Decomposition of Complete Bipartite Graphs

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Abstract

Conditions are given for decomposing $K_{m,n}$ into edge-disjoint copies of a bipartite graph G by translating its vertices in the bipartition of the vertices of $K_{m,n}$. A construction of the bipartite adjacency matrix of the d-cube Q_d is given leading to a convenient α-valuation and a proof that $K_{d2^{d-2},d2^{d-1}}$ can be decomposed into copies of Q_d for $d > 1$.

1 Cyclic Decomposition of $K_{m,n}$

In [5] A. Rosa showed that the complete graph K_{2n+1} can be cyclically decomposed into edge-disjoint copies of a graph G with n edges if the vertices of G have a certain numbering, called a *ρ-valuation*. If Z_{2n+1} is taken to be the vertex set of K_{2n+1}, then the decomposition consists of the iterates of ϕ applied to G (regarded as a subgraph of K_{2n+1}), where ϕ is the graph isomorphism induced by the vertex permutation $i \longrightarrow i+1$ of Z_n. The proof depends on the fact that the definition of a ρ-valuation guarantees that the *lengths* $|i-j|$ run from 1 to n as $\{i,j\}$ runs through the edges of G.

Consider the complete bipartite graph $K_{m,n}$ to have the edge set $Z_m \times Z_n$, and let $\phi_{r,s}$ be the map from $Z_m \times Z_n$ into itself sending (i,j) into $(i+r, j+s)$. Since the map $i \longrightarrow i+r$ has order $m/\gcd(m,r)$ on Z_m, the map $\phi_{r,s}$ has order $q = \operatorname{lcm}(m/\gcd(m,r), n/\gcd(n,s))$. In fact for any edge (i,j) the order of $\phi_{r,s}$ is also the minimal k such that $\phi^k_{r,s}(i,j) = (i,j)$, and so each orbit of the map $\phi_{r,s}$ has exactly q elements.

If a bipartite graph G has exactly one edge in each orbit, then these do not overlap as $\phi_{r,s}$ is applied successively, yielding a decomposition of $K_{m,n}$ into copies of G. To apply this we need to choose m, n, r, and s so that the number mn/q of orbits equals the number of edges of G. Since r and s only enter into the formula for q in $\gcd(r,m)$ and $\gcd(s,n)$, we assume that $r|m$ and $s|n$. Then the number of orbits is $mn/\operatorname{lcm}(m/r, n/s) = \gcd(ms, nr)$. The following lemma tells when two edges fall in the same orbit, thus giving an analog to Rosa's "length".

Lemma 1 *Let* $r|m$, $s|n$, $d = \gcd(r, s)$, $R = r/d$, $S = s/d$, *and* $k = \gcd(Sm, Rn)$. *Define* $\psi : Z_m \times Z_n \to Z_k \times Z_d$ *by* $\psi(i,j) = (Si - Rj, \lfloor i/R \rfloor)$. *Then* (i,j) *and* (I,J) *are in the same orbit with respect to* $\phi_{r,s}$ *iff* $\psi(i,j) = \psi(I,J)$.

Proof: First assume that (i,j) and (I,J) are in the same orbit. Then for some integer t

$$I \equiv i + rt \pmod{m} \tag{1}$$

and

$$J \equiv j + st \pmod{n}. \tag{2}$$

We must prove

$$SI - RJ \equiv Si - Rj \pmod{k} \tag{3}$$

and

$$\lfloor I/R \rfloor \equiv \lfloor i/R \rfloor \pmod{d}. \tag{4}$$

From (1) and (2)

$$S(I - i) \equiv Srt \pmod{Sm}$$

and

$$R(J - j) \equiv Rst \pmod{Rn}.$$

But $Srt = Rst$ and k divides both moduli, so

$$S(I - i) \equiv R(J - j) \pmod{k},$$

yielding (3). Also from (1) there exists an integer x such that $I = i + rt + mx$. Then

$$\lfloor I/R \rfloor = \lfloor i/R + rt/R + mx/R \rfloor = \lfloor i/R \rfloor + dt + (m/r)dx,$$

proving (4).

