) = 1$, $f'(w) = 0$ and $e_{f'}(0) = e_{f'}(1) + 1$, $v_{f'}(0) = v_{f'}(1)$.

4.d) Assume $f(u) = 1$ and let $f'(u,w) = 1$. Then $f'(u) = 0$, $f'(w) = 1$ and $e_{f'}(1) = e_{f'}(0) + 1$, $v_{f'}(0) = v_{f'}(1)$.
This completes the induction step, and thus the proof. We conclude that any tree T_n, $n \not\equiv 2 \pmod 4$ can have an E-cordial labelling. □

3 Complete Graphs and Complete Bipartite Graphs

Theorem 3.1 *The complete graph K_n is E-cordial for all $n \not\equiv 2 \pmod 4$.*

Proof: Necessity follows from Theorem 1.1.
For sufficiency, the induction step is given as follows;

Case a) Let f be the E-cordial labelling of K_n , when $n \equiv 3 \pmod 4$, i.e. $n = 4k + 3$. By Corollary 1.2, we have $v_f(1) = v_f(0) + 1$ and we can assume w.l.o.g. that $e_f(1) = e_f(0) + 1$.
Let

$$f(v_i) = \begin{cases} 0 & \text{i=1,2,...,2k+1} \\ 1 & \text{i=2k+2,...,4k+3} \end{cases}$$

Add a new vertex v_{n+1}, adjacent to each vertex of K_n, thus obtaining K_{n+1} . Let f' be a binary labelling of K_{n+1}, such that:

$$f'(v_i, v_{n+1}) = \begin{cases} 1 & \text{i=1,3,...,4k+3} \\ 0 & \text{i=2,4,...,4k+2} \end{cases}$$

and

$$f'(v_i) = \begin{cases} 1 & \text{i=1,3,...,2k+1 and i=2k+2,...,4k+2} \\ 0 & \text{i=2,4,...,2k and i=2k+3,...,4k+3} \end{cases}$$

and it follows that $f'(v_{n+1}) = 0$ and $v_{f'}(0) = v_{f'}(1)$, $e_{f'}(0) = e_{f'}(1)$ .Therefore f' is an E-cordial labelling of K_{n+1}, where $n + 1 \equiv 0 \pmod 4$.

Case b) Let f be the E-cordial labelling of K_n, when $n \equiv 0 \pmod 4$, i.e. $n = 4k$. We have $v_f(0) = v_f(1)$ and $e_f(0) = e_f(1)$.Let,

$$f(v_i) = \begin{cases} 0 & \text{i=1,2,...,2k} \\ 1 & \text{i=2k+1,...,4k} \end{cases}$$

Add a new vertex v_{n+1}, adjacent to each vertex of K_n, thus obtaining K_{n+1}.Let f' be binary labelling of K_{n+1}, such that;

$$f'(v_i, v_{n+1}) = \begin{cases} 1 & \text{i=1,3,...,4k-1} \\ 0 & \text{i=2,4,...,4k} \end{cases}$$

and

$$f'(v_i) = \begin{cases} 1 & \text{i=1,3,...,2k-1 and i=2k+2,2k+4,...,4k} \\ 0 & \text{i=2,4,...,2k and i=2k+1,2k+3,...,4k-1} \end{cases}$$

So it follows that $f'(v_{n+1}) = 0$ and $v_{f'}(0) = v_{f'}(1) + 1$, $e_{f'}(0) = e_{f'}(1)$. Therefore f' is an E-cordial labelling K_{n+1} , where $n + 1 \equiv 1 \pmod 4$.

