A FAN-TYPE RESULT FOR REGULAR FACTORS

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ABSTRACT. Let G be a connected graph of order n and let k be a positive integer with kn even and $n \geq 8k^2 + 12k + 6$. We show that if $\delta(G) \geq k$ and $\max\{d(u), d(v)\} \geq n/2$ for each pair of vertices u, v at distance two, then G has a k-factor. Thereby a conjecture of Nishimura is answered in the affirmative.

1. INTRODUCTION

All graphs considered here are simple, that is, undirected without loops or multiple edges. Let G be a graph of order $n = |V(G)|$, where $V(G)$ is the vertex set of G. For a vertex $v \in V(G)$ the neighborhood and the degree in G are denoted by $N_G(v)$ and $d_G(v)$, respectively. If no ambiguity can occur, we write $N(v)$ instead of $N_G(v)$ and $d(v)$ instead of $d_G(v)$. The graph G is k-regular, if $d(v) = k$ for every $v \in V(G)$. By $\delta(G)$ we denote the minimum degree and we let $\sigma_2(G) = \min\{d(u) + d(v)\}$, where the minimum is taken over all pairs of nonadjacent vertices $u, v \in V(G)$. The distance, denoted by $dist_G(u, v)$ or just $dist(u, v)$, between any two vertices $u, v \in V(G)$ is the minimum length of a $u - v$ path. A subgraph H of G with $V(H) = V(G)$ is called a factor of G. If H is a k-regular factor of G, then H is called a k-factor of G.

Ore [9] showed that every graph G with $\sigma_2(G) \geq n \geq 3$ has a hamilton cycle, and therefore in particular a 2-factor. This degree condition guarantees the existence of many other regular factors as the following result shows.

Theorem 1. (Iida, Nishimura [5]) *Let G be a graph of order n and let k be a positive integer with kn even and $n \geq 4k-5$. If $\delta(G) \geq k$ and $\sigma_2(G) \geq n$, then G has a k-factor.*

This theorem improved a minimum degree condition due to Katerinis [6] and Egawa and Enomoto [3] in the same way as Ore's result improved the well-known theorem of Dirac [2].

We will say that a graph G is of *Fan type*, if every pair of vertices $u, v \in V(G)$ with $dist(u, v) = 2$ satisfies $\max\{d(u), d(v)\} \geq n/2$, since Fan [4] proved that for 2-connected graphs this degree condition is sufficient for being hamiltonian. His result generalized Ore's theorem in two directions; first, by weakening the degree condition and, second, by restricting the condition only to pairs of vertices at distance two. It was shown by Nishimura that for k-factors a generalization of Theorem 1 in the first direction is also possible.

Theorem 2. (Nishimura [8]) *Let G be a connected graph of order n and $\delta(G) \geq k$, where $k \geq 3$ is an integer with kn even and $n \geq 4k-3$. If all nonadjacent vertices $u, v \in V(G)$ satisfy*

$$\max\{d_G(u), d_G(v)\} \geq \frac{n}{2},$$

then G has a k-factor.

Moreover, Nishimura conjectured in the same paper that at least in a weak sense a generalization in both directions is possible.

Conjecture. [8] *Let G be a connected graph of order n with $\delta(G) \geq k$, where k is a positive integer with kn even. If G is of Fan type and if n is sufficiently large compared to k, then G has a k-factor.*

The aim of this paper is to answer this conjecture in the affirmative by the following theorem.

Theorem 3. *Let G be a connected graph of order n with $\delta(G) \geq k$, where k is a positive integer with kn even and $n \geq 8k^2 + 12k + 6$. If G is of Fan type, then G has a k-factor.*

2. Preliminary results

We need some further notation. Let G be a graph and let $S \subseteq V(G)$. For convenience we write $d_G(S)$ instead of $\sum_{x \in S} d_G(x)$. By $G[S]$ we denote the subgraph of G induced by S. If $u \in V(G) - S$, then $e_G(u, S)$ denotes the number of edges joining u to a vertex in S. If $T \subseteq V(G) - S$, then we write $e_G(T, S)$ instead of $\sum_{u \in T} e_G(u, S)$. By $\omega(G)$ we denote the number of components of G.

