The seven graphs whose H-transformations are uniquely determined

Kiyoshi Ando and Hideo Komuro
University of Electro-Communications
Tokyo, Japan

ABSTRACT. H-transformation on a simple 3-connected cubic planar graph G is the dual operation of flip flop on the triangulation G^* of the plane, where G^* denotes the dual graph of G. We determine the seven 3-connected cubic planar graphs whose H-transformations are uniquely determined up to isomorphism.

1 Introduction

In this paper all graphs are simple and planar. Let G be a graph. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For a vertex x of G, we denote the neighbourhood of x by $N_G(x)$. The number of edges incident to x is called the degree of x, and is denoted by $deg(x)$. A graph G is said to be cubic if $deg(x) = 3$ for all $x \in V(G)$. An edge e of a 3-connected graph G is said to be *contractible* if the contraction of e results in a 3-connected graph.

A plane graph is an embedding of a planar graph into the sphere. Let G be a plane graph and let F be a face of G. We denote the boundary of F by ∂F. We write $V(F)$ and $E(F)$ for $V(\partial F)$ and $E(\partial F)$, respectively. Let F and F' be two distinct faces of G. If $e \in E(F) \cap E(F')$ we say that F is adjacent to F' along e. An edge $e \in E(G)$ is said to be a triangle edge if there is a triangle T whose boundary contains e, otherwise we call it a non-triangle edge. We denote the set of non-triangle edges of G by $\tilde{E}(G)$.

Let G be a 3-connected cubic planar graph. We denote the face set of G by $\mathcal{F}(G)$. Then Euler's formula $|V(G)| - |E(G)| + |\mathcal{F}(G)| = 2$ holds. Since G is cubic, $3|V(G)| = 2|E(G)|$. Then, from the Euler's formula together with this equality, we get $-|E(G)| + 3|\mathcal{F}(G)| = 6$.

By Whitney's theorem, the embedding of a 3-connected planar graph into the sphere is unique and we can regard G as a plane graph. Let $e = xy$ be a non-triangle edge of G and let F_1 and F_2 be two adjacent faces which share the edge e in common. Let R_1 and R_2 be the faces such that $V(R_1) \cap V(e) = \{x\}$ and $V(R_2) \cap V(e) = \{y\}$. Write $N_G(x) = \{x_1, x_2, y\}$ and write $N_G(y) = \{y_1, y_2, x\}$. Since neither F_1 nor F_2 is a triangle, all six vertices x, x_1, x_2, y, y_1 and y_2 are distinct. We may assume $x_1, y_1 \in V(F_1)$ and $x_2, y_2 \in V(F_2)$. Now we define a local operation on G as follows (see Figure 1.1):

(I) In the interior of F_1 and F_2, add new vertices u and v, respectively.

(II) From G, delete the vertices x, y and all five edges incident to x or y.

(III) Add new edges x_1u, y_1u, x_2v, y_2v and uv.

We note that the faces F_1, F_2, R_1 and R_2 are distinct because G is 3-connected.

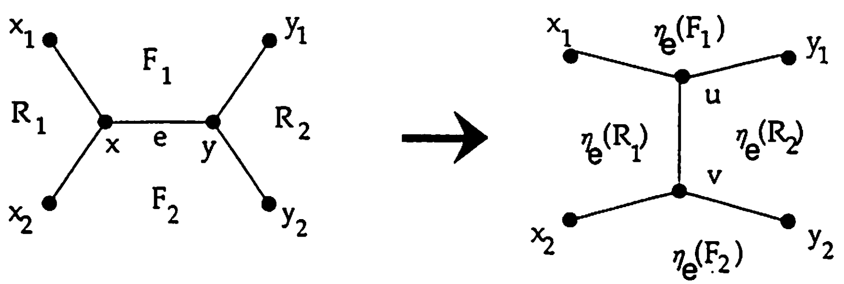


Figure 1.1

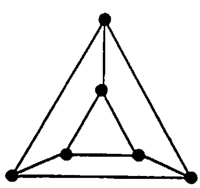
We call this local operation on G, an *H-transformation* around $e = xy$ and we denote the resulting graph by $\eta_e(G)$. The resulting graph $\eta_e(G)$ is said to be an H-transform of G. We denote the face of $\eta_e(G)$ corresponding to F by $\eta_e(F)$. "H-transformations" were originally introduced by Tsukui [3] as a local operation on cubic graphs which are not necessarily 3-connected nor planar. Because we deal with 3-connected cubic planar graphs in this paper, we adopt the definition (1.1) which is slightly different from the original one. By the definition (1.1), an H-transform of a 3-connected cubic planar graph is cubic and planar, but not necessary 3-connected. But lemma 2.2 in the next section assures us that $\eta_e(G)$ remains 3-connected if e is contractible in G. We note that there is no H-transformation around a triangle edge.

The dual graph G^* of a 3-connected cubic planar graph G is a triangulation of the plane. We note that an H-transformation of G around a contractible edge is just the dual operation of a diagonal flip on the triangulation G^*.

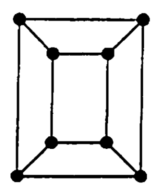
It was shown that each triangulation can be transformed to any other triangulation of the same order by finite number of diagonal flips (Wagner[4]). In other words, each 3-connected cubic planar graph can be transformed to any other 3-connected cubic planar graph of the same order by finite number of H-transformations.