Now assume that $\psi(i,j) = \psi(I,J)$, so that (3) and (4) hold. Then (3) yields $R(J - j) - S(I - i) \equiv 0 \pmod{\gcd(Sm, Rn)}$, so there exist integers x and y such that

$$Smx + Rny = R(J - j) - S(I - i).$$

Then $S(mx + I - i) = R(-ny + J - j)$, and so R divides $mx + I - i$. Let $mx + I - i = Ru$, so $-ny + J - j = Su$. These equations yield

$$I \equiv i + Ru \pmod{m}$$

and

$$J \equiv j + Su \pmod{n},$$

so to prove (1) and (2) it suffices to show that d divides u. But we have

$$u \;=\; u + \lfloor i/R \rfloor - \lfloor i/R \rfloor = \lfloor u + i/R \rfloor - \lfloor i/R \rfloor =$$

$$\lfloor (m/R)x + I/R \rfloor - \lfloor i/R \rfloor = (m/r)dx + \lfloor I/R \rfloor - \lfloor i/R \rfloor,$$

and this is a multiple of d by (4). □

Consider the edges of $K_{m,n}$ to be $Z_m \times Z_n$. We say an edge-disjoint decomposition of $K_{m,n}$ into a set Γ of graphs is **r, s-cyclic** in case whenever G and G' are in Γ, then $\phi^t_{r,s}(G) = G'$ for some integer t. This is stronger than Rosa's definition of cyclic, but necessary to make the theorem that follows if and only if.

Theorem 1 *Let G be a bipartite graph with vertex bipartition (V_1, V_2) and edge set E. Suppose that m and n are positive integers and r, and s are integers such that $r|m$, $s|n$, and $|E| = \gcd(ms, nr)$, and let d, k, and ψ be as in Lemma 1. Then there exists an r, s-cyclic decomposition of $K_{m,n}$ into copies of G if and only if there exist one-to-one functions N_1 and N_2 from V_1 and V_2 into Z_m and Z_n, respectively, such that the function $\theta : E \rightarrow Z_k \times Z_d$ defined by $\theta(v_1, v_2) = \psi(N_1(v_1), N_2(v_2))$ is one-to-one.*

Proof: First assume that N_1 and N_2 exist that make θ one-to-one. Notice that since E and $Z_k \times Z_d$ contain the same number $\gcd(ms, nr) = dk$ of elements, if θ is one-to-one it must also be onto. We consider G as a subgraph of $Z_m \times Z_n$ by identifying the edge $\{v_1, v_2\}$ of G with the edge $\{N_1(v_1), N_2(v_2)\}$ of $Z_m \times Z_n$. Since θ is one-to-one, by Lemma 1 each edge of G is in a distinct orbit with respect to $\phi_{r,s}$. Thus applying $\phi_{r,s}$ to the subgraph G $\operatorname{lcm}(m/r, n/s)$ times gives an r, s-cyclic decomposition of $Z_m \times Z_n$ into copies of G.

Now suppose that $K_{m,n}$ (considered to have edge set $Z_m \times Z_n$) has an r, s-cyclic decomposition into copies of G. Let $G^* = (V_1^* \cup V_2^*, E^*)$ be one of the subgraphs of the decomposition, and let τ be a graph isomorphism from G to G^* taking V_1 into V_1^* and V_2 into V_2^*. Define N_1 and N_2 to be the restrictions of τ to V_1 and V_2, respectively. We must show that the function θ defined in the theorem is one-to-one. If not, then there are distinct edges e_1 and e_2 of G^* with the same image under the function ψ. Thus by Lemma 1 e_1 and e_2 are in the same orbit with respect to the function $\phi_{r,s}$, and so there exists an integer t such that $\phi^t_{r,s}(e_1) = e_2$. Since $\phi_{r,s}$ has order $\operatorname{lcm}(m/r, n/s)$ and the decomposition is r, s-cyclic, the graphs $\phi^i_{r,s}(G^*)$, $i = 1, \ldots, \operatorname{lcm}(m/r, n/s)$ are exactly the edge-disjoint graphs of the decomposition. They are disjoint because $\operatorname{lcm}(m/r, n/s)|E| = mn$. Thus we must have $\operatorname{lcm}(m/r, n/s)|t$, and so $e_1 = e_2$, contrary to the assumption. □