Let now $D, S \subseteq V(G)$ be disjoint sets. For a positive integer k we call a component of $G - (D \cup S)$ an *odd component* (of G with respect to (D, S, k)), if $k|V(C)| + e_G(C, S)$ is odd, and by $q_G(D, S, k)$ we denote the number of odd components. Let $\Theta_G(D, S, k) = k|D| - k|S| + d_{G-D}(S) - q_G(D, S, k)$.

The following theorem is a special case of *Tutte's f-factor Theorem* [11], which was first proved by Belck [1].

Theorem 4. (k-Factor Theorem) *Let G be a graph of order n and let k be a non-negative integer with kn even. Then the following statements hold.*
(i) [11] *$\Theta_G(D, S, K)$ is even for any disjoint sets $D, S \subseteq V(G)$;*
(ii) [1], [11] *G does not have a k-factor if and only if G has a k-*Tutte-pair*, that is a pair of disjoint subsets (D, S) of $V(G)$ with $\Theta_G(D, S, k) \leq -2$.*

It is easy to see that $\Theta_G(D, S, k)$ cannot be lowered by adding edges to G, and hence the following holds.

Lemma 5. *Let G be a graph and let k be a non-negative integer. Then $\Theta_H(D, S, k) \leq \Theta_G(D, S, k)$ for every factor H of G and all disjoint sets $D, S \subseteq V(G)$.* ∎

Lemma 6. *Let G be a graph without k-factor, where $k \geq 2$ is an integer. If G has a $(k-2)$-factor, then for every k-Tutte-pair (D, S) of G it holds $|S| \geq |D| + 1$.*

Proof. Let (D, S) be a k-Tutte-pair of G, that is $\Theta_G(D, S, k) \leq -2$. Since G has a $(k-2)$-factor, we have $\Theta_G(D, S, k-2) \geq 0$ by the k-factor theorem. With $q_G(D, S, k) = q_G(D, S, k-2)$ we obtain

$$-2 \geq \Theta_G(D, S, k) - \Theta_G(D, S, k-2) = 2|D| - 2|S|,$$

and thus $|S| \geq |D| + 1$. ∎

We call a graph *k-maximal*, if it has no k-factor and is edge-maximal with respect to this property. Clearly, every graph without k-factor is a factor of a k-maximal graph. A k-Tutte-pair (D, S) of a graph G is called *tight*, if $\Theta_G(D, S, k) = -2$.

The following theorem is proved in Niessen [7].

Theorem 7. (k-Triple Theorem) *Let G be a graph of order n and let k be an integer with $1 \leq k \leq n-1$ and kn even. If G is k-maximal with $\delta(G) \geq k$, then there exists a triple (D, S, S') of subsets of $V(G)$ with $S' \subseteq S$ and $D \cap S = \emptyset$ such that the following statements hold.*
K0: *(D, S) and (D, S') are tight k-Tutte-pairs of G;*
K1: *$d_{G-D}(x) \geq k+1$ for every vertex $x \in V(G) - (D \cup S)$;*
K2: *$e_G(x, S) \leq k-1$ for every vertex $x \in V(G) - (D \cup S)$;*
K3: *$|V(C)| \geq \max\{3, k+2-|S|\}$ for every component C of $G - (D \cup S)$;*
K4: *$d_{G-D}(X) \leq k|X| - 2 + c(X) \leq k|X| - 2 + q_G(D, S', k)$ for every $\emptyset \neq X \subseteq S'$, where $c(X)$ denotes the number of odd components C of G with respect to (D, S', k) with $N_G(X) \cap V(C) \neq \emptyset$;*