If G is 3-connected cubic planar and $|V(G)| \geq 6$, then $\tilde{E}(G) \neq \phi$. Thus there are $|\tilde{E}(G)|$ H-transforms of G. In general they are not mutually isomorphic. G is said to be *HU-graph* if all H-transforms of it are mutually isomorphic, i.e., $\eta_e(G) \cong \eta_{e'}(G)$ for all e, $e' \in \tilde{E}(G)$. The complete graph with 4 vertices , K_4 is the *trivial* HU-graph. In this paper, we give the complete list of non-trivial HU-graphs. Our result is the following.

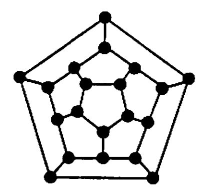
Main Theorem A non-trivial HU-graph is one of the following seven graphs:



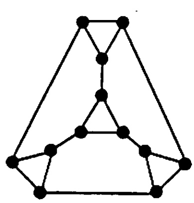
P:prism



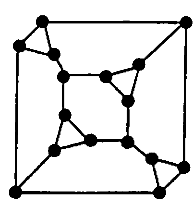
Q:cube



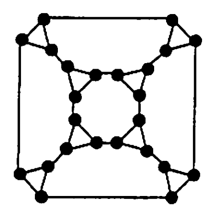
D:dodecahedron



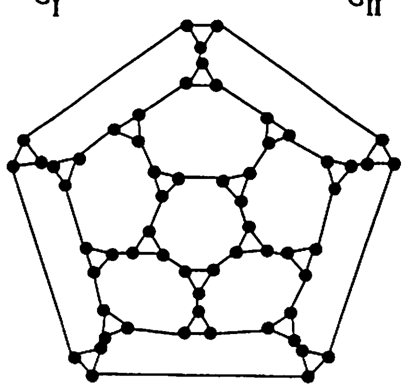
G_{I}



G_{II}



G_{III}



G_{IV}

2 Preliminaries

In this section we give some notations and preliminary results. Let G be a 3-connected graph and S be a vertex 3-cut of G. A non-empty set A of components of $G-S$ is said to be a *fragment* of $G-S$ if $V(G)-S-A \neq \phi$. On the number of contractible edges in a 3-connected graph, the following fundamental result is known.

Lemma 2.1. (Ando et al. [1]) *Let G be a 3-connected graph with $|V(G)| \geq 5$. Then, G has at least $|V(G)|/2$ contractible edges.* □

Lemma 2.2. *Let G be a 3-connected cubic planar graph and let $e = xy$ be an edge of G. Then, $\eta_e(G)$ is 3-connected if and only if e is contractible in G.*

Proof: We consider H-transformation around e.

We show that e is non-contractible if and only if $\eta_e(G)$ has 2-cut. We first assume that e is non-contractible, then there is a 3-cut S of G which includes $V(e) = \{x, y\}$. Write $S = \{x, y, z\}$, and in the notation of Fig.2.1, $S' = \{u, z\}$ is a 2-cut of $\eta_e(G)$. (see Figure 2.1)

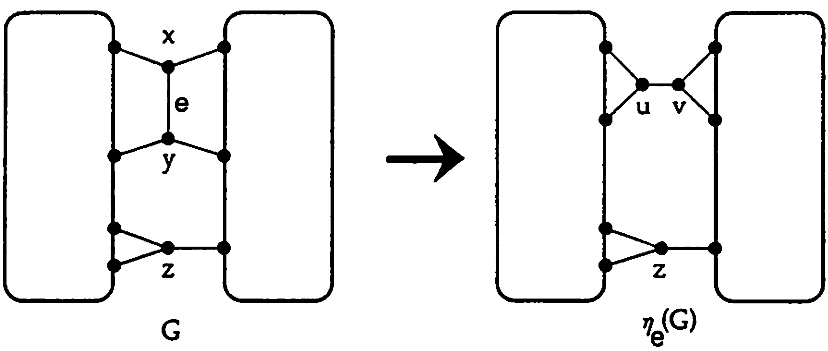


Figure 2.1

Next we prove that if $\eta_e(G)$ has a 2-cut then e is non-contractible. Assume that $\eta_e(G)$ has a 2-cut S'. If $S' \cap \{u, v\} = \phi$, then S' can be regarded as a subset of $V(G)$. Since $\eta_e(G) - S'$ is disconnected, $G - S'$ is also disconnected, which contradicts the fact that G is 3-connected. Hence $S' \cap \{u, v\} \neq \phi$. By the definition (1.1), we get $G - \{x, y\} \cong \eta_e(G) - \{u, v\}$. Since G is 3-connected, $G - \{x, y\} \cong \eta_e(G) - \{u, v\}$ is connected, and this implies that $S' - \{u, v\} \neq \phi$.

Write $S' - \{u, v\} = \{z\}$. Then, since $S' \cap \{u, v\} \neq \phi$, z is a cut vertex of $G - \{x, y\} \cong \eta_e(G) - \{u, v\}$, which implies that $G - \{x, y, z\}$ is disconnected. Thus, we get a 3-cut $\{x, y, z\}$ of G which includes $\{x, y\}$, and this means that the edge $e = xy$ is non-contractible. Now lemma 2.2 is proved. □

Corollary 2.3. *Let G be a 3-connected cubic planar graph. If G is an HU-graph, then each non-triangle edge of G is contractible.*

Proof: If $|V(G)| = 4$, then G is isomorphic to K_4 which has no non-triangle edge. Hence we may assume that $|V(G)| \geq 6$. Then, by lemma 2.1, there is a contractible edge e of G. By way of contradiction, suppose that there is a non-triangle edge e' which is not contractible. Then $\eta_{e'}(G)$ is not 3-connected, however, $\eta_e(G)$ is 3-connected which means that $\eta_{e'}(G)$ and $\eta_e(G)$ are not isomorphic, a contradiction. □

We note that if G is an HU-graph, then R_1 and R_2 in Fig 1.1 are not adjacent. Because if R_1 and R_2 are adjacent, then xy is non-contractible in G, which contradicts corollary 2.3.