2 Cubes

The d-dimensional cube is usually defined as the graph Q_d with vertex set Z_2^d, where $\{x, y\}$ is an edge if and only if x and y differ in a single component. It is

$$[\,1\,],\begin{bmatrix}1&1\\1&1\end{bmatrix},\begin{bmatrix}1&1&0&1\\1&1&1&0\\0&1&1&1\\1&0&1&1\end{bmatrix},\begin{bmatrix}1&1&0&1&0&0&0&1\\1&1&1&0&0&0&1&0\\0&1&1&1&0&1&0&0\\1&0&1&1&1&0&0&0\\0&0&0&1&1&1&0&1\\0&0&1&0&1&1&1&0\\0&1&0&0&0&1&1&1\\1&0&0&0&1&0&1&1\end{bmatrix}$$

Figure 1: Bipartite adjacency matices of the first four cubes

bipartite, since every edge joins vertices with odd and even numbers of nonzero components. A definition more convenient for our purposes is that Q_d is the graph product of K_2 with itself d times. This leads to a construction of the bipartite adjacency matrix of the cube based on the recursion $Q_{d+1} = Q_d \times K_2$. The cases with d =1, 2, 3, and 4 are shown in Figure 1. In fact we *define* the cube Q_d by the positions of the 1's in these adjacency matrices. (This is an abuse of language; properly the d-cube is the graph with vertex set $\{r_1, r_2, \ldots, r_{2^{d-1}}\} \bigcup \{c_1, c_2, \ldots, c_{2^{d-1}}\}$ with $\{r_i, c_j\}$ an edge iff $(i, j) \in Q_d$.) Namely, let $Q_1 = \{(1, 1)\}$, and for $d \geq 1$ let

$$Q_{d+1} = Q_d \bigcup (Q_d + (2^{d-1}, 2^{d-1})) \bigcup \{(i, 2^d + 1 - i) : 1 \leq i \leq 2^d\}.$$

It is easily proved by induction that Q_d is a subset of $\{1, 2, ..., 2^{d-1}\}^2$ with $d2^{d-1}$ elements. The next theorem will allow us to compute directly whether an ordered pair is in Q_d. If n is a positive integer, let $t(n) = \gcd(n, 2^n)$.

Theorem 2 *If $1 \leq i, j \leq 2^{d-1}$, then $(i, j) \in Q_d$ if and only if $|i-j| \leq t(i+j-1)$.*

Proof: The proof will be by induction on d. If $d = 1$, the statement says $(1, 1) \in Q_1$ iff $0 \leq t(1)$, which is true.

Now assume that if $1 \leq i, j \leq 2^{k-1}$, then $(i, j) \in Q_k$ iff $|i-j| \leq t(i+j-1)$. We must prove this with k replaced by $k + 1$. Assume $1 \leq i, j \leq 2^k$.

Case 1: $1 \leq i, j \leq 2^{k-1}$.

Then the conclusion is clear since $\{(i, j) \in Q_{k+1} : 1 \leq i, j \leq 2^{k-1}\} = Q_k$.

Case 2: $2^{k-1} < i, j \leq 2^k$.

