(a,d)-ANTIMAGIC PARACHUTES II

R. BODENDIEK and G. WALTHER
Institute of Mathematics and its Didactics
University of Kiel
Olshausenstraße 75
D–24118 Kiel

Abstract

In [2, 3], the authors dealt with the problem of determining the set $\Gamma(G)$ of all (a,d)-antimagic graphs, $a,d \in \mathbb{N}$, where the concept of an (a,d)-antimagic graph is a variation of the concept of an antimagic graph given in [4]. A connected graph $G = (V,E) \in \Gamma =$ set of all finite undirected graphs without loops and multiple edges on $n = |V| \geq 3$ vertices and $m = |E| \geq 2$ edges is said to be (a,d)-antimagic iff its edges can be assigned mutually distinct nonnegative integers from $\{1,2,\ldots,m\}$ so that the values of the vertices obtained as the sums of the numbers assigned to the edges incident to them can be arranged in the arithmetic progression $a, a+d, \ldots, a+(n-1)d$. In [2], the authors obtained some interesting general results on (a,d)-antimagic graphs from $\Gamma(G)$ by applying the theory of linear Diophantine equations and other number theoretical topics. Applying these general results to wheels $W_{g,b} = 1 * C_{g+b}$, $g \geq 3$, $b \geq 1$, $C_{g+b} =$ cycle of order $g+b$, and parachutes $P_{g,b}$ as the spanning subgraph of W_{g+b} arising from W_{g+b} by removing b successive spokes of W_{g+b} we succeeded in proving that every wheel W_{g+b} cannot be (a,d)-antimagic and, for every $g \geq 3$ or $g \geq 4$, there are the five integers $b_1 = 2g^2 - 3g - 1$, $b_2 = g^2 - 2g - 1$, $b_3 = g-1$, $b_4 = g-3$ and $b_5 = \frac{1}{2}(g^2 - 3g - 2)$ with the property that the corresponding parachute P_{g,b_i}, $i = 1,2,\ldots,5$, can be (a,d)-antimagic. If $\Gamma_i(P)$ denotes the set $\Gamma_i(P) = \{P_{g,b_i} \in \Gamma(P) \mid g \geq 3\}$, $i = 1,2,\ldots,5$, the main result in [2] says that $\Gamma_3(P)$ and $\Gamma_4(P)$ are finite, $\Gamma_3(P) = \{P_{3,2}, P_{4,3}, \ldots, P_{8,7}, P_{10,9}, P_{11,10}\}$ and $\Gamma_4(P) = \{P_{4,1}, P_{5,2}, \ldots, P_{10,7}\}$. Concerning $\Gamma_1(P)$, $\Gamma_2(P)$ and $\Gamma_5(P)$ the authors conjecture that they are infinite. Here, we continue [2] and prove

the conjecture given in [2] for $\Gamma_1(P)$ and $\Gamma_2(P)$. Instead of $\Gamma_5(P)$ we prove the infiniteness of $\Gamma'(P) = \{P_{g,\frac{1}{3}(2g^2-5g-3)} \in \Gamma(P) \mid g \equiv 0(3)$ or $g \equiv 1(3)\}$. Furthermore, we succeed in showing the existence of minimum integers $b_{min} \in \{\frac{g^2-3g-2}{2}, \frac{g^2-4g-3}{3}, \frac{g^2-5g-4}{4}, \frac{2g^2-7g-5}{5}\}$ with respect to $g \geq 26$ with the property that the parachute $P_{g,b}$ is not (a,d)-antimagic for each positive integer $b < b_{min}$. The immediate consequence of this fact is that for every $g \geq 26$ there are at most 8 different integers $b \geq b_{min}$ such that the corresponding parachute $P_{g,b}$ could be (a,d)-antimagic.

1 Introduction

In [2, 3], the authors introduce the concept of an (a,d)-antimagic parachute, $a,\ d \in \mathbb{N}$, where a parachute $P_{g,\,b}$ arises from a wheel $W_{g+b} = (V_W,\, E_W) = \{v\}*C_{g+b},\, g \geq 3$ and b elements in $\mathbb{N}$, $V_W = \{v, x_1, x_2, \ldots, x_g, v_1, v_2, \ldots, v_b\}$, $E_W = \{h_i = \{v, x_i\} \mid i = 1,\, 2,\, \ldots,\, g\} \cup \{k_i = \{v,\, v_i\} \mid i = 1,\, 2,\, \ldots,\, b\} \cup E\left(C_{g+b}\right)$ — C_{g+b} is the cycle on $g+b$ vertices $x_1, x_2, \ldots, x_g, v_1, v_2, \ldots, v_b$ and $g+b$ edges $e_i = \{v_i,\, v_{i+1}\}$, $i = 1,\, 2,\, \ldots,\, b-1$, $e'_j = \{x_j,\, x_{j+1}\}$, $j = 1,\, 2,\, \ldots,\, g-1$, $e = \{x_1,\, v_1\}$ and $e' = \{x_g,\, v_b\}$ — by removing the b edges $\{v,\, v_i\}$, $i = 1,\, 2,\, \ldots,\, b$. Figures 1a, 1b and 1c show a $(33,2)$-antimagic labeling of the parachute $P_{15,\,54}$, a $(74,1)$-antimagic labeling of $P_{15,\,68}$ and a $(84,1)$-antimagic labeling of $P_{15,\,89}$. As exactly done in [2], a connected graph $G = (V,E) \in \Gamma$ (= set of all finite undirected graphs without loops and multiple edges) of order $n = |V| \geq 3$ is said to be (a,d)-antimagic iff there exist positive integers $a, d \in \mathbb{N}$ and a bijective mapping $f : E \longrightarrow \{1, 2, \ldots, |E|\}$ such that the mapping g_f induced by f and defined by