Case c) Let f be the E-cordial labelling of K_n, when $n \equiv 1 \pmod 4$. i.e. $n = 4k + 1$. We have by Corollary 1.1, $v_f(0) = v_f(1) + 1$ and $e_f(0) = e_f(1)$.
Let

$$f(v_i) = \begin{cases} 0 & \text{i=1,2,...,2k+1} \\ 1 & \text{i=2k+2,2k+3,...,4k+1} \end{cases}$$

Add a new vertex v_{n+1}, adjacent to each vertex of K_n, thus obtaining K_{n+1} . Let f' be a binary labelling of K_{n+1}, such that;

$$f'(v_i, v_{n+1}) = \begin{cases} 1 & \text{i=1,3,...,4k+1} \\ 0 & \text{i=2,4,...,4k} \end{cases}$$

and,

$$f'(v_i) = \begin{cases} 1 & \text{i=1,3,...,2k+1 and i=2k+2,2k+4,...,4k} \\ 0 & \text{i=2,4,...,2k and i=2k+3,2k+5,...,4k+1} \end{cases}$$

So it follows that $f'(v_{n+1}) = 1$ and $v_{f'}(1) = v_{f'}(0) + 2$, $e_{f'}(1) = e_{f'}(0) + 1$
It is clearly seen that f' is not an E-cordial labelling of K_{n+1}, where

$n+1 \equiv 2 \pmod 4$, as Theorem 1.1 implies.

Case d) Let f be the binary labelling of K_n , when $n \equiv 2 \pmod 4$, i.e. $n = 4k+2$, with $v_f(1) = v_f(0) + 2$ and $e_f(1) = e_f(0) + 1$ as obtained in *Case c*. Let,

$$f(v_i) = \begin{cases} 0 & \text{i=1,2,...,2k} \\ 1 & \text{i=2k+1,2k+2,...,4k+2} \end{cases}$$

Add a new vertex v_{n+1} , adjacent to each vertex of K_n , thus obtaining K_{n+1} .Let f' be binary labelling of K_{n+1}, such that;

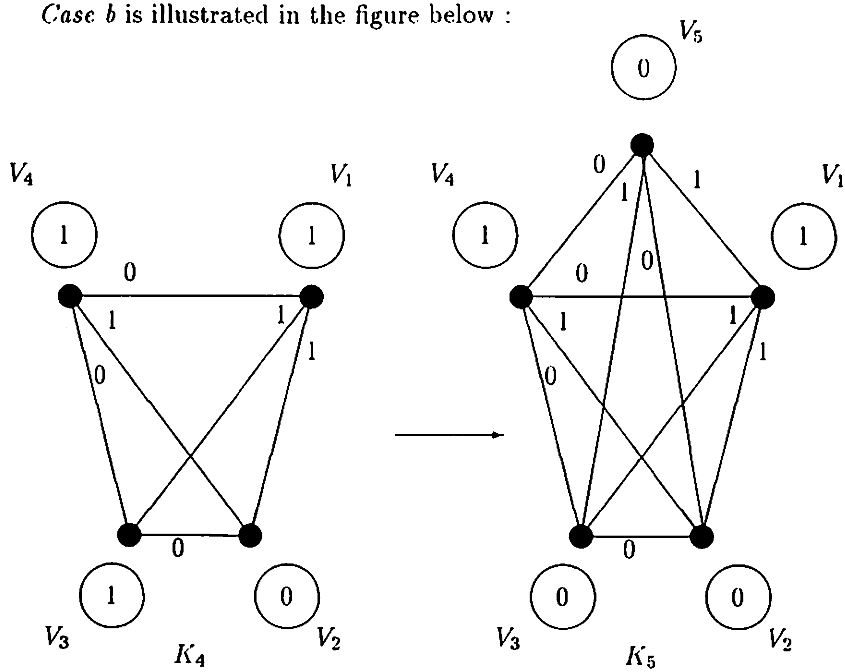
$$f'(v_i, v_{n+1}) = \begin{cases} 1 & \text{i=1,3,...,4k+1} \\ 0 & \text{i=0,2,...,4k+2} \end{cases}$$

and

$$f'(v_i) = \begin{cases} 1 & \text{i=1,3,...,2k-1 and i=2k+2,2k+4,...,4k+2} \\ 0 & \text{i=2,4,...,2k and i=2k+1,2k+3,...,4k+1} \end{cases}$$