Then $(i, j) \in Q_{k+1}$ iff $(i - 2^{k-1}, j - 2^{k-1}) \in Q_k$ iff $|i-j| \leq t(i+j-1-2^k)$. Thus it suffices to show $t(i+j-1-2^k) = t(i+j-1)$. Note that $0 < i+j-1-2^k < i+j-1 < 2^{k+1}$. Thus if $t(i+j-1-2^k) = 2^r$ and $t(i+j-1) = 2^s$, we have $r \leq k$ and $s \leq k$. Then $2^r | (i+j-1-2^k) + 2^k = i+j-1$, so $r \leq s$, and $2^s | (i+j-1) - 2^k$, so $s \leq r$. Thus $r = s$.

Case 3: $1 \leq i \leq 2^{k-1} < j \leq 2^k$.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$N_1(r_i)$	1	2	4	3	6	5	7	8	14	15	13	16	11	10	12	9
$N_2(c_i)$	9	10	12	11	1	4	2	3	7	8	6	5	13	16	14	15

Table 1:

($\Rightarrow$) Suppose $(i, j) \in Q_{k+1}$. Then $i+j = 2^k+1$. Thus $t(i+j-1) = t(2^k) = 2^k$, and clearly $|i-j| \leq 2^k$.

($\Leftarrow$) Suppose $|i-j| \leq t(i+j-1) = 2^r$. Clearly $r \leq k$. Let $i = 2^{k-1} - a$, $a \geq 0$, and $j - 1 = 2^{k-1} + b$, $b \geq 0$. Now $2^r | i+j-1 = 2^k + b - a$, so $2^r | b - a$. But $2^r \geq |i-j| = j - i = 2^{k-1} + b + 1 - 2^{k-1} + a = a + b + 1 > |b - a|$. Thus $a = b$. Then $i + j = 2^k + 1$ and so $(i, j) \in Q_{k+1}$. □

We now use our characterization of Q_d in an application of Theorem 1.

Theorem 3 *The complete bipartite graph $K_{20,20}$ can be decomposed into five copies of Q_5.*

Proof: We apply Theorem 1 with $m = n = 20$ and $r = s = 4$. Then $d = 4$, $R = S = 1$, and $k = 20$. Note that $\gcd(ms, nr) = 80$, which is the number of edges of Q_5. The map $\psi : Z_{20} \times Z_{20} \to Z_{20} \times Z_4$ works out to $\psi(i, j) = (i-j, i)$. We define the functions N_1 and N_2 on the vertex partition $\{r_1, r_2, \ldots r_{16}\} \bigcup \{c_1, c_2, \ldots c_{16}\}$ of Q_5 as in Table 1. To complete the proof it only remains to check that the values of $(N_1(r_i) - N_2(c_j), N_1(r_i))$ are distinct in $Z_{20} \times Z_4$ for (i, j) in Q_5, that is for $|i-j| \leq t(i+j-1)$, $1 \leq i, j \leq 16$. □

3 An α-valuation of Q_d

Lemma 2 *If $1 \leq r < 2^d$, then $r + t(r) \leq 2^d$.*

Proof: Let $2^d = r + s$, where $0 < s < 2^d$. Then $t(s) < 2^d$, and so $r + t(r) = 2^d - s + t(2^d - s) = 2^d - s + t(s) \leq 2^d - s + s = 2^d$. □

Lemma 3 *Let $2 \leq k \leq 2^d$. Then $|\{(i, j) \in Q_d : i + j = k\}| = t(k-1)$.*

Proof: Fix k with $2 \leq k \leq 2^d$, and suppose $i + j = k$. Then $(i, j) \in Q_d$ iff

$$1 \leq i, j \leq 2^{d-1}, \tag{5}$$

and

$$|i - j| = |2i - k| \leq t(i + j - 1) = t(k - 1). \tag{6}$$

n	1	2	3	4	5	6	7	8	9	10
$t(n)$	1	2	1	4	1	2	1	8	1	2
$T(n)$	0	1	3	4	8	9	11	12	20	21
$T(n)+n$	1	3	6	8	13	15	18	20	29	31