$$g_f : \begin{cases} V \longrightarrow \mathbb{N} \\ v \longrightarrow g_f(v) = \sum\limits_{e \in I(v)} f(e), \quad v \in V \end{cases}$$

is injective and $g_f(V) = \{a, a+d, \ldots, a+(|V|-1)d\}$, where $I(v) = \{e \in E \mid e$ is incident to $v\}$ for each $v \in V$. If $G = (V,E)$ is (a,d)-antimagic and $f : E \longrightarrow \{1, 2, \ldots, |E|\}$ is a corresponding bijective mapping of G, then f is said to be an (a,d)-antimagic labeling of G. Figures 1a, 1b, 1c

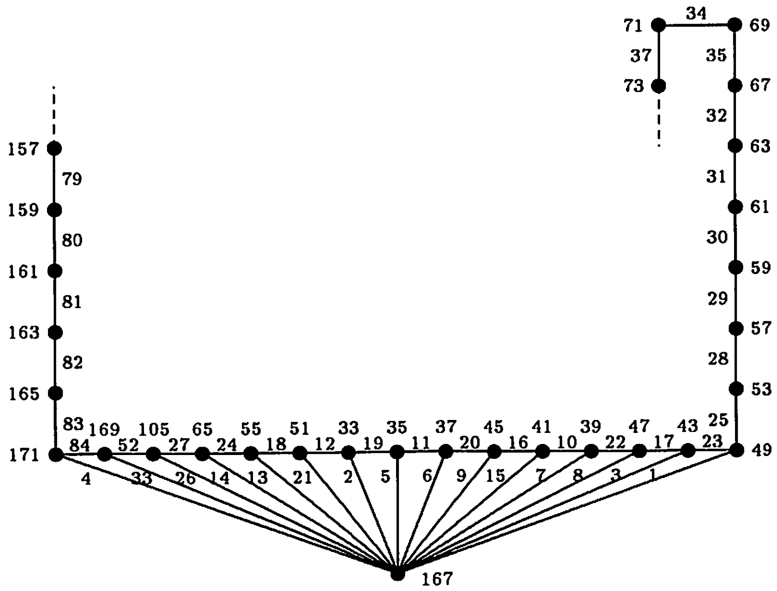

Figure 1a: $P_{15,54}$ is (33,2)-antimagic

indicate that the parachutes $P_{15,54}$, $P_{15,68}$ and $P_{15,89}$ belong to the set of all (a,d)-antimagic parachutes in Γ denoted by $\Gamma(P)$.

The aim explained in [2] is to determine the set $\Gamma(P)$ by means of number theoretical facts and results concerning linear Diophantine equations and properties of rectangular numbers. It could be shown that if a parachute $P_{g,b}$ is (a,d)-antimagic, i. e. if $P_{g,b}$ is an element of $\Gamma(P)$, then (a,d) is a solution of the linear Diophantine equation

$$(1) \qquad (2g+b)(2g+b+1) = (g+b+1)a + \frac{d}{2}(g+b)(g+b+1)$$

By means of (1) we could even show in [2] that for every $g \in \mathbb{N} - \{1,2\}$ there are at most finitely many integers $b = t - g - 1 \leq 2g^2 - 3g - 1$ such that $P_{g,b} \in \Gamma(P)$ where $t \mid 2g(g-1)$. The main result given in [2] is that $\Gamma(P)$ has the five subsets $\Gamma_1(P), \Gamma_2(P), \ldots, \Gamma_5(P)$ defined as $\Gamma_1(P) = \{P_{g,2g^2-3g-1} \mid g \geq 3\}$, $\Gamma_2(P) = \{P_{g,g^2-2g-1} \mid g \geq 4\}$, $\Gamma_3(P) = \{P_{g,g-1} \mid g \geq 3\}$, $\Gamma_4(P) = \{P_{g,g-3} \mid g \geq 4\}$ and $\Gamma_5(P) = \{P_{g,\frac{1}{2}(g^2-3g-2)} \mid g \geq 4\}$,