K5: *the subgraph induced by S' in G has maximum degree at most $k-2$;*
K6: $d_G(y) = n-1$ *for every vertex $y \in D$;*
K7: *every component of $G-(D \cup S)$ or $G-(D \cup S')$ is complete;*
K8: *every component of $G-(D \cup S)$ or $G-(D \cup S')$ is an odd component of G with respect to (D, S, k) or (D, S', k), respectively;*
K9: $k-1 \leq d_{G-D}(x) \leq k$ *for every vertex $x \in S - S'$;*
K10: *for every component C' of $G-(D \cup S')$ it holds either $V(C') = V(C) \cup M$, where C is a component of $G-(D \cup S)$ and $M \subseteq \{x \in S - S' \mid d_{G-D}(x) = k\}$, or $V(C') = \{y\}$, where $y \in S - S'$ with $d_{G-D}(y) = k-1$;*
K11: $q_G(D, S', k) = q_G(D, S, k) + |\{x \in S - S' \mid d_{G-D}(x) = k-1\}|$.

Lemma 8. *Let G be a connected graph of Fan type. Then it holds $\omega(G-A) \leq |A|+1$ for every $A \subset V(G)$.*

Proof. The proof is by contradiction. Therefore we suppose that there exists a set $A \subset V(G)$ with $\omega(G-A) \geq |A|+2$. Let $\omega = \omega(G-A)$ and denote by $C_1, C_2, \ldots, C_\omega$ the components of $G-A$. Without loss of generality we may assume that $|V(C_1)| \leq |V(C_2)| \leq \ldots \leq |V(C_\omega)|$ holds. Since G is connected and $\omega \geq 2$, there exists a vertex $x_i \in V(C_i)$ with $N(x_i) \cap A \neq \emptyset$ for every $i \in \{1, 2, \ldots, \omega\}$. So we can find vertices $x_j, x_l \in \{x_1, x_2, \ldots, x_{|A|+1}\}$ with $dist(x_j, x_l) = 2$. Thus at least one of these vertices, say x_j, has degree at least $n/2$ in G. This yields

$$\frac{n}{2} \leq d_G(x_j) \leq |V(C_j)| - 1 + |A|.$$

Therefore, we have for every $i \in \{j, j+1, \ldots, \omega\}$

$$|V(C_i)| \geq \frac{n}{2} - |A| + 1.$$

Since $j \leq |A|+1$, we obtain

$$n \geq |A| + \sum_{i=1}^{|A|+2} |V(C_i)| \geq 2|A| + 2\left(\frac{n}{2} - |A| + 1\right) = n+2,$$

a contradiction. ∎

3. Proof of Theorem 3

The proof is by contradiction. We suppose that G is a graph without k-factor, where G and k satisfy the hypotheses of the theorem and k is chosen as small as possible.

If $k=1$, then it follows from Tutte's 1-factor Theorem [10] that there exists a set $A \subset V(G)$ such that $o(G-A) > |A|$, where $o(G-A)$ denotes the number of components of $G-A$ having odd order. Since G is of even order, it follows that $o(G-A) \geq |A|+2$. This contradicts Lemma 8.

Let now $k \geq 2$. We call the vertices of G having degree at least $n/2$ *rich* vertices.

Our main goal in the first four claims will be to find a k-Tutte-pair (D, S') of G such that S' contains *no* rich vertex. This enables us to show in Claim 5 that the number of edges joining vertices of D with vertices of S' is relatively small, that is, $e_G(D, S') \leq (k-1)|D|$.

G is a factor of a k-maximal graph G_1. The graph G_1 satisfies the hypotheses of the k-triple Theorem, and so there exists a triple (D, S, S') of subsets of $V(G_1) = V(G)$ with $D \cap S = \emptyset$ and $S' \subseteq S$ such that the statements K0-K11 hold with respect to G_1. By Lemma 5 we obtain $\Theta_G(D, S, k) \leq \Theta_{G_1}(D, S, k)$ and $\Theta_G(D, S', k) \leq \Theta_{G_1}(D, S', k)$. Therefore, (D, S) and (D, S') are k-Tutte-pairs of G by K0.