Let G be a plane graph. Let $N_G[x] = N_G(G) \cup \{x\}$ and we call it the closed neighbourhood of $x \in V(G)$. For disjoint subsets S and S' of $V(G)$, let $E_G(S, S')$ denote the set of edges between S and S', namely, $E_G(S, S') = \{xy \in E(G) \mid x \in S \text{ and } y \in S'\}$. Furthermore, let $e_G(S, S') = |E_G(S, S')|$. A face F of G is said to be an i-face if $|V(F)| = i$. We denote the set of i-faces of G by $\mathcal{F}_i(G)$. Let $f_i(G) = |\mathcal{F}_i(G)|$, i.e., $f_i(G)$ denotes the number of i-faces of G. If there is no ambiguity, we write $\mathcal{F}_i$ and f_i for $\mathcal{F}_i(G)$ and $f_i(G)$, respectively. We denote the maximum value of $|V(F)|$ by $M(G)$, i.e., $M(G) = \max\{\, i \mid \mathcal{F}_i(G) \neq \phi\}$. For a face F of G, let $g_G(F; i)$ denote the number of i-faces of G which are adjacent to F, i.e., $g_G(F; i) = |\{\, F' \in \mathcal{F}_i(G) \mid E(F) \cap E(F') \neq \phi \,\}|$. A vertex $x \in V(G)$ is said to be a triangle vertex if there is a triangle $T \in \mathcal{F}_3(G)$ whose boundary contains x, otherwise we call it a non-triangle vertex. We denote the set of triangle vertices of G and the set of non-triangle vertices of G by $W(G)$ and $\tilde{W}(G)$, respectively. Furthermore, let $w(G) = |W(G)|$ and $\tilde{w}(G) = |\tilde{W}(G)|$.

Let G be a 3-connected cubic planar graph. Then, for adjacent distinct edges e, $e' \in G$, there exists just one face of G whose boundary contains both e and e'. For adjacent distinct edges e, $e' \in G$, we denote $F(e, e')$ the face of G uniquely determined by e and e'. For a non-triangle edge e of G and a given integer i, we define

$$\varphi_i(e) = f_i(\eta_e(G)) - f_i(G), \tag{2.1}$$

i.e., $\varphi_i(e)$ denotes the difference between the number of i-faces of G and the number of i-faces of $\eta_e(G)$. Let G be an HU-graph, then, by definition for any $e, e' \in \tilde{E}(G)$, $\varphi_i(e) = \varphi_i(e')$, i.e., $\varphi_i(e)$ is independent from the choice of e. We denote this constant by $\varphi_i(G)$. Let F be a face of G and let i be an integer. We define $\sigma_i(F)$ and $\rho_i(F)$ as follows:

$$\sigma_i(F) = \begin{cases} 1 & \text{if } F \in \mathcal{F}_{i-1} \\ -1 & \text{if } F \in \mathcal{F}_i \\ 0 & \text{otherwise} \end{cases} \qquad \rho_i(F) = \begin{cases} 1 & \text{if } F \in \mathcal{F}_{i+1} \\ -1 & \text{if } F \in \mathcal{F}_i \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

By definition, if $i \geq M+2$, then $\sigma_i(F) = \rho_i(F) = 0$ for each $F \in \mathcal{F}(G)$. Furthermore, we observe that $\rho_{M+1}(F) = 0$, $\sigma_{M+1}(F) \neq -1$ and $\rho_M(F) \neq 1$ for each $F \in \mathcal{F}(G)$. If we do not specify the integer i, we write $\varphi_*(e)$, $\sigma_*(F)$ and $\rho_*(F)$ for $\varphi_i(e)$, $\sigma_i(F)$ and $\rho_i(F)$, respectively.

Let $e = xy$ be a non-triangle edge of a 3-connected cubic planar graph and F_1 and F_2 be the faces which share e in common. Furthermore let R_1 and R_2 be the faces such that $V(R_1) \cap V(e) = \{x\}$ and $V(R_2) \cap V(e) = \{y\}$ (see Fig 1.1). In this situation, from (1.1), (2.1) and (2.2), we get

$$\varphi_i(e) = \sigma_i(R_1) + \sigma_i(R_2) + \rho_i(F_1) + \rho_i(F_2) \tag{2.3}$$

Lemma 2.4. *Let G be an HU-graph. Then,*

(i) *No two triangles in G have an edge in common.*

(ii) *If a triangle is adjacent to a quadrangle in G, then $G \cong P$.*

Proof: (i) Immediately follows from the fact that G is 3-connected.

We show (ii). Assume that a triangle T is adjacent to a quadrangle F_1. Write $V(T) = \{x_1, x_2, x_3\}$ and write $V(F_1) = \{x_1, x_2, y_2, y_1\}$. Moreover write $N_G(x_3) = \{x_1, x_2, y_3\}$. From (i), these six vertices are distinct. If $V(G) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$, then since G is cubic, both y_1 and y_2 are adjacent to y_3, and hence, we get $G \cong P$. By way of contradiction, suppose that $|V(G)| \geq 7$. If $y_1y_3 \in E(G)$, then $\{y_2, y_3\}$ is a two-cut of G since $|V(G)| \geq 7$, and this contradicts the fact that G is 3-connected, and hence, we get $y_1y_3 \notin E(G)$. By symmetry, we get $y_2y_3 \notin E(G)$. Write $N_G(y_1) = \{x_1, y_2, w\}$ and write $N_G(y_3) = \{x_3, z_1, z_2\}$. Let $F_2 = F(x_1x_3, x_3y_3)$, $F_3 = F(x_2x_3, x_3y_3)$, $F_4 = F(wy_1, y_1y_2)$ and $F_5 = F(z_1y_3, y_3z_2)$ (see Figure 2.2).