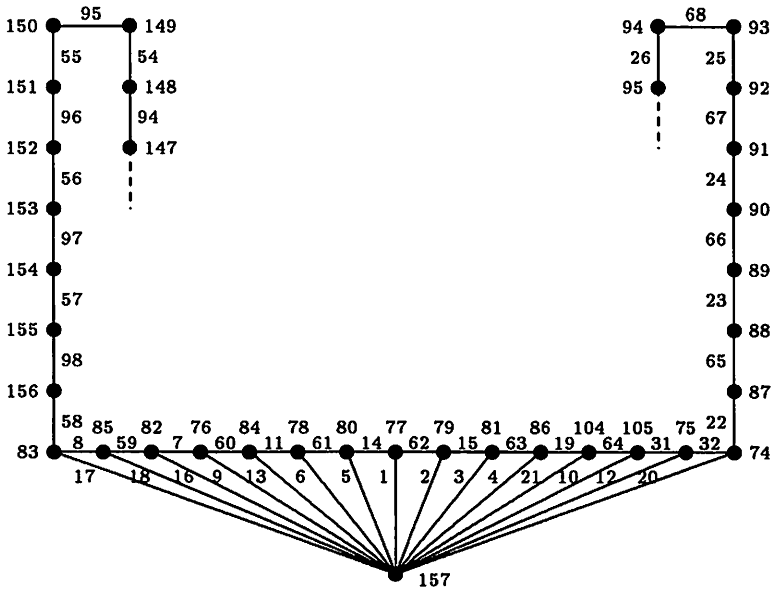

Figure 1b: $P_{15,68}$ is $(74,1)$-antimagic

with the properties $\Gamma_3 = \{P_{3,2}, P_{4,3}, P_{5,4}, P_{6,5}, P_{7,6}, P_{8,7}, P_{10,9}, P_{11,10}\}$ and $\Gamma_4 = \{P_{4,1}, P_{5,2}, P_{6,3}, P_{7,4}, P_{8,5}, P_{9,6}, P_{10,7}\}$. At the end of [2], the authors conjecture that $\Gamma_1(P)$, $\Gamma_2(P)$ and $\Gamma_5(P)$ are infinite.

Here, we are turning towards this conjecture and prove that $\Gamma_1(P)$ and $\Gamma_2(P)$ are infinite. Furthermore, we succeed in showing that a further subset $\Gamma'(P) = \{P_{g,\frac{1}{3}(2g^2-5g-3)} \mid g \equiv 0(3) \vee g \equiv 1(3),\ g \geq 15\}$ is an infinite subset of $\Gamma(P)$ and that there exists a minimum integer b_{min} such that for every positive integer $b < b_{min}$ the corresponding parachute $P_{g,b}$ is not (a,d)-antimagic for every $g \geq 26$. Hence for every $g \geq 26$ there are at most 8 different values for b, such that the parachutes $P_{g,b}$ are (a,d)-antimagic.

2 Infiniteness of $\Gamma_1(P)$ and $\Gamma_2(P)$

In order to show the infiniteness of $\Gamma_1(P) = \{P_{g,2g^2-3g-1} \mid g \geq 3\}$ we at first prove

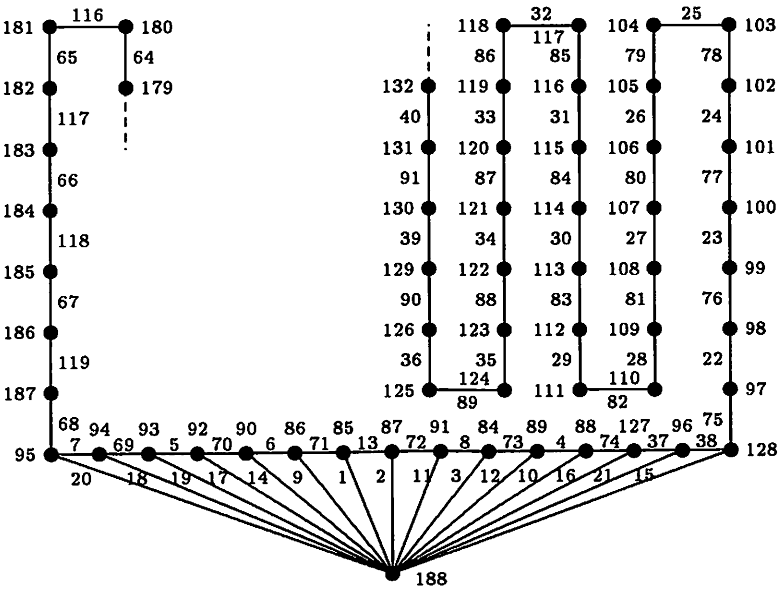

Figure 1c: $P_{15,89}$ is $(84, 1)$-antimagic

Proposition 1:

$$\bigwedge_{\substack{g \in \mathbb{N} \\ g \geq 3}} \left[P_{g,2g^2-3g-1} \in \Gamma(P) \quad \longrightarrow \quad P_{g,2g^2-3g-1} \text{ is } (g^2+g, 1)\text{-antimagic} \right]$$

Proof: Assume $P_{g,2g^2-3g-1}$ is an element of $\Gamma(P)$ for an arbitrary $g \geq 3$. Putting $b = 2g^2 - 3g - 1$ the Diophantine equation (1) becomes

(2) $\quad 2a + \left(2g^2 - 2g - 1\right) d = 4g^2 - 1.$

(2) implies that d is equal to 1 because of $a \geq 3$. Putting $d = 1$ we obtain $a = g^2 + g = g(g+1)$.

Figure 2a, 2b, 2c depict $P_{3,8}$, $P_{5,34}$, $P_{13,298} \in \Gamma_1(P)$ and corresponding $(12,1)$-antimagic and $(30,1)$-antimagic and $(182,1)$-antimagic labelings, respectively, and imply

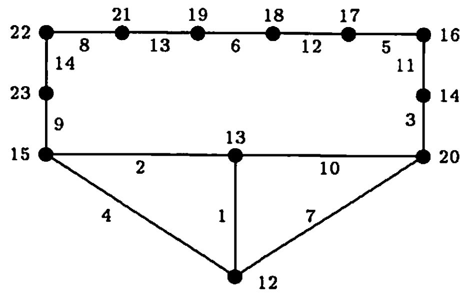

Figure 2a: $P_{3,8}$ is $(12,1)$-antimagic

Theorem 1:

$$\bigwedge_{g\in\mathbb{N}\setminus\{1,2\}} P_{g,2g^2-3g-1} \text{ is } (g^2+g,1)\text{-antimagic}$$