So it follows that $f'(v_{n+1}) = 1$ and $v_{f'}(1) = v_{f'}(0) + 1$, $e_{f'}(1) = e_{f'}(0) + 1$.Therefore f' is an E-cordial labelling of K_{n+1} , where $n+1 \equiv 3 \pmod 4$. And this completes the proof. □

Case b is illustrated in the figure below :



The number of vertices in the complete bipartite graph $K_{n,m}$ is $v = n + m$, and the number of edges is $e = n \cdot m$. Let
$N = \{u_1, u_2, \ldots, u_n\}$,
$M = \{w_1, w_2, \ldots, w_m\}$.
The vertices of $K_{n,m}$ are labelled as follows:

$$f(u_i) = \sum_{j=1}^{m} f(e_{i,j}) \pmod 2, i = 1, 2, \ldots, n$$

$$f(w_j) = \sum_{i=1}^{n} f(e_{j,i}) \pmod 2, j = 1, 2, \ldots, m.$$

Let L be an $n \times m$ matrix, consisting of 0's and 1's, such that :
for $i = 1, \ldots, n$,

$$\sum_{j=1}^{m} L[i,j] = f(u_i)$$

and for $j = 1, \ldots, m$,

$$\sum_{i=1}^{n} L[i,j] = f(w_j)$$

i.e.

$$L = \begin{bmatrix} f(e_{1,1}) & f(e_{1,2}) & \cdots & f(e_{1,m}) \\ f(e_{2,1}) & f(e_{2,2}) & \cdots & f(e_{2,m}) \\ f(e_{3,1}) & \cdots & & \\ \vdots & & & \\ f(e_{n,1}) & f(e_{n,2}) & \cdots & f(e_{n,m}) \end{bmatrix}$$

We will use this matrix for denoting the labelling of $K_{n,m}$. And for standardization we will assume $n \geq m$.

Lemma 3.1 *The complete bipartite graph $K_{n,n}$ is E-cordial iff $n \equiv 0 \pmod 2$.*

Proof: Necessity follows from Theorem 1.1. For sufficiency we present an edge labelling algorithm.
Let

$$L[i,j] = \begin{cases} 1 & \text{if } i < j \\ 0 & \text{if } i > j \end{cases}$$

and

$$L[i,i] = \begin{cases} 1 & \text{if } i \le \frac{n}{2} \\ 0 & \text{if } i > \frac{n}{2} \end{cases}$$

and the resulting matrix L will represent an E-cordial labelling of $K_{n,n}$ with $e_f(0) = e_f(1)$ and $v_f(0) = v_f(1)$. □

Theorem 3.2 *The complete bipartite graph $K_{m,n}$ is E-cordial for all m,n such that $m + n \not\equiv 2 \pmod 4$.*

Proof: Necessity follows from Theorem 1.1. For sufficiency we present an edge labelling algorithm. There are 4 possible cases for n and m:
1) $n = odd$, $m = odd$;
2) $n = even$, $m = even$;
3) $n = odd$, $m = even$;
4) $n = even$, $m = odd$.
Let us examine E-cordiality of $K_{n,m}$ in each case:

Case 1: Both n and m are *odd*, therefore it is required that $v_f(0) = v_f(1)$, for f to be an E-cordial labelling of $K_{n,m}$. Assume w.l.o.g. that $n > m$ ($n \neq m$ by Lemma 3.1). Label the edges of $K_{n,m}$ as follows:
Let

$$L[i,j] = \begin{cases} 0 & i = 1,\ldots,\frac{n-1}{2}\ ,\ j = 1,\ldots,m \\ 1 & i = \frac{n+3}{2},\ldots,n\ ,\ j = 1,\ldots,m \end{cases}$$

and if $n \equiv 1 \pmod 4$

$$L[\frac{n+1}{2},j] = \begin{cases} 0 & j = 1,\ldots,\frac{m+1}{2} \\ 1 & j = \frac{m+3}{2},\ldots,m \end{cases}$$

if $n \equiv 3 \pmod 4$

$$L[\frac{n+1}{2},j] = \begin{cases} 0 & j = 1,\ldots,\frac{m-1}{2} \\ 1 & j = \frac{m+1}{2},\ldots,m \end{cases}$$

The resulting induced vertex labels will give $v_f(0) = v_f(1)$.