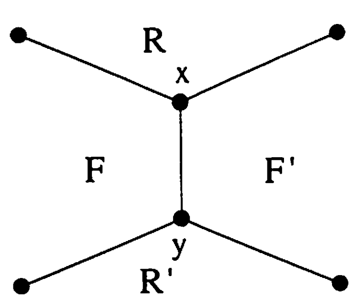


Figure 2.2

Since neither y_1 nor y_2 is adjacent to y_3, $|V(F_2)|$, $|V(F_3)| \geq 5$, and hence we get $\rho_3(F_2) = \rho_3(F_3) = 0$. Hence, from (2.3), we get

$$\begin{aligned} \varphi_3(x_1y_1) &= \sigma_3(T) + \sigma_3(F_4) + \rho_3(F_1) + \rho_3(F_2) \\ &= -1 + \sigma_3(F_4) + 1 + 0 = \sigma_3(F_4) \end{aligned} \tag{2.4}$$

Also, from (2.3), we get

$$\begin{aligned}\varphi_3(x_3y_3) &= \sigma_3(T) + \sigma_3(F_5) + \rho_3(F_2) + \rho_3(F_3) \\ &= -1 + \sigma_3(F_5) + 0 + 0 = \sigma_3(F_5) - 1\end{aligned} \tag{2.5}$$

From (2.4) and (2.5), we obtain $\sigma_3(F_4) = \sigma_3(F_5) - 1$. This together with the fact that $\sigma_3(F) = 0$ or -1 for each face $F \in \mathcal{F}(G)$ implies that $\sigma_3(F_4) = -1$, which means that F_4 is a triangle. Thus $y_2w \in E(G)$ and we see that $\{x_3, w\}$ is a two-cut of G which contradicts the fact that G is 3-connected. This contradiction completes the proof of lemma 2.4. □

Recall that $W(G)$ denotes the set of triangle vertices of G. An HU-graph G is called *triangle-type* if $V(G) = W(G)$ and called *triangle-free* if $W(G) = \phi$. If G is neither triangle-type nor triangle-free, then it is called *mixed-type*.

Lemma 2.5. *Let G be a mixed-type HU-graph. Then*

$$\tilde{E}(G) = E_G(W(G), \tilde{W}(G)).$$

Proof: Let $e = xy \in E(G)$. If either x or y is a non-triangle vertex, then xy is non-triangle, and hence $\tilde{E}(G) \supset E_G(W(G), \tilde{W}(G))$.

We show that $\tilde{E}(G) \subset E_G(W(G), \tilde{W}(G))$. Note that G is not isomorphic to the prism P which is not mixed-type. Firstly we show that $\varphi_3(G) = -1$. Take an edge $e = x_1y_1 \in E_G(W(G), \tilde{W}(G))$ such that $x_1 \in W(G)$ and $y_1 \in \tilde{W}(G)$. By lemma 2.4 (i), there is just one triangle containing x_1, say T. Write $V(T) = \{x_1, x_2, x_3\}$ and write $N_G(y_1) = \{x_1, z_1, z_2\}$. From lemma 2.4 (i), these six vertices are distinct. Let $F_1 = F(y_1x_1, x_1x_2)$, $F_2 = F(y_1x_1, x_1x_3)$ and $F_3 = F(z_1y_1, y_1z_2)$ (see Figure 2.3).

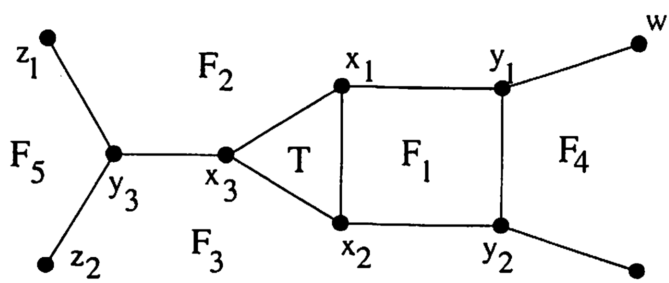


Figure 2.3

From the choice of y_1, F_3 is not triangle. By lemma 2.4(i), (ii) together with the fact that G is not isomorphic to P, neither a triangle nor a quadrangle is adjacent to T and this implies that $\rho_3(F_1) = \rho_3(F_2) = 0$. Thus, from (2.3), we get

$$\begin{aligned}\varphi_3(G) &= \varphi_3(x_1y_1)\\ &= \sigma_3(T) + \sigma_3(F_3) + \rho_3(F_1) + \rho_3(F_2)\\ &= -1 + 0 + 0 + 0 = -1.\end{aligned}$$