Proof: The proof is by construction a bijective mapping $f : E_{g,b} \longrightarrow \{1,2,\ldots,|E_{g,b}|\}$, $b = 2g^2-3g-1$, and showing that f is a $(g^2+g,1)$-antimagic labeling of $P_{g,b}$. It turns out that it is useful to distinguish the two cases $g \geq 3$ odd and $g \geq 4$ even. Let P_{g,g^2-3g-1}, $g \geq 3$ odd, be an arbitrary parachute in $\Gamma_1(P)$ with $E_{g,b} = \{h_1, h_2, \ldots, h_g, e, e', e'_1, e'_2, \ldots, e'_{g-1}, e_1, e_2, \ldots, e_{b-1}\}$ and $V_{g,b} = \{v, x_1, x_2, \ldots, x_g, v_1, v_2, \ldots, v_b\}$, $b = 2g^2-3g-1$. Observing the two properties (a) and (b) with

(a)
$$\bigwedge_{\substack{g\in\mathbb{N}\setminus\{1\}\\ g \text{ odd}}} g(g+1) = \left(1+2+\ldots+\frac{g-1}{2}\right) + (g+1) + \\ + \left(\frac{3g+5}{2} + \frac{3g+7}{2} + \ldots + \\ + 2g + (2g+1)\right)$$

(b) $\quad a_{max} = |E_{g,b}| + g^2, \quad$ where $a_{max} = a + |V_{g,b}| - 1,$

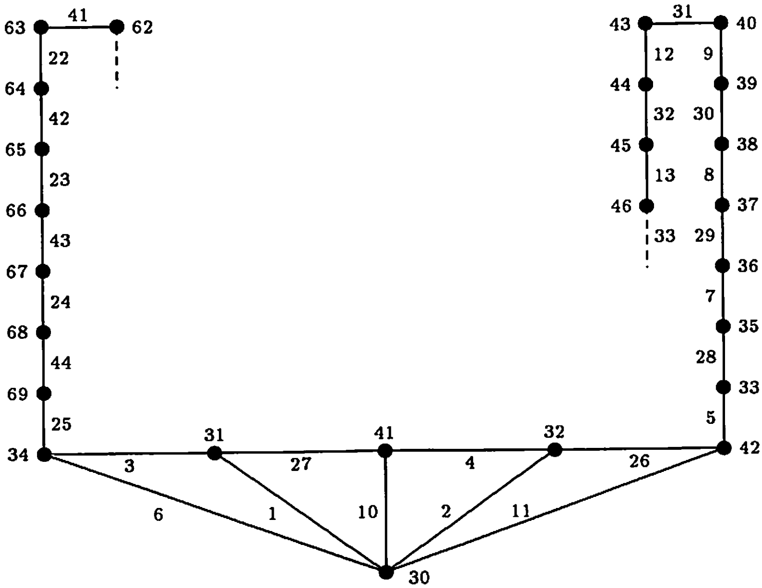

Figure 2b: $P_{5,34}$ is $(30,1)$-antimagic

we define the bijective mapping f in the following way

$$f : \begin{cases} E_{g,b} \longrightarrow \{1, 2, \ldots, |E_{g,b}| = 2g^2 - g - 1\} \\ h_1 \longrightarrow g + 1 \\ h_{2i} \longrightarrow i, \quad i = 1, 2, \ldots, \frac{g-1}{2} \\ h_{2i+1} \longrightarrow \frac{3g+5}{2} + (i-1), \quad i = 1, 2, \ldots, \frac{g-1}{2} \\ e \longrightarrow g^2 \\ e' \longrightarrow g \\ e'_{2i+1} \longrightarrow \frac{g+1}{2} + i, \quad i = 0, 1, \ldots, \frac{g-3}{2} \\ e'_{g-2i+1} \longrightarrow g^2 + i, \quad i = 1, 2, \ldots, \frac{g-1}{2} \\ e_{2i+1} \longrightarrow 2g^2 - g - 1 - i, \quad i = 0, 1, 2, \ldots, \frac{b-2}{2} \\ e_{2i} \longrightarrow g^2 - i, \quad i = 1, 2, \ldots, i_0 = \frac{b-(g+3)}{2} = g^2 - 2g - 2 \\ e_{2i_0+2+2i} \longrightarrow \frac{3}{2}(g+1) - i, \quad i = 0, 1, \ldots, \frac{g-1}{2} \end{cases}$$