Case 2: Both n and m are *even*, therefore it is required that $v_f(0) = v_f(1)$ and $e_f(0) = e_f(1)$, for f to be an E-cordial labelling of $K_{n,m}$. The case $n = m$ was proven to be E-cordial in Lemma 3.1. Now label the edges of $K_{n,m}$ as follows:

(i) If $n \equiv m \equiv 2 \pmod 4$, then let $k = \frac{n+m}{4}$;
for $i = 1, \ldots, k, j = 1, \ldots, m$ $L[i,j] = 1$,
for $i = k+1, \ldots, \frac{n}{2}$;

$$L[i,j] = \begin{cases} 1 & j = 1, \ldots, \frac{m}{2} \\ 0 & j = \frac{m}{2}+1, \ldots, m \end{cases}$$

for $i = \frac{n}{2}+1, \ldots, n-k$;

$$L[i,j] = \begin{cases} 0 & j = 1, \ldots, \frac{m}{2} \\ 1 & j = \frac{m}{2}+1, \ldots, m \end{cases}$$

for $i = n-k+1, j = 1, \ldots, m$ $L[i,j] = 0$.
Such a labelling will result in $v_f(0) = v_f(1)$ and $e_f(0) = e_f(1)$.

(ii) If $n \equiv m \equiv 0 \pmod 4$, initialize L as follows :
for $i = 1, \ldots, \frac{n}{2}$,

$$L[i,j] = \begin{cases} 1 & j = 1, \ldots, \frac{m}{2} \\ 0 & j = \frac{m}{2}+1, \ldots, m \end{cases}$$

for $i = \frac{n}{2}+1, \ldots, n$,

$$L[i,j] = \begin{cases} 0 & j = 1, \ldots, \frac{m}{2} \\ 1 & j = \frac{m}{2}+1, \ldots, m \end{cases}$$

Then choose an arbitrary entry of L, namely $L[a,c]$, and switch the parities of $L[a,c]$ and $L[n-a+1,c]$. This technique is called *Column Parity Generator*. Finally, apply the *Row Parity Generator Technique* on all entries of a row, that is to say, choose an arbitrary row r of L, and change the parities of all entries $L[r,j]$, $j = 1, \ldots, m$.
The resulting matrix L will represent an E-cordial labelling f, with $v_f(0) = v_f(1)$ and $e_f(0) = e_f(1)$.

Case 3: n is *odd* and m is *even*, therefore it is required that $e_f(0) = e_f(1)$ and $| v_f(0) - v_f(1) | = 1$, for f to be an E-cordial labelling of $K_{n,m}$.
Apply the labelling as follows :
Initialize L as

$$L[i,j] = \begin{cases} 1 & i = 1, \ldots, \frac{n-1}{2}, \ j = 1, \ldots, m \\ 0 & i = \frac{n+3}{2}, \ldots, n, \ j = 1, \ldots, m \end{cases}$$

for $i = \frac{n+1}{2}$,

$$L[i,j] = \begin{cases} 1 & j = 1, \ldots, \frac{m}{2} \\ 0 & j = \frac{m}{2} + 1, \ldots, m \end{cases}$$

Then select an arbitrary column c of L, and let k denote the number of entries on which we apply the *column parity generator* technique.

(*i*) If $m \equiv 0 \pmod 4$ and $n + m \equiv 1 \pmod 4$ then $k = \frac{n-1}{4}$;
(*ii*) If $m \equiv 0 \pmod 4$ and $n + m \equiv 3 \pmod 4$ then $k = \frac{n+1}{4}$;
(*iii*) If $m \equiv 2 \pmod 4$ and $n + m \equiv 1 \pmod 4$ then $k = \frac{n-3}{4}$;
(*iv*) If $m \equiv 2 \pmod 4$ and $n + m \equiv 3 \pmod 4$ then $k = \frac{n-1}{4}$.