Now we show that each non-triangle edge belongs to $E_G(W(G), \tilde{W}(G))$. By way of contradiction, suppose that there is a non-triangle edge $uv \notin E_G(W(G), \tilde{W}(G))$. Then, either $u, v \in W(G)$ or $u, v \in \tilde{W}(G)$. We first assume that $u, v \in W(G)$. By the definition of $W(G)$, there are triangles T_u and T_v such that $u \in V(T_u)$ and $v \in V(T_v)$. Because the edge uv is non-triangle, T_u and T_v are distinct. Let F_1 and F_2 be the faces which share the edge uv in common. By lemma 2.4 (ii) together with the fact that G is not isomorphic to P, neither F_1 nor F_2 is a quadrangle and this implies that $\rho_3(F_1) = \rho_3(F_2) = 0$. Thus, from (2.3), we get

$$\begin{aligned}\varphi_3(uv) &= \sigma_3(T_u) + \sigma_3(T_v) + \rho_3(F_1) + \rho_3(F_2)\\ &= (-1) + (-1) + 0 + 0 = -2,\end{aligned}$$

and this contradicts the fact that $\varphi_3(G) = -1$. If both u and v belong to $\tilde{W}(G)$, then we clearly get $\varphi_3(uv) \geq 0$ which again contradicts the fact that $\varphi_3(G) = -1$. These contradictions complete the proof of lemma 2.5. □

Lemma 2.6. *Let G be a triangle-type HU-graph. Then*

$$\mathcal{F}(G) = \mathcal{F}_3(G) \cup \mathcal{F}_{M(G)}(G).$$

Proof: We write M for $M(G)$. By way of contradiction, suppose $\mathcal{F}(G) \neq \mathcal{F}_3(G) \cup \mathcal{F}_M(G)$. Then, since $\mathcal{F}(P) = \mathcal{F}_3(P) \cup \mathcal{F}_4(P)$, G is not isomorphic to P. If G has a quadrangle, then, since G is triangle-type, G has a triangle which is adjacent to the quadrangle, and hence, by lemma 2.4 (ii), G is isomorphic to P , a contradiction. Consequently, G has no quadrangle, and which implies that $M \geq 6$. Take a face $F_1 \in \mathcal{F}(G) - (\mathcal{F}_3(G) \cup \mathcal{F}_M(G))$ so that F_1 is adjacent to a face $F_2 \in \mathcal{F}_M(G)$ along an edge x_1y_1. Since G is triangle-type, there is the triangle T containing the vertex x_1. Write $V(T) = \{x_1, x_2, x_3\}$. Moreover write $N_G[x_i] = \{x_1, x_2, x_3, y_i\}$ $(1 \leq i \leq 3)$. Then, by lemma 2.4 (i), both x_2y_2 and x_3y_3 are non-triangle. We may assume $F(y_1x_1, x_1x_3) = F_1$ and $F(y_1x_1, x_1x_2) = F_2$. We denote the face $F(y_3x_3, x_3x_2)$ by F_3. Let T_i be the triangle containing the vertex y_i $(1 \leq i \leq 3)$ (see Figure 2.4).

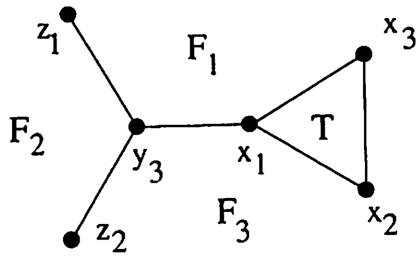


Figure 2.4

By lemma 2.4 (i), the six vertices x_1, x_2, x_3, y_1, y_2 and y_3 are distinct, and hence, the faces T, F_1, F_2 and F_3 are distinct. Furthermore, since each F_i is not quadrangle, T_1, T_2 and T_3 are also distinct. Since $M \geq 6$, $\sigma_M(T) = \sigma_M(T_2) = \sigma_M(T_3) = 0$, and since $|V(F_1)| < M$ and $|V(F_2)| = M$, $\rho_M(F_1) = 0$ and $\rho_M(F_2) = -1$. Consequently, we get

$$\begin{aligned}\varphi_M(G) &= \varphi_M(x_2y_2)\\ &= \sigma_M(T) + \sigma_M(T_2) + \rho_M(F_2) + \rho_M(F_3),\\ &= 0 + 0 + (-1) + \rho_M(F_3) = \rho_M(F_3) - 1\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\varphi_M(G) &= \varphi_M(x_3y_3)\\ &= \sigma_M(T) + \sigma_M(T_3) + \rho_M(F_1) + \rho_M(F_3).\\ &= 0 + 0 + 0 + \rho_M(F_3) = \rho_M(F_3)\end{aligned} \tag{2.7}$$

Then (2.6) and (2.7) contradict each other and this contradiction completes the proof of lemma 2.6. □

Lemma 2.7. *Let G be a mixed-type HU-graph. Then*

$$\mathcal{F}(G) = \mathcal{F}_3(G) \cup \mathcal{F}_{M(G)}(G).$$

Proof: We write M for $M(G)$. By way of contradiction, suppose $\mathcal{F}(G) \neq \mathcal{F}_3(G) \cup \mathcal{F}_M(G)$. If G has a quadrangle, then, by lemma 2.5, there is a triangle which is adjacent to the quadrangle, and hence, by lemma 2.4 (ii), G is isomorphic to P which is not mixed-type. Hence, G has no quadrangle, and which implies that $M \geq 6$. Take a face $F_1 \in \mathcal{F}(G) - (\mathcal{F}_3(G) \cup \mathcal{F}_M(G))$ so that F_1 is adjacent to a face $F_2 \in \mathcal{F}_M(G)$ along an edge xx_1. The edge xx_1 is non-triangle, and hence, by lemma 2.5, we see $xx_1 \in E_G(W(G), \tilde{W}(G))$. Without loss of generality, we assume that $x \in \tilde{W}(G)$. Write $N_G(x) = \{x_1, x_2, x_3\}$. Then xx_1, xx_2 and xx_3 are non-triangle and, again by lemma 2.5, $x_1, x_2, x_3 \in W(G)$. Let T_i be the triangle which contains x_i and write $V(T_i) = \{x_i, y_i, z_i\}$ $(1 \leq i \leq 3)$. We may assume that $y_1x_1, x_1x, xx_3, x_3z_3 \in$

$E(F_1)$ and $z_1x_1, x_1x, xx_2, x_2y_2 \in E(F_2)$. Let $F_3 = F(x_2x, xx_3)$ (see Figure 2.5).