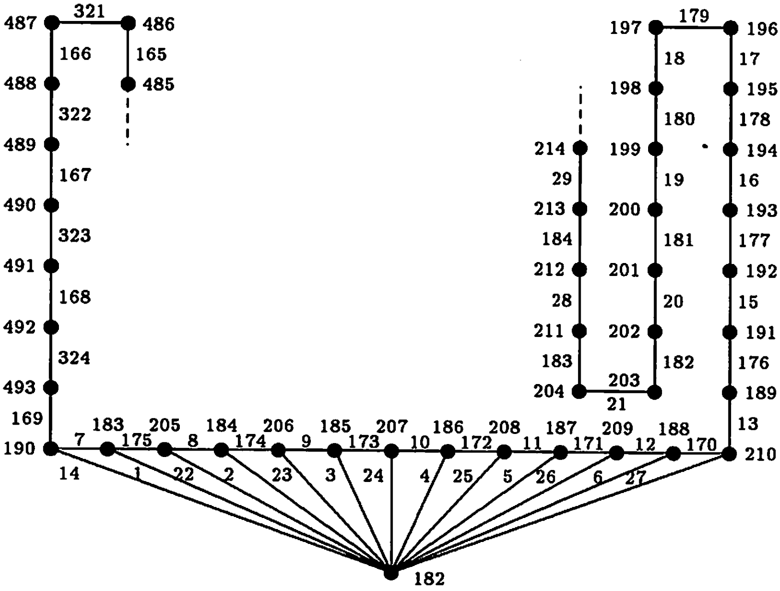

Figure 2c: $P_{13,298}$ is $(182,1)$-antimagic

In order to show $g_f(V_{g,b}) = \{a, a+1, \ldots, a_{max}\}$ we successively compute $g_f(v)$, $g_f(x_i)$, $i = 1, 2, \ldots, g$, and $g_f(v_i)$, $i = 1, 2, \ldots, b = 2g^2 - 3g - 1$. Because of (a) we know $g_f(v) = g(g+1) = a$. It is a matter of routine checking that $\{g_f(x_i) \mid i = 1, 2, \ldots, g\} = \{a+1 = g^2+g+1, a+2 = g^2+g+2, \ldots, g^2 + \frac{3g-1}{2}, g^2 + \frac{3g+3}{2}, g^2 + \frac{1}{2}(5g+7), \ldots, g^2 + 3g + 2\}$. Furthermore, we obtain $\{g_f(v_i) \mid i = 1, 2, \ldots, 2i_0+1 = 2g^2 - 4g - 3\} = \{a_{max} = 3g^2 - g - 1,\ a_{max} - 1, \ldots, g^2 + 3g + 3\}$, and, finally $\{g_f(x_i) \mid i = 2i_0 + 2, 2i_0 + 3, \ldots, b\} = \{g^2 + \frac{5}{2}(g+1), g^2 + \frac{1}{2}(5g+3), \ldots, g^2 + \frac{3g+5}{2}, g^2 + \frac{3g+1}{2}\}$. This proves that f is a $(g^2+g, 1)$-antimagic labeling of $P_{g,2g^2-3g-1}$.

Now it remains to prove the corresponding statement for even $g \geq 4$. Here, we use the property that the rectangular number $g(g+1)$, $g \equiv 0(2)$, can be represented as a sum of g summands such that

$$\text{(A)} \quad \bigwedge_{\substack{g \in \mathbb{N}\setminus\{1,2\} \\ g \geq 4, \text{ even}}} g(g+1) = \left(1 + 2 + \ldots + \frac{g}{2}\right) + \left(\left(\frac{3}{2}g + 2\right) + \right.$$

$$+ \quad \left(\frac{3}{2}g+3\right)+\ldots+2g+1\Big).$$

Observing the $(g(g+1),1)$-antimagic labelings of $P_{g,2g^2-3g-1}$, $g=4,6,8,10$, in Figure 3a, 3b, 3c, 3d we are able to construct an $(g^2+g,1)$-antimagic labeling of $P_{g,2g^2-3g-1}$ in the general case in the following way

$$f:\begin{cases} E_{g,b} \longrightarrow \{1,\,2,\,\ldots,\,|E_{g,b}|=2g^2-g-1\} \\ h_{2i-1} \longrightarrow i, \quad i=1,\,2,\,\ldots,\,\frac{g}{2} \\ h_{2i} \longrightarrow \frac{3}{2}g+1+i, \quad i=1,\,2,\,\ldots,\,\frac{g}{2} \\ e \longrightarrow g^2 \\ e' \longrightarrow g^2+\frac{g}{2} \\ e'_{2i} \longrightarrow g^2+i, \quad i=1,\,2,\,\ldots,\,\frac{g}{2}-1 \\ e'_{2i-1} \longrightarrow \frac{g}{2}+\left(\frac{g}{2}+1-i\right)=g+1-i, \quad i=1,\,2,\,\ldots,\,\frac{g}{2} \\ e_{2i} \longrightarrow g^2-i, \quad i=1,\,2,\,\ldots, \quad i_0=g^2-2g-2 \\ e_{2i_0+2(i+1)} \longrightarrow \frac{3}{2}g+1-i, \quad i=0,\,1,\,\ldots,\,\frac{g}{2} \\ e_{2i+1} \longrightarrow 2g^2-g-1-i, \quad i=0,\,1,\,\ldots,\,\frac{b-3}{2} \end{cases}$$

In order to show that this bijective mapping f is a $(g^2+g,1)$-antimagic labeling of $P_{g,2g^2-3g-1}$ we show that $g_f(V_{g,b})=\{a,\,a+1,\,\ldots,\,a_{max}\}$.

In order to do this we successively compute $g_f(v)$, $g_f(x_i)$, $i=1,\,2,\,\ldots,\,g$, and $g_f(v_i)$, $i=1,\,2,\,\ldots,\,b=2g^2-3g-1$. At first, we obtain $g_f(v)=g(g+1)=a$ because of (A). Then it is true that $\{g_f(v_i)\mid i=1,\,2,\,\ldots,\,2i_0+1=2g^2-4g-3\}=\{a_{max}=3g^2-g-1,\,a_{max}-1=3g^2-g-2,\,\ldots,\,g^2+3(g+1)\}$ and $\{g_f(v_i)\mid i=2i_0+2,\,\ldots,\,b\}=\{g^2+\frac{5}{2}g+2,\,g^2+\frac{5}{2}g+1,\,\ldots,\,g^2+\frac{3}{2}g+1\}$ and $\{g_f(x_i)\mid i=1,\,2,\,\ldots,\,g\}=\{g^2+3g+2,\,g^2+3g+1,\,\ldots,\,g^2+\frac{5}{2}g+3\}\cup\{g^2+\frac{3}{2}g,\,g^2+\frac{3}{2}g-1,\,\ldots,\,a+1=g^2+g+1\}$, whose proofs is a matter of routine checking, each. This completes the proof of Theorem 1 and the infiniteness of $\Gamma_1(P)$.