After transposing the parities of k entries on column c with their symmetrics, the resulting L matrix will represent an E-cordial labelling of $K_{n,m}$ with $e_f(0) = e_f(1)$ and $v_f(0) = v_f(1) + 1$ if $n + m \equiv 1 \pmod 4$; $v_f(1) = v_f(0) + 1$ if $n + m \equiv 3 \pmod 4$.

Case 4: n is *even* and m is *odd*, therefore it is required that $e_f(0) = e_f(1)$ and $| v_f(0) - v_f(1) | = 1$, for f to be an E-cordial labelling of $K_{n,m}$. Apply the labelling as follows :
Initialize L as
for $i = 1, \ldots, n$,

$$L[i,j] = \begin{cases} 1 & j = 1, \ldots, \frac{m-1}{2} \\ 0 & j = \frac{m+3}{2}, \ldots, m \end{cases}$$

for $j = \frac{m+1}{2}$,

$$L[i,j] = \begin{cases} 1 & i = 1, \ldots, \frac{n}{2} \\ 0 & j = \frac{n}{2} + 1, \ldots, n \end{cases}$$

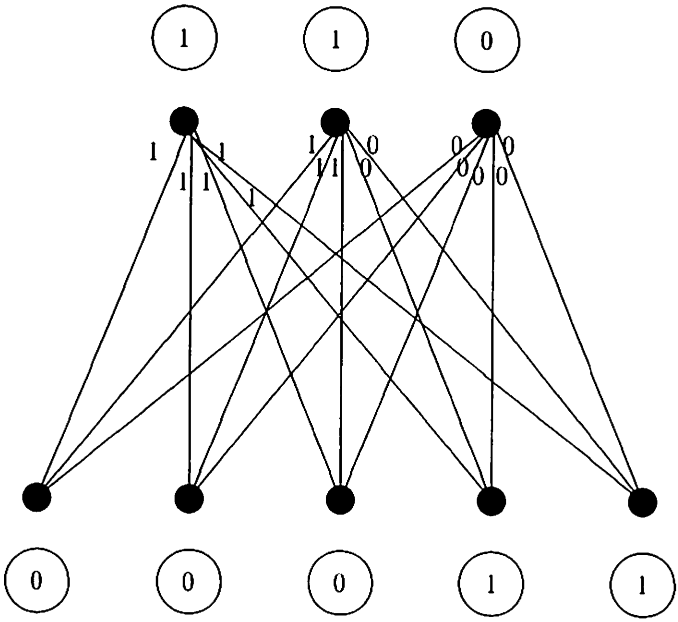
Then select an arbitrary row r of L, and let k denote the number of entries on which we apply the *row parity generator* technique.

(*i*) If $n \equiv 0 \pmod 4$ and $n + m \equiv 1 \pmod 4$ then $k = \frac{m-1}{4}$;
(*ii*) If $n \equiv 0 \pmod 4$ and $n + m \equiv 3 \pmod 4$ then $k = \frac{m+1}{4}$;
(*iii*) If $n \equiv 2 \pmod 4$ and $n + m \equiv 1 \pmod 4$ then $k = \frac{m-3}{4}$;
(*iv*) If $n \equiv 2 \pmod 4$ and $n + m \equiv 3 \pmod 4$ then $k = \frac{m-1}{4}$.

After transposing the parities of k entries on row r with their symmetrics, the resulting L matrix will represent an E-cordial labelling of $K_{n,m}$ with $e_f(0) = e_f(1)$ and $v_f(0) = v_f(1) + 1$ if $n + m \equiv 1 \pmod 4$; $v_f(1) = v_f(0) + 1$ if $n + m \equiv 3 \pmod 4$.
So we conclude that $K_{n,m}$ is E-cordial $\forall n, m : n + m \not\equiv 2 \pmod 4$, and this completes the proof.□

Case 1 is illustrated below for $K_{3,5}$:

$$L = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



$K_{3,5}$

4 Cycles and Perfect Matchings

The cycle of length n, C_n, is the graph with n vertices $v_1, v_2, \ldots, v_n$ and the edges $e_{1,2}, e_{2,3}, \ldots, e_{n,1}$.