Next we show that G has a $(k-2)$-factor. This is obvious, if $k = 2$. For $k \geq 3$ it follows by the choice of k, since G and $k-2$ satisfy the hypotheses of the theorem. So, Lemma 6 can be applied to the k-Tutte-pair (D, S') of G, and thus

$$|S| \geq |S'| \geq |D| + 1. \tag{1}$$

CLAIM 1. $|D| < (n-4k)/2$.
Suppose that $|D| \geq (n-4k)/2$, that is, $n - 2|D| \leq 4k$. This yields

$$|S| - |D| = n - 2|D| - |V(G) - (D \cup S)| \leq 4k - q_G(D, S, k). \tag{2}$$

Since (D, S) is a k-Tutte-pair of G, we have by (2)

$$\begin{aligned} d_{G-D}(S) &\leq k|S| - k|D| + q_G(D, S, k) - 2 \\ &\leq k(4k - q_G(D, S, k)) + q_G(D, S, k) - 2 \\ &\leq 4k^2 - 2. \end{aligned} \tag{3}$$

Let $d = d_{G-D}(S)/|S|$ (note that $|S| > 0$ by (1)). By (1), (3) and our assumption we obtain

$$d = \frac{d_{G-D}(S)}{|S|} \leq \frac{4k^2 - 2}{|D| + 1} \leq \frac{8k^2 - 4}{n - 4k + 2} \leq \frac{k-1}{k}, \tag{4}$$

where the last estimation follows from $n \geq 8k^2 + 12k + 6$.

Let now $T = \{x \in S \mid d_{G-D}(x) = 0\}$. Then it holds $d_G(x) \leq |D| < n/2$ for every $x \in T$ by (1). Since T is an independent set in G and since G is of Fan type, it follows thereby that the neighborhoods of vertices in T are disjoint. These neighborhoods are subsets of D and hence

$$|D| \geq \left| \bigcup_{x \in T} N_G(x) \right| \geq \delta(G)|T| \geq k|T|. \tag{5}$$

Moreover, we obtain $|T| \geq |S|/k$ with (4) by

$$\frac{k-1}{k}|S| \geq d|S| = d_{G-D}(S) \geq |S| - |T|,$$

and therefore it holds with (5) $\quad |D| \geq k|T| \geq |S|$, contradicting (1).

CLAIM 2. $e_G(y, S') \leq k-1$ for every vertex $y \in V(G) - (D \cup S')$.

Let $y \in V(G) - (D \cup S')$. If $y \in V(G) - (D \cup S)$, then it holds $e_G(y, S') \leq e_G(y, S) \leq e_{G_1}(y, S) \leq k-1$ by K2. If $y \notin V(G) - (D \cup S)$, then $y \in S - S'$. By K9 we have $k-1 \leq d_{G_1 - D}(y) \leq k$. So, if $d_{G_1 - D}(y) = k-1$, we have already $e_G(y, S') \leq d_{G_1 - D}(y) \leq k-1$. Finally, if $d_{G_1 - D}(y) = k$, then y is in G_1 adjacent to at least one vertex in $V(G) - (D \cup S)$ by K10, and so we have again $e_G(y, S') \leq d_{G_1 - D}(y) - 1 = k-1$.

We call a component of $G - (D \cup S')$ a *rich component*, if it contains a rich vertex of G. Furthermore, we let p denote the number of rich components.

CLAIM 3. Every rich component contains at least $n/2 - |D| - k + 2$ vertices, and $p \leq 3$.

Let C be a rich component containing the rich vertex y. By Claim 2 we obtain

$$\begin{aligned} |V(C)| &\geq d_G(y) + 1 - (|D| + e_G(y, S')) \\ &\geq \frac{n}{2} + 1 - |D| - (k-1) = \frac{n}{2} - |D| - k + 2. \end{aligned} \tag{6}$$

This proves the first statement of this claim.

Suppose now that $p \geq 4$. Then we obtain with (6) and (1)

$$n \geq |D| + |S'| + 4\left(\frac{n}{2} - |D| - k + 2\right) \geq 2n - 2|D| - 4k + 9,$$

and therefore $|D| \geq (n - 4k + 9)/2$. This contradicts Claim 1.

CLAIM 4. It holds $d_G(x) < n/2$ for every $x \in S'$.