Now we turn towards the infiniteness of $\Gamma_2(P)$, for which a proof analogous to that of Theorem 1 can be given. For sake of brevity we merely state Proposition 2 which corresponds to Proposition 1.

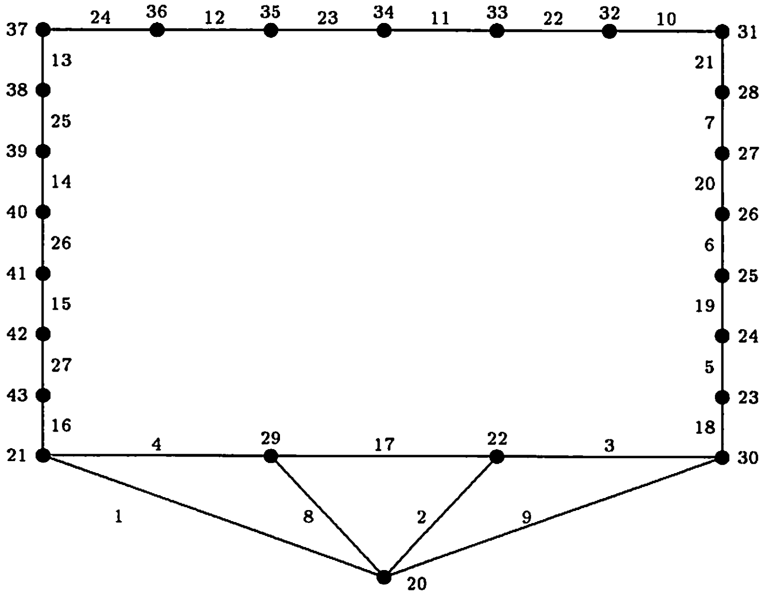

Figure 3a: $P_{4,19}$ is (20,1)-antimagic

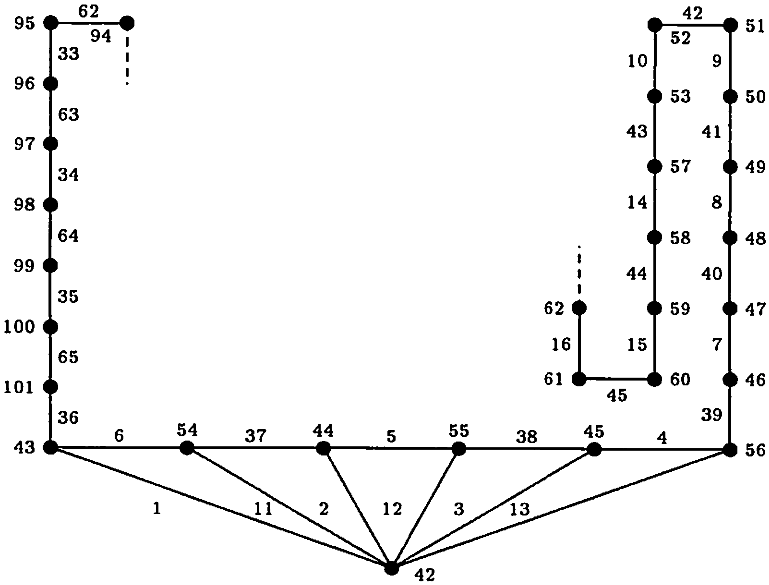

Figure 3b: $P_{6,53}$ is $(42, 1)$-antimagic

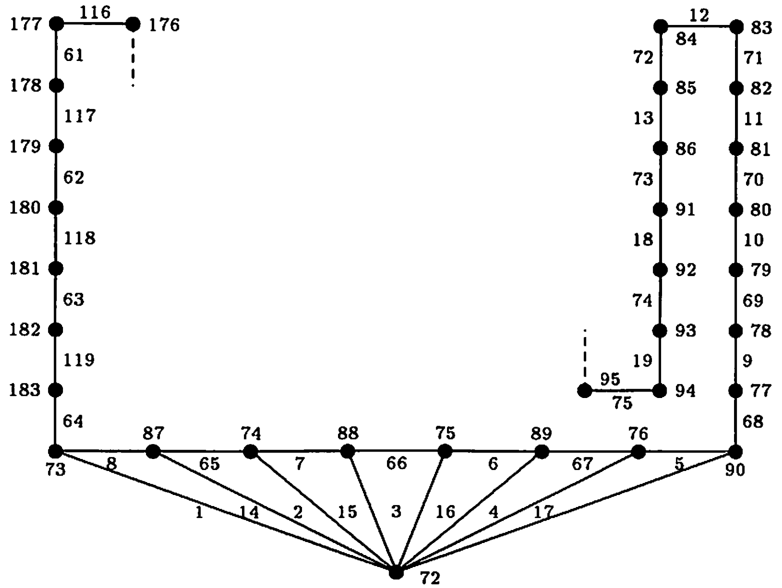

Figure 3c: $P_{8,103}$ is (72,1)-antimagic

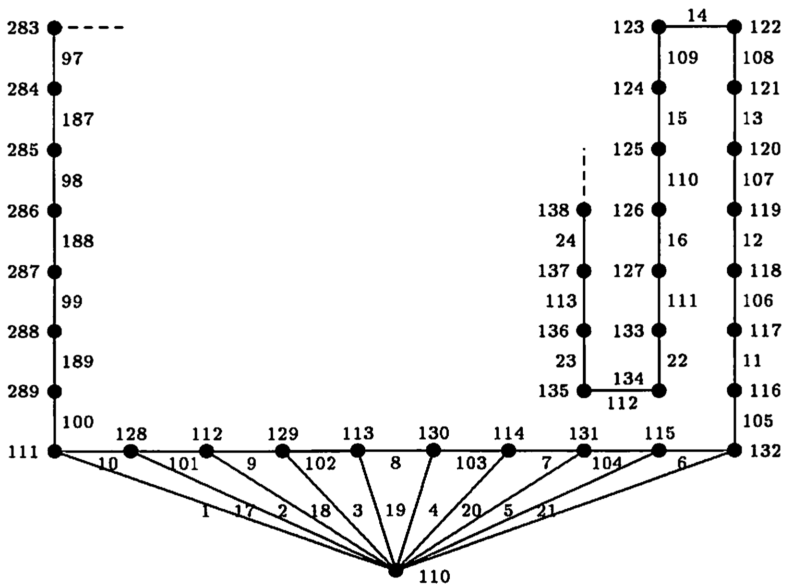

Figure 3d: $P_{10,169}$ is $(110, 1)$-antimagic

For the set $\Gamma_2(P) = \{P_{g,b} \in \Gamma(P) \mid b = g^2 - 2g - 1,\ g \geq 4\}$ of parachutes the following is true.

Proposition 2:

$$\bigwedge_{\substack{g \in \mathbb{N} \\ g \geq 4}} \left[P_{g,g^2-2g-1} \in \Gamma_2(P) \longrightarrow P_{g,g^2-2g-1} \text{ is } (2g+1,2)\text{-anti-magic and } a_{max} = 2g^2 - 1\right]$$

Now the following theorem holds. We omit details of its proof and confine ourselves on sketching its main ideas.

Theorem 2:

$$\bigwedge_{\substack{g \in \mathbb{N} \\ g \geq 4}} \left[P_{g,g^2-2g-1} \text{ is } (2g+1,2)\text{-antimagic}\right]$$

Proof: Again it is appropriate to distinguish the two cases a) g is odd and b) g is even. Let P_{g,g^2-2g-1}, $g \geq 5$, $g \equiv 1(2)$, be a parachute in $\Gamma_2(P)$. The proof is by construction a