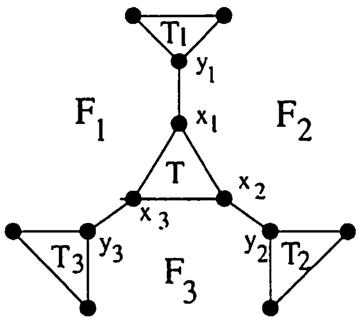


Figure 2.5

By the definition, $\rho_{M+1}(F) = 0$ for each $F \in \mathcal{F}(G)$ and, since $M \geq 6$, $\sigma_{M+1}(T) = 0$. Furthermore, since $|V(F_1)| < M$, we get $\sigma_{M+1}(F_1) = 0$. Consequently, we get

$$\begin{aligned} \varphi_{M+1}(G) &= \varphi_{M+1}(x_3x) \\ &= \sigma_{M+1}(T_3) + \sigma_{M+1}(F_2) + \rho_{M+1}(F_1) + \rho_{M+1}(F_3)\,. \\ &= 0 + 1 + 0 + 0 = 1 \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \varphi_{M+1}(G) &= \varphi_{M+1}(x_2x) \\ &= \sigma_{M+1}(T_2) + \sigma_{M+1}(F_1) + \rho_{M+1}(F_2) + \rho_{M+1}(F_3)\,. \\ &= 0 + 0 + 0 + 0 = 0 \end{aligned} \tag{2.9}$$

(2.8) and (2.9) contradict each other and this contradiction completes the proof of lemma 2.7. □

A 3-connected planar graph G is said to be regular polyhedron graph if all faces of G have the same size. There are two cubic triangle-free regular polyhedron graphs, the cube Q and the dodecahedron D[2].

Lemma 2.8. *A triangle-free HU-graph G is a regular polyhedron graph.*

Proof: We write M for $M(G)$. We show that $\mathcal{F}(G) = \mathcal{F}_M(G)$. By way of contradiction, suppose $\mathcal{F}(G) - \mathcal{F}_M(G) \neq \phi$. Then $M \geq 5$. Let $\tilde{\mathcal{F}}$ be the set of faces which are not M-faces and adjacent to an M-face, i.e., $\tilde{\mathcal{F}} = \{F \in \mathcal{F}(G) - \mathcal{F}_M(G) \mid E(F) \cap E(F') \neq \phi \text{ for some } F' \in \mathcal{F}_M(G)\}$. Take a face $F_1 \in \tilde{\mathcal{F}}$ with minimum size and let F_1 be adjacent to a face $F_2 \in \mathcal{F}_M(G)$ along an edge xx_1. Let $|V(F_1)| = M'$. Write $N_G(x) = \{x_1, x_2, x_3\}$ and write $N_G(x_i) = \{x, y_i, z_i\}$ $(1 \leq i \leq 3)$. Without loss of generality, we may assume that $y_1x_1, x_1x, xx_3, x_3z_3 \in E(F_1)$ and $z_1x_1, x_1x, xx_2, x_2y_2 \in E(F_2)$.

Let $F_3 = F(x_2x, xx_3)$ and let $R_i = F(y_ix_i, x_iz_i)$ $(1 \le i \le 3)$ (see Figure 2.6).

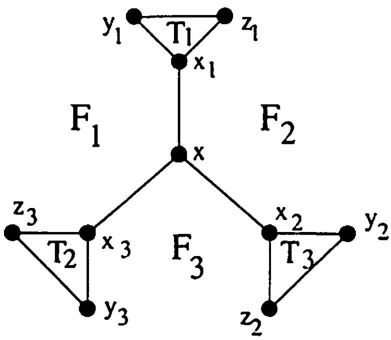


Figure 2.6

Clearly F_1, F_2 and F_3 are distinct. From $\varphi_*(x_1x) = \sigma_*(R_1) + \sigma_*(F_3) + \rho_*(F_1)+\rho_*(F_2)$, $\varphi_*(x_2x) = \sigma_*(R_2)+\sigma_*(F_1)+\rho_*(F_2)+\rho_*(F_3)$ and $\varphi_*(x_3x) = \sigma_*(R_3) + \sigma_*(F_2) + \rho_*(F_1) + \rho_*(F_3)$ we get

$$\sigma_*(R_1) + \sigma_*(F_3) + \rho_*(F_1) = \sigma_*(R_2) + \sigma_*(F_1) + \rho_*(F_3). \qquad (2.10)$$

and

$$\sigma_*(R_2) + \sigma_*(F_1) + \rho_*(F_2) = \sigma_*(R_3) + \sigma_*(F_2) + \rho_*(F_1). \qquad (2.11)$$

Firstly we show that $R_2 \in \mathcal{F}_M(G)$ and

$$\sigma_*(F_1) + \rho_*(F_2) = \sigma_*(R_3) + \rho_*(F_1). \qquad (2.12)$$

If $R_2 \in \mathcal{F}_M(G)$, then (2.12) follows from (2.11). Therefore it is enough to show that $R_2 \in \mathcal{F}_M(G)$. From (2.11), since $\sigma_{M+1}(F_1) = 0$, $\sigma_{M+1}(F_2) = 1$ and $\rho_{M+1}(F) = 0$ for each $F \in \mathcal{F}(G)$, we get $\sigma_{M+1}(R_3) + 1 = \sigma_{M+1}(R_2)$, and this equality together with the fact that $\sigma_{M+1}(F) = 0$ or 1 for each $F \in \mathcal{F}(G)$ implies that $\sigma_{M+1}(R_3) = 0$ and $\sigma_{M+1}(R_2) = 1$, and the latter equality means that $R_2 \in \mathcal{F}_M(G)$.