Let $x \in S'$. Then

$$d_{G_1 - D}(x) \leq k - 2 + c(\{x\}) \tag{7}$$

by K4, where $c(\{x\})$ denotes the number of odd components C of G_1 with respect to (D, S', k) such that $N_{G_1}(x) \cap V(C) \neq \emptyset$.

Let now c_x denote the number of components C of $G - (D \cup S')$ with $N_G(x) \cap V(C) \neq \emptyset$. Note that $c_x \leq p + 1 \leq 4$ by Claim 3, since G is of Fan type.

Since every component of $G_1 - (D \cup S')$ is an odd component of G_1 with respect to (D, S', k) by K8, it follows

$$c(\{x\}) - c_x \leq d_{G_1 - D}(x) - d_{G-D}(x).$$

This yields together with (7) and $c_x \leq 4$

$$d_{G-D}(x) \leq d_{G_1 - D}(x) - c(\{x\}) + c_x \leq k - 2 + c_x \leq k + 2.$$

Finally, we obtain with Claim 1 and $k \geq 2$

$$d_G(x) \leq |D| + d_{G-D}(x) < \frac{n - 4k}{2} + k + 2 = \frac{n}{2} - k + 2 \leq \frac{n}{2},$$

as required.

CLAIM 5. $e_G(D,S') \leq (k-1)|D|$.
Let $u \in D$. Since G is of Fan type, Claim 4 implies that $G[N_G(u) \cap S']$ is complete. Therefore it follows with K5

$$e_G(u,S') \leq \Delta(G[S']) + 1 \leq \Delta(G_1[S']) + 1 \leq k-1,$$

and hence $e_G(D,S') \leq (k-1)|D|$, as required.

Next we will obtain lower and upper bounds for $q_G(D,S',k)$. Therefore we let $A^+ = \{x \in A \mid d_G(x) \geq k+1\}$ for every $A \subseteq V(G)$.

Since (D,S') is a k-Tutte-pair of G, we obtain with Claim 5 and $\delta(G) \geq k$

$$\begin{aligned} q_G(D,S',k) &\geq k|D| - k|S'| + d_{G-D}(S') + 2 \\ &= k|D| - k|S'| + d_G(S') - e_G(D,S') + 2 \\ &\geq |D| - k|S'| + d_G(S') + 2 && (8) \\ &\geq |D| + |(S')^+| + 2. && (9) \end{aligned}$$

Let now C be a component of $G-(D \cup S')$ containing no rich vertex of G. We will show that $U_C = N_G(V(C)) \cap (D \cup S')^+ \neq \emptyset$. Therefore we suppose that U_C is empty and we consider the graph $G[V(C) \cup N_G(V(C))]$. Since U_C is empty, all vertices of this graph have degree less than $n/2$ in G, and so this graph is complete, since it is connected and G is of Fan type. Furthermore, every vertex of C has degree at least k in this subgraph, and so the minimum degree of this graph is at least k. Moreover, since G is connected, there exists a vertex z in this subgraph that is adjacent to at least one vertex outside. Clearly, z has degree at least $k+1$ in G and cannot belong to $V(C)$. But this means that $z \in U_C$, contradicting our assumption.

For the following estimation observe that $\omega(G-(D \cup S'))-p$ components of $G-(D \cup S')$ contain no rich vertex of G and that the corresponding sets U_C of these components are disjoint, since G is of Fan type. Hence we obtain

$$|(D \cup S')^+| + p \geq \omega(G-(D \cup S')) \geq q_G(D,S',k). \tag{10}$$

Now we can show that

$$p \geq 2, \quad \text{and} \quad d_G(x) \leq k+2 \quad \text{for every} \quad x \in S'. \tag{11}$$

The first statement of (11) follows immediatly from (9) and (10). To verify the second statement we assume that there exists a vertex in S' having degree at least $k+3$ in G. Then we find with (10), (8) and $\delta(G) \geq k$

$$\begin{aligned} |(D \cup S')^+| + p &\geq q_G(D,S',k) \geq |D| - k|S'| + d_G(S') + 2 \\ &\geq |D| - k|S'| + (k|S'| + |(S')^+| + 2) + 2 \\ &= |D| + |(S')^+| + 4, \end{aligned}$$

and thus $p \geq 4$, contradicting Claim 3.