bijective mapping $f : E_{g,b} \longrightarrow \{1, 2, \ldots, |E_{g,b}|\}$, $b = g^2 - 2g - 1$, and showing that f is a $(2g+1,2)$-antimagic labeling of $P_{g,b} \in \Gamma_2(P)$. Since this proof is very similar to the proof given in Theorem 1, details are omitted. But it is remarkable that we use the identity (B) by

$$\text{(B)} \quad \bigwedge_{\substack{g \in \mathbb{N} \\ g \geq 5,}} \left[g^2 = \sum_{i=3}^{g}(2i-1) + 4 = 5 + 7 + \cdots + (2g-1) + 4\right]$$

instead of identity (a) mentioned above. The bijective mapping $f : E_{g,b} \longrightarrow$

$\{1, 2, \ldots, |E_{g,b}|\}$is defined as

$$f : \begin{cases} E_{g,b} \longrightarrow \{1, 2, \ldots, |E_{g,b}| = g^2 - 1\} \\ h_1 \longrightarrow g^2 - 1 \\ h_2 \longrightarrow 2g - 3 \\ h_3 \longrightarrow 4 \\ h_i \longrightarrow 7 + 2(i - 4), \quad i = 4, 5, \ldots, g - 2 \\ h_{g-1} \longrightarrow 2g - 1 \\ h_g \longrightarrow 5 \\ e'_1 \longrightarrow 1 \\ e'_2 \longrightarrow 3 \\ e'_{g-1} \longrightarrow 2 \\ e'_{3+i} \longrightarrow (2g - 4) - 2i, \quad i = 0, 1, \ldots, g - 5 \\ e \longrightarrow g^2 - 3 \\ e' \longrightarrow 2g - 2 \\ e_{2i-1} \longrightarrow g^2 - 2i, \quad i = 1, 2, \ldots, \frac{b}{2} \\ e_{2i} \longrightarrow g^2 - 3 - 2i, \quad i = 1, 2, \ldots, \frac{b-2}{2} \end{cases}$$

Figure 4 shows a $(23, 2)$-antimagic labeling of $P_{11,98}$.