Theorem 4.1 *The cycle C_n is E-cordial if and only if $n \not\equiv 2 \pmod 4$.*

Proof: Necessity follows from Theorem 1.1. For sufficiency, label the edges of C_n as follows:

$$f(e_{i,j}) = \begin{cases} 0 & \text{if } i \equiv 1,2 \pmod 4 \\ 1 & \text{if } i \equiv 0,3 \pmod 4 \end{cases}$$

It can easily be verified that f is E-cordial, since we have for $n \equiv 0 \pmod 4$ $v_f(0) = v_f(1)$ and $e_f(0) = e_f(1)$; for $n \equiv 1 \pmod 4$ $v_f(0) = v_f(1) + 1$ and $e_f(0) = e_f(1) + 1$; and for $n \equiv 3 \pmod 4$ $v_f(1) = v_f(0) + 1$ and $e_f(0) = e_f(1) + 1$. □

Theorem 4.2 *The regular graph of degree 1 on $2n$ vertices, $L(2n)$, is E-cordial if and only if $n \not\equiv 1 \pmod 2$.*

Proof: Necessity follows from Theorem 1.1. It is not possible to have an E-cordial labelling of $L(2n)$ when $2n \equiv 2 \pmod 4$, i.e. $n \equiv 1 \pmod 2$. For each of the n edges in $L(2n)$, the two vertices incident with that edge takes the same label as it. Therefore we always have $v_f(0) = 2e_f(0)$ and $v_f(1) = 2e_f(1)$.
If $n \equiv 0 \pmod 2$, then we have $e_f(0) = e_f(1)$ and $v_f(0) = v_f(1)$. However, when $n \equiv 1 \pmod 2$ we either have $e_f(0) = e_f(1) + 1$ and $v_f(0) = v_f(1) + 2$, or $e_f(1) = e_f(0) + 1$ and $v_f(1) = v_f(0) + 2$. □

5 Friendship Graphs, Fans, and Wheels

The friendship graph F_n consists of n triangles with a common vertex. F_n has $v = 2n + 1$ vertices and $e = 3n$ edges.

Theorem 5.1 *The friendship graph F_n is E-cordial for all $n \geq 1$.*

Proof: The necessary condition for E-cordiality, $v \not\equiv 2 \pmod 4$ (Theorem 1.1), always holds for F_n since it has $2n + 1 \equiv 1 \pmod 2$ vertices. For sufficiency, consider the frienship graph as the union of a *star* S_{2n} and n *1-factors*. Label the edges of S_{2n} as:

$$f(e_i) = \begin{cases} 1 & i = 1, 3, 5, \dots \\ 0 & i = 2, 4, 6, \dots \end{cases}$$

Then starting with the one adjacent to e_1, label the 1-factors with $1, 0, 1, 0, 1, 0, \dots$ in clockwise direction. This will obviously result in an E-cordial labelling of F_n. □

The fan f_n ($n \geq 2$) is obtained by joining all vertices of the path P_n to a further vertex called the center. f_n has $v = n+1$ vertices and $e = 2n-1$ edges.

Theorem 5.2 *The fan f_n is E-cordial if and only if $n \not\equiv 1 \pmod 4$.*

Proof: Necessity follows from Theorem 1.1. For sufficiency carry out the labelling in the following manner:
Let $e_1, e_2, \ldots, e_{n-1}$ be the edges on P_n; label these edges as:

$$f(e_i) = \begin{cases} 1 & i = 1,3,5,\ldots \\ 0 & i = 2,4,6,\ldots \end{cases}$$

Then let $e_n, e_{n+1}, \ldots, e_{2n-1}$ be the edges of star S_n, and label these edges with $1,0,1,0,1,0,\ldots$ in order, starting with e_n. Such a labelling will result in an E-cordial labelling of f_n for $n \not\equiv 1 \pmod 4$. □

The wheel W_n is obtained by joining all vertices of cycle C_n to the center. W_n contains $v = n+1$ vertices and $e = 2n$ edges.