Next we show that $R_3 \in \mathcal{F}_{M'-1}(G)$. Since $M' < M$, we get $\rho_{M'-1}(F_2) = 0$. Hence, from (2.12), we have $\sigma_{M'-1}(F_1) = \sigma_{M'-1}(R_3) + \rho_{M'-1}(F_1)$. Since $F_1 \in \mathcal{F}_{M'}(G)$, $\sigma_{M'-1}(F_1) = 0$ and $\rho_{M'-1}(F_1) = 1$. Hence, we get $\sigma_{M'-1}(R_3) = -1$ which means that $R_3 \in \mathcal{F}_{M'-1}(G)$.

If F_3 is M-face, then the fact that F_3 is adjacent to R_3 which is $(M'-1)$-face contradicts the coice of F_1. Therefore F_3 is not M-face and we get $\sigma_{M+1}(F_3) = 0$, and since $\rho_{M+1}(F_1) = \rho_{M+1}(F_3) = \sigma_{M+1}(F_1) = 0$, from (2.10), we get $\sigma_{M+1}(R_1) = \sigma_{M+1}(R_2) = 1$. This implies that R_1 is M-face. Since both F_2 and R_1 are M-faces, the (2.10) implies

$$\sigma_*(F_3) + \rho_*(F_1) = \sigma_*(F_1) + \rho_*(F_3). \qquad (2.13)$$

The equality (2.13) implies that two faces F_1 and F_3 have the same size, M'. Now we get all sizes of the six faces; $F_2, R_2, R_1 \in \mathcal{F}_M(G)$, $F_1, F_3 \in \mathcal{F}_{M'}(G)$ and $R_3 \in \mathcal{F}_{M'-1}(G)$.

Let $e = x_1x$ and $e' = x_3x$ and we consider H-transformations around e and e'. Let $H = \eta_e(G)$ and $H' = \eta_{e'}(G)$. In view of the assumption that G is an HU-graph, H is isomorphic to H'. Recall that $g_G(F;i)$ denotes the number of i-faces of G which are adjacent to F, and observe that $\eta_e(R_1)$ is the only $(M+1)$-face of H and $\eta_{e'}(F_2)$ is the only $(M+1)$-face of H'. Hence, the equality

$$g_H(\eta_e(R_1);i) = g_{H'}(\eta_{e'}(F_2);i) \tag{2.14}$$

holds for each integer i. Now we count the numbers of $(M'-1)$-faces around $\eta_e(R_1)$ and $\eta_{e'}(F_2)$. By the choice of F_1, there is no $(M'-1)$-face around R_1 in G, i.e., $g_G(R_1; M'-1) = 0$. By the same reason, we get $g_G(F_2; M'-1) = 0$. Since $\eta_e(F_3) \in \mathcal{F}_{M'}(H)$, $\eta_e(F_1) \in \mathcal{F}_{M'-1}(H)$ and $\eta_e(F_2) \in \mathcal{F}_{M-1}(H)$, there is one $(M'-1)$-face among $\eta_e(F_3)$, $\eta_e(F_1)$ and $\eta_e(F_2)$. This implies that

$$g_H(\eta_e(R_1); M'-1) = g_G(R_1; M'-1) + 1 = 1. \tag{2.15}$$

However, since both $\eta_{e'}(F_1)$ and $\eta_{e'}(F_3)$ are $(M'$-1)-faces of H', we get

$$g_{H'}(\eta_{e'}(F_2); M'-1) = g_G(F_2; M'-1) + 2 = 2. \tag{2.16}$$

(see Figure 2.7). (2.15) and (2.16) contradict each other.

This is the final contradiction and the proof of lemma 2.8 is completed. □

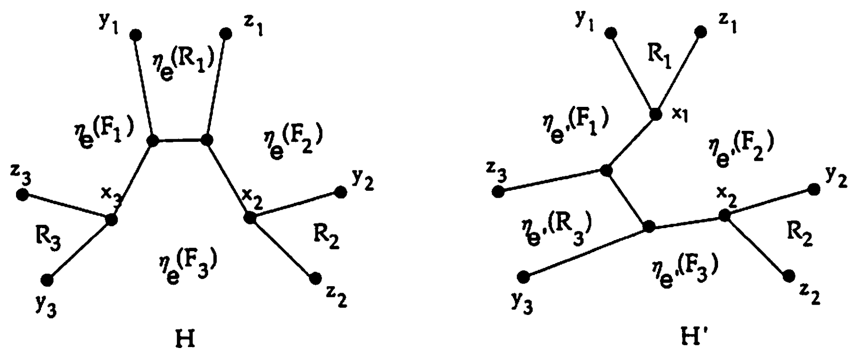


Figure 2.7

3 Proof of main theorem

In this section we give a proof of the theorem. Let G be an HU-graph. If G is triangle-free, then, by lemma 2.8, either $G \cong Q$ or $G \cong D$. Therefore we may assume that G is either mixed-type or triangle-type. Let $n = |V(G)|$, $m = |E(G)|$ and $f = |\mathcal{F}(G)|$. We write M for $M(G)$.