Table 2:

Inequality (6) is equivalent to

$$k - t(k-1) \leq 2i \leq k + t(k-1). \tag{7}$$

Note that both extremes of (7) are odd, the left side is positive, and from Lemma 2 (with $r = k - 1$) the right side is $\leq 2^d + 1$. Thus (7) implies $0 < 2i \leq 2^d$, and so that i satisfies (5); by symmetry the same goes for j. But the number of i satisfying (7) is $[k + t(k-1) - (k - t(k-1))]/2 = t(k-1)$. □

We define T by $T(n) = \sum_{0<i<n} t(i)$. Thus $T(n+1) = T(n) + t(n)$ for $n \geq 1$. See Table 2. Note that by Lemma 3 we have $T(2^d) = |Q_d| = d2^{d-1}$.

Lemma 4 *If* $1 \leq i \leq 2^d$, *then (a)* $T(i) + T(2^d + 1 - i) = T(2^d)$, *and (b)* $T(i) + T(2^d + 1) = T(i + 2^d)$.

Proof: The proofs of (a) and (b) will be by induction on i. If $i = 1$ both parts hold since $T(1) = 0$. Now assume (a) and (b) hold for a particular value of i, $1 \leq i < 2^d$. Then (a) $T(i+1) + T(2^d + 1 - (i+1)) = T(i) + t(i) + T(2^d + 1 - i) - t(2^d - i) = T(2^d)$ by the induction hypothesis and since $t(i) = t(2^d - i)$. Likewise (b) $T(i+1) + T(2^d + 1) = T(i) + t(i) + T(2^d + 1) = T(i + 2^d) + t(i)$ by the induction hypothesis. But this equals $T(i + 2^d + 1) - t(i + 2^d) + t(i) = T(i + 2^d + 1)$ since $t(i + 2^d) = t(i)$. □

Lemma 5 *If* $(i, j) \in Q_d$, *then*

$$2T(i) + 2T(j) + i + j = 2T(i + j - 1) + t(i + j - 1) + 1. \tag{8}$$

Proof: We use induction on d. If $d = 1$, then $i = j = 1$ and (8) is easily checked. Assume the result for d, and let $(i, j) \in Q_{d+1}$, so that $1 \leq i, j \leq 2^d$.

Case 1: $1 \leq i, j \leq 2^{d-1}$.

Then $(i, j) \in Q_d$ and we can use the induction hypothesis.

Case 2: $2^{d-1} < i, j \leq 2^d$.

Then $(i - 2^{d-1}, j - 2^{d-1}) \in Q_d$, so we have

$$2T(i-2^{d-1}) + 2T(j-2^{d-1}) + i + j - 2^d = 2T(i+j-2^d-1) + t(i+j-2^d-1) + 1 \tag{9}$$

by the induction hypothesis. But by Lemma 4(b) we have

$$T(i - 2^{d-1}) + T(2^{d-1} + 1) = T(i),$$

$$T(j - 2^{d-1}) + T(2^{d-1} + 1) = T(j),$$

and

$$T(i + j - 1 - 2^d) + T(2^d + 1) = T(i + j - 1).$$

Thus (using (9) for the second equality)

$$2T(i) + 2T(j) + i + j = 2T(i - 2^{d-1}) + 2T(j - 2^{d-1}) + 4T(2^{d-1} + 1) + i + j =$$

$$2T(i + j - 2^d - 1) + t(i + j - 2^d - 1) + 1 - (i + j - 2^d) + 4T(2^{d-1} + 1) + i + j =$$

$$2T(i + j - 1) - 2T(2^d + 1) + t(i + j - 2^d - 1) + 1 + 2^d + 4T(2^{d-1} + 1).$$