The remainder of the proof can be explained as follows. By (11) we know that $D \cup S'$ separates at least two rich components, say C_1 and C_2, where we assume without loss of generality that $|V(C_1)| \leq |V(C_2)|$. Thereby it follows

$$|V(C_1)| \leq \frac{1}{2}(n - |D| - |S'|),$$

and hence we see with (1) that every rich vertex y of C_1 is joined to at least two vertices of S' by

$$e_G(y, S') \geq \frac{n}{2} - (|V(C_1)| - 1) - |D| \geq \frac{|S'| - |D|}{2} + 1 \geq \frac{3}{2}.$$

Thus, if we let $W = \{y \in V(C_1) \mid d_G(y) \geq n/2\}$, we obtain with (11)

$$2|W| \leq e_G(V(C_1), S') \leq (k+2)|S'|. \tag{12}$$

Our final aim is to obtain a lower bound for $|W|$ and an upper bound for $|S'|$, which can be used to show that (12) is impossible.

Let $y \in W$ and $x \in S' \cap N_G(y)$ and consider the set $(V(C_1) \cap N_G(y)) - N_G(x)$. The vertices in this set belong to C_1 and have distance two from x. Since $d_G(x) \leq k + 2 < n/2$ by (11) and since G is of Fan type, this set is a subset of W. So, we have with Claim 2 and (11)

$$\begin{aligned} |W| + |D| &\geq |(V(C_1) \cap N_G(y)) - N_G(x)| + |D| \\ &\geq |V(C_1) \cap N_G(y)| - d_G(x) + |D| \\ &\geq d_G(y) - (|D| + e_G(y, S')) - d_G(x) + |D| \\ &\geq \frac{n}{2} - (k-1) - (k+2) = \frac{n}{2} - 2k - 1. \end{aligned} \tag{13}$$

Now it follows with (13), Claim 3 and (9)

$$\begin{aligned} n &\geq |W| + |D| + |S'| + |V(C_2)| + q_G(D, S', k) - 2 \\ &\geq \left(\frac{n}{2} - 2k - 1\right) + |S'| + \left(\frac{n}{2} - |D| - k + 2\right) + |D| \\ &= n - 3k + 1 + |S'|, \end{aligned}$$

and hence we have

$$|S'| \leq 3k - 1. \tag{14}$$

Thereby it follows with (13) and (1)

$$|W| \geq \frac{n}{2} - 2k - 1 - |D| \geq \frac{n}{2} - 2k - 1 - (|S'| - 1) \geq \frac{n}{2} - 5k + 1.$$

This yields together with (12) and (14)

$$n - 10k + 2 \leq 2|W| \leq (k+2)|S'| \leq (k+2)(3k-1),$$

contradicting $n \geq 8k^2 + 12k + 6$. ∎

Remark. Here we would like to discuss the condition $n \geq 8k^2 + 12k + 6$ of Theorem 3. As the proof shows, this condition is not necessary for $k = 1$. Moreover, also for $k = 2$ this condition can be dropped (this follows by a simple investigation of graphs with connectivity 1, since otherwise the result is implied by Fan's result). For $k \geq 3$, Nishimura [8] presented the graphs $K_{2k-3} + (K_1 \cup (k-1)K_2)$ and $K_{2k-4} + (K_1 \cup C_{2k-1})$, where $+$ denotes join, $\cup$ denotes union, and K_n and C_n denote the complete graph and the cycle of order n, respectively. These graphs show that the condition $n \geq 4k-3$ in Theorem 2 is best possible and also that for $k \geq 3$ the condition $n \geq 8k^2 + 12k + 6$ in Theorem 3 cannot be replaced by a condition weaker than $n \geq 4k-3$.

Since we expect that Theorem 3 holds with a much smaller bound for the order, we made no efforts to obtain small improvements of the given bound.

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