Theorem 5.3 *The wheel W_n is E-cordial if and only if $n \not\equiv 1 \pmod 4$.*

Proof: Necessity follows from Theorem 1.1.
For sufficiency apply the labelling as follows:
Let $e_1, e_2, \ldots, e_n$ be the spoke edges of W_n, and $e_{1,2}, e_{2,3}, \ldots, e_{n,1}$ be the edges of the cycle C_n ($C_n \subset W_n$). Define the labelling f as follows :

$$f(e_i) = \begin{cases} 1 & \text{if } 1 \leq i \leq \frac{n+1}{2} \\ 0 & \text{if } \frac{n-1}{2} < i \leq n \end{cases}$$

and

$$f(e_{i,i+1}) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 2 \\ 0 & \text{if } i \equiv 1 \pmod 2 \end{cases}$$

It can easily be verified that, under the labelling f
if $n \equiv 0 \pmod 4$ we have $v_f(0) = v_f(1) + 1$,
if $n \equiv 2 \pmod 4$ we have $v_f(1) = v_f(0) + 1$,
if $n \equiv 3 \pmod 4$ we have $v_f(1) = v_f(0)$.
Hence, W_n is E-cordial *iff* $n \not\equiv 1 \pmod 4$, and this completes the proof.
□

6 Generalization of the Problem and the Final Remarks

In [5], I.Cahit defined a natural generalization of cordial labelling, called *k-equitable* labelling.

Definition 6.1 *A labelling $f : V(G) \rightarrow \{0, 1, \ldots, k-1\}$ is called k-equitable if the conditions $\mid v_f(i) - v_f(j) \mid \leq 1$ and $\mid e_{\bar{f}}(i) - e_{\bar{f}}(j) \mid \leq 1$, $i \neq j$, $i, j = 0, 1, \ldots, k-1$ are satisfied, where $v_f(x)$ and $e_{\bar{f}}(x)$, $x \in \{0, 1, \ldots, k-1\}$, are the number of vertices and edges of G respectively with label x, and the induced edge-labelling $\bar{f}$ is given by $\bar{f}(u, v) = \mid f(u) - f(v) \mid$*

.

Now we combine the k-equitable labelling and the edge-graceful labelling of graphs and we define a new graph labelling technique, called E_k-*Cordial* labelling.

Definition 6.2 *Let f be an edge labelling of graph $G = \{V, E\}$, such that $f : E(G) \rightarrow \{0, 1, 2, ..., k-1\}$, and the induced vertex labelling is given as $f(v) = \sum_{\forall u} f(u, v) \pmod{k}$, where $v \in V$ and $\{u, v\} \in E$. f is called an E_k-cordial labelling of G, if the following conditions are satisfied for $i, j = 0, 1, \ldots, k-1$, $i \neq j$:*

1) $\mid e_f(i) - e_f(j) \mid \leq 1$,
2) $\mid v_f(i) - v_f(j) \mid \leq 1$;

where $e_f(i)$, $e_f(j)$ denote the number of edges, and $v_f(i)$, $v_f(j)$ denote the number of vertices labelled with i's and j's respectively.
The graph G is called E_k-cordial if it admits an E_k-cordial labelling.

The case $k = 2$ is the *E-cordial* case which forms the core of this study. In [13] the E_3-cordial labelling is defined and the E_3-cordiality of some special graphs is discussed.

The E_k-cordial labelling approaches to the edge-graceful labelling, as k increases. Edge-graceful graphs have been attracting graph theorists for the last decade, a number of conjectures have been proposed, and many problems related to the topic remain unsolved.

We expect that, a stepwise study(increasing k one by one) of E_k-cordial labelling of graphs will give us a better vision of edge-graceful graphs. It may also help the proof of some conjectures related to edge-graceful graphs.

◇

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