Now $1 \le i + j - 2^d - 1 \le 2^d - 1$, so $t(i + j - 2^d - 1) = t(i + j - 1)$. Also $T(2^d + 1) = 2^d + d2^{d-1}$ and $T(2^{d-1} + 1) = 2^{d-1} + (d - 1)2^{d-2}$ by the remark before Lemma 4. Thus

$$2T(i) + 2T(j) + i + j =$$

$$2T(i + j - 1) - 2(2^d + d2^{d-1}) + t(i + j - 1) + 1 + 2^d + 2^{d+1} + (d - 1)2^d =$$

$$2T(i + j - 1) + t(i + j - 1) + 1.$$

Case 3: $0 < i \le 2^{d-1} < j \le 2^d$ or $0 < j \le 2^{d-1} < i \le 2^d$.

Then since $(i, j) \in Q_{d+1}$ we have $i + j = 2^d + 1$. Thus

$$\begin{aligned} 2T(i) + 2T(j) + i + j &= 2T(i) + 2T(2^d + 1 - i) + 2^d + 1 \\ &= 2T(2^d) + 2^d + 1 = 2T(2^d) + t(2^d) + 1 \\ &= 2T(i + j - 1) + t(i + j - 1) + 1, \end{aligned}$$

where the 2nd equality uses Lemma 4(a). □

We define $\alpha : Q_d \to Z^+$ by $\alpha(i, j) = T(i) + T(j) + j$.

Lemma 6 *Let* $2 \le k \le 2^d$, *and let* $j_0 = \min\{j : (i, j) \in Q_d, i + j = k\}$. *Then* $j_0 = (k - t(k - 1) + 1)/2$. *Furthermore, if* $(i, j) \in Q_d$ *and* $i + j = k$, *then* $\alpha(i, j) = T(k - 1) + j - j_0 + 1$.

Proof: By Lemma 3 there are $t(k - 1)$ elements of Q_d with $i + j = k$. By the construction of Q_d these range consecutively from (i_0, j_0) to (j_0, i_0), where $i_0 + j_0 = k$. Thus we have $i_0 - j_0 + 1 = k - j_0 - j_0 + 1 = t(k - 1)$, and so $2j_0 = k + 1 - t(k - 1)$.

For the second statement note that by Lemma 5 we have

$$2\alpha(i, j) = 2T(i) + 2T(j) + 2j = 2T(i + j - 1) + t(i + j - 1) + 1 + j - i$$

	$\Omega(c_j)$	32	30	27	25	20	18	15	13
	$T(j)+j$	1	3	6	8	13	15	18	20
	0	1	3	*	8	*	*	*	20
	1	2	4	7	*	*	*	19	*
$T(i) = \Omega(r_i)$	3	*	6	9	11	*	18	*	*
	4	5	*	10	12	17	*	*	*
	8	*	*	*	16	21	23	*	28
	9	*	*	15	*	22	24	27	*
	11	*	14	*	*	*	26	29	30
	12	13	*	*	*	25	*	31	32

Figure 2: An α-valuation of Q_4

$$
\begin{aligned}
&= 2T(k-1)+t(k-1)+1+j-i \\
&= 2T(k-1)+t(k-1)+1-i-j+2j \\
&= 2T(k-1)+t(k-1)-1-k+2j+2 \\
&= 2T(k-1)-2j_0+2j+2.
\end{aligned}
$$

□

Figure 2 illustrates the following theorem for $d = 4$.

Theorem 4 *If we define a $2^{d-1} \times 2^{d-1}$ array with i,j-entry $\alpha(i,j)$ whenever $(i,j) \in Q_d$, then the entries run consecutively from 1 to $d2^{d-1}$ as j increases along the diagonals $i+j = 2,3,\ldots,2^d$.*

Proof: Note that $\alpha(1,1) = 1$. Lemma 3 and the definition of T imply that there are $T(k-1)$ elements of Q_d in the diagonals $i+j = 2,3,\ldots,k-1$, and Lemma 6 tells us that the entries with $i+j = k$ start at $T(k-1)+1$ and increment by 1 as j does. □