Claim 1. If G is either mixed-type or triangle-type, then

$$3f_3 + (6 - M)f_M = 12 \qquad (3-1)$$

Proof: Recall $-m + 3f = 6$. Since G is either mixed-type or triangle-type, lemmas 2.6 and 2.7 assure us that $f = f_3 + f_M$, and hence, we have $-m + 3f_3 + 3f_M = 6$. Since each edge belongs to two faces, $2m = 3f_3 + Mf_M$. From these two equalities, we obtain the desired equality $3f_3 + (6 - M)f_M = 12$. □

Claim 2. A mixed-type HU-graph is isomorphic to G_{II}.

Proof: Let G be a mixed-type HU-graph.

Firstly we show that $M \equiv 0 \pmod 3$. Let F be an M-face of G. Then ∂F, the boundary of F, is isomorphic to M-cycle. Let $V_T(F)$ and $V_N(F)$ be the set of triangle vertices of $V(F)$ and the set of non-triangle vertices of $V(F)$, respectively. Then lemma 2.5 assures us that $G[V_T(F)]$ is a one-factor of ∂F and $V_N(F)$ is an independent set of ∂F. Consequently, $|V_T(F)| = 2|V_N(F)|$, and hence, $M = |V_T(F)| + |V_N(F)| = 3|V_N(F)| \equiv 0 \pmod 3$.

Next we prove that $M = 6$. Recall that $w(G)$ and $\tilde{w}(G)$ are the number of triangle vertices and the number of non-triangle vertices of G. We write w and $\tilde{w}$ for $w(G)$ and $\tilde{w}(G)$, respectively. By lemma 2.4(i), we get $w = 3f_3$. By lemma 2.5, $\tilde{W}(G)$ is an independent set of G, and this implies that $e_G(\tilde{W}(G), W(G)) = 3|\tilde{W}(G)| = 3\tilde{w}$. On the other hand, since $G[W(G)]$ is a union of triangles, we observe that $e_G(W(G), \tilde{W}(G)) = |W(G)| = w$, and hence $w = 3\tilde{w}$. Since each triangle vertex belongs to two M-faces and each non-triangle vertex belongs to three M-faces, $Mf_M = 2w + 3\tilde{w}$. From $w = 3\tilde{w}$, $w = 3f_3$ and $Mf_M = 2w + 3\tilde{w}$, we get $Mf_M = 9f_3$. From this equality together with (3-1), we obtain $(9-M)f_M = 18$, which implies that $5 \le M \le 8$. This inequality together with the fact that $M \equiv 0 \pmod 3$ implies that $M = 6$.

From the above argument, we get $f_M = 18/(9-M) = 6$, $f_3 = Mf_M/9 = 4$, $\tilde{w} = f_3 = 4$, $w = 3\tilde{w} = 12$ and $n = w + \tilde{w} = 16$. Let $\tilde{G}$ be the graph obtained from G by contracting all four triangles of it. Then $\tilde{G}$ is cubic planar and $|V(\tilde{G})| = n - 2f_3 = 8$. Furthermore, since each 6-face of G has two triangle edges, the size of each face of $\tilde{G}$ is $M - 2 = 4$, and hence, $\tilde{G} \cong Q$. The four vertices of $\tilde{G}$ corresponding to the four triangles of G are independent in $\tilde{G}$. Hence the set of these four vertices is a maximum independent set of $\tilde{G} \cong Q$. Since a maximum independent set of Q is unique, we can conclude that $G \cong G_{II}$. Now Claim 2 is proved. □

Claim 3. A triangle-type HU-graph is isomorphic to one of G_I, G_{III} and G_{IV}.

Proof: Let G be a triangle-type HU-graph.

We show that $M = 6, 8$ or 10. Let F be an M-face of G. Then, lemma 2.4(i) assures us that the number of triangle edges and the number of non-triangle edges in the boundary of F are the same, and this means that M is an even integer. Since each vertex belongs to one triangle and to two M-faces, we get $3f_3 = n$ and $Mf_M = 2n$, and which imply that $Mf_M = 6f_3$. From this equality together with (3-1), we obtain $(12 - M)f_M = 24$, which implies that $5 \leq M \leq 11$. From this inequality together with the fact that M is even, we obtain that $M = 6, 8$ or 10.

Let $G^{(M)}$ be a triangle-type HU-graph with $M(G) = M$. Then, by the above argument, we get the following table.

	M	$f_M = 24/(12 - M)$	$f_3 = Mf_M/6$	$n = 3f_3$
$G^{(6)}$	6	4	4	12
$G^{(8)}$	8	6	8	24
$G^{(10)}$	10	12	20	60

Let $\tilde{G}^{(M)}$ be the graph obtained from $G^{(M)}$ by contracting all their triangles. Then $\tilde{G}^{(M)}$ is cubic planar and $|V(\tilde{G}^{(M)})| = n/3$. Furthermore, since each M-face of $G^{(M)}$ has $M/2$ triangle edges, the size of each face of $\tilde{G}^{(M)}$ is $M/2$, i.e., $G^{(M)}$ is the regular polyhedoron graph with the face size= $M/2$. Hence, we have $\tilde{G}^{(6)} \cong K_4$, $\tilde{G}^{(8)} \cong Q$ and $\tilde{G}^{(10)} \cong D$. Concequently, we can conclude that $G^{(6)} \cong G_I$, $G^{(8)} \cong G_{III}$ and $G^{(10)} \cong G_{IV}$. Now Claim 3 is proved and the proof of the Thorem is completed.

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