Rosa [5] defines an **α-valuation** of a graph G with e edges to be a one-to-one map Ω from its vertex set to $\{0,1,\ldots,e\}$ such that (1) $\{|\Omega(x)-\Omega(y)| : \{x,y\}$ an edge of $G\} = \{1,2,\ldots,e\}$, and (2) there exists an integer λ such that if $\{x,y\}$ is an edge of G, then $\min\{\Omega(x),\Omega(y)\} \leq \lambda < \max\{\Omega(x),\Omega(y)\}$. We can see that Q_d has an α-valuation by taking $\Omega(r_i) = T(i)$ and $\Omega(c_j) = d2^{d-1}+1-T(j)-j$. Condition (1) follows from Theorem 4, and it is easily checked that $\Omega(r_i) \leq \lambda < \Omega(c_j)$ with $\lambda = (d-1)2^{d-2}$. The $d = 4$ case is also illustrated in Figure 2. This result is not new; Kotzig [3] and Maheo [4] showed that the d-cube has an α-valuation, and another proof appears in [1, pp. 65-67]. In these proofs the map Ω is defined recursively and is harder to compute than in the above formulation. For a fixed d that is not too large the easiest plan is to start with the numbering of the edges of Q_d guaranteed by Theorem 4, set the label on row 1 equal to 0, and then deduce the remaining values of $T(i)$ and $T(j) + j$ (and so Ω) from the definition of $\alpha(i, j)$.

4 Decompositions of $K_{m,n}$ into Cubes

The following theorem is proved in [2]; here is a proof using Theorems 1 and 4.

Theorem 5 *If $d \geq 1$ then $K_{d2^{d-1},d2^{d-1}}$ can be decomposed into edge-disjoint copies of Q_d.*

Proof: We apply Theorem 1 with $m = n = d2^{d-1}$, $G = Q_d$, $r = -1$, and $s = 1$. Then $|E| = d2^{d-1} = \gcd(sm, rn)$. Note that $d = \gcd(r, s) = 1$, $R = -1$, $S = 1$, and $k = d2^{d-1}$. Also $\psi(i, j) = (Si - Rj, \lfloor i \rfloor) = (i + j, 0)$ in $Z_k \times Z_d$ since $d = 1$. Take $N_1(i) = T(i)$ and $N_2(j) = T(j) + j$. Then $N_1(i) + N_2(j)$ is one-to-one in Z_k on Q_d by Theorem 4. □

A slight adjustment of the numbering of the vertices of Q_d produces the following improvement of the last result.

Theorem 6 *If $d \geq 2$ then $K_{d2^{d-2},d2^{d-1}}$ can be decomposed into edge-disjoint copies of Q_d.*

Proof: We apply Theorem 1 with $m = d2^{d-2}$, $n = d2^{d-1}$, $G = Q_d$, $r = -1$, and $s = 2$. Then $|E| = d2^{d-1} = \gcd(sm, rn)$. Note that $d = \gcd(r, s) = 1$, $R = -1$, $S = 2$, and $k = d2^{d-1}$. Also $\psi(i, j) = (Si - Rj, \lfloor i \rfloor) = (2i + j, 0)$ in $Z_k \times Z_d$ since $d = 1$. Take $N_1(i) = T(i)$ and $N_2(j) = 2(T(j) + j) - e_j$, where e_j is 0 if $j \leq 2^{d-2}$ and 1 if $j > 2^{d-2}$. Then $2N_1(i) + N_2(j) = 2\alpha(i, j) - e_j$. Note that by Theorem 4 the values of $\alpha(i, j)$ for (i, j) in Q_d run from 1 to $k/2$ for $e_j = 0$ and from $k/2 + 1$ to k for $e_j = 1$. Thus $2\alpha(i, j) - e_j$ yields exactly all even elements of Z_k for $e_j = 0$ and all odd for $e_j = 1$, and so Theorem 1 applies. □

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