In case b), $g \equiv 0(2)$, the auxiliary equation (C) by

$$\text{(C)} \qquad \bigwedge_{\substack{g \in \mathbb{N} \\ g \geq 4}} g^2 - 2 = 3 + 5 + \ldots + (2g - 3) + 2g - 2$$

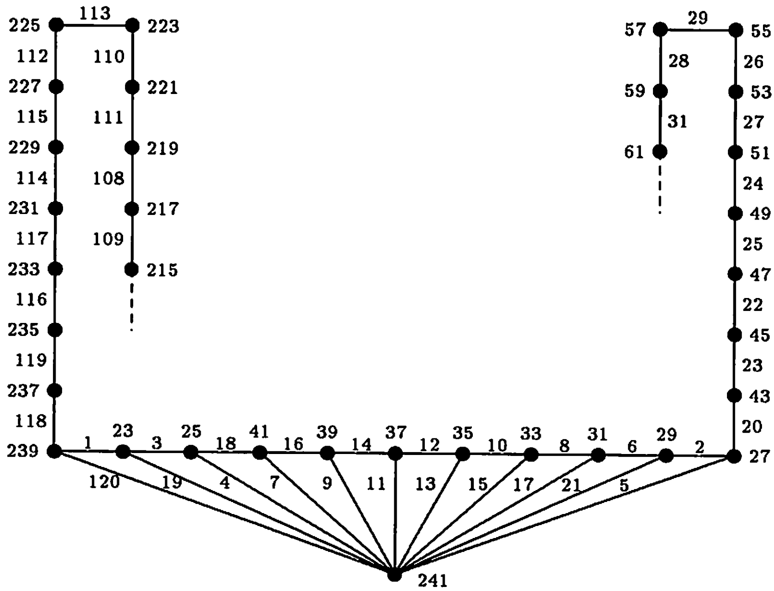

Figure 4: $P_{11,98}$ is $(23, 2)$-antimagic

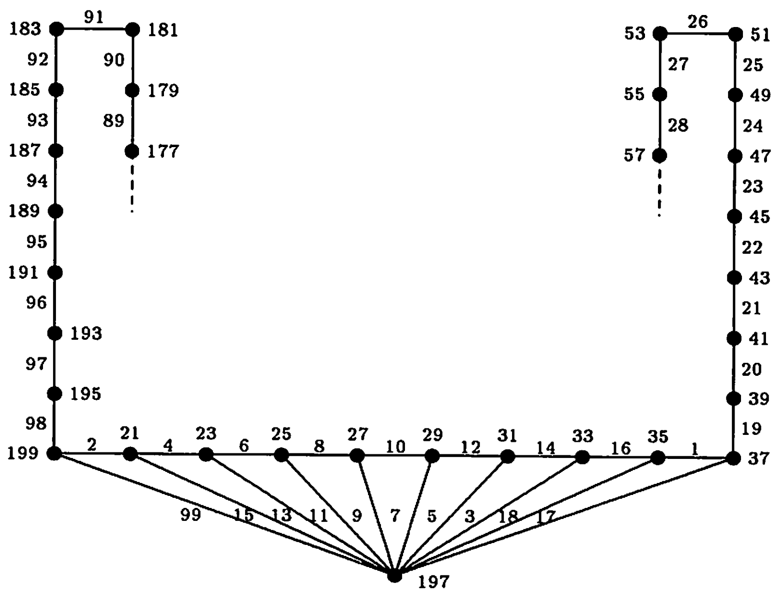

Figure 5: $P_{10,79}$ is $(21, 2)$-antimagic

is used to show that the bijective mapping $f : E_{g,b} \longrightarrow \{1, 2, \ldots, |E_{g,b}|\}$ defined as

$$f : \begin{cases} E_{g,b} \longrightarrow \{1, 2, \ldots, |E_{g,b}| = g^2 - 1\} \\ h_1 \longrightarrow g^2 - 1 \\ h_{g-1} \longrightarrow 2g - 2 \\ h_g \longrightarrow 2g - 3 \\ h_i \longrightarrow 2g - 5 - 2(i - 2), \quad i = 2, 3, \ldots, g - 2 \\ e'_{g-1} \longrightarrow 1 \\ e'_i \longrightarrow 2i, \quad i = 1, 2, \ldots, g - 2 \\ e \longrightarrow g^2 - 2 \\ e' \longrightarrow 2g - 1 \\ e_{2i-1} \longrightarrow g^2 - (2i + 1), \quad i = 1, 2, \ldots, \frac{b-1}{2} \\ e_{2i} \longrightarrow g^2 - 2(i + 1), \quad i = 1, 2, \ldots, \frac{b-1}{2} \end{cases}$$

is in fact an $(2g+1, 2)$-antimagic labeling of P_{g,g^2-2g-1}. Figure 5 shows an $(21, 2)$-antimagic labeling of $P_{10,79}$.

3 Infiniteness of $\Gamma'(P)$

Now we consider the set $\Gamma'(P) = \{P_{g,b} \in \Gamma(P) \mid b = \frac{1}{3}\,(2g^2 - 5g - 3),\ g \geq 9\}$. In order to show that each parachute $P_{g,b} \in \Gamma'(P)$, with $g \equiv 0(3)$ or $g \equiv 1(3)$, $g \geq 15$, is (a, d)-antimagic, we firstly prove

Proposition 3:

$$\bigwedge_{\substack{g \in \mathbb{N} \\ g \geq 7 \\ g \not\equiv 2(3)}} \left[P_{g,\frac{1}{3}(2g^2-5g-3)} \in \Gamma'(P) \longrightarrow P_{g,\frac{1}{3}(2g^2-5g-3)} \text{ is } \left(\frac{g^2+5g+3}{3}, 1\right)\text{-antimagic and } a_{max} = g(g+1) \right]$$

Proof:
Let $P_{g,\frac{1}{3}(2g^2-5g-3)}$ be a parachute in $\Gamma'(P)$. Since $\left|V\left(P_{g,\frac{1}{3}(2g^2-5g-3)}\right)\right| = \frac{2}{3}g(g-1)$ and $\left|E\left(P_{g,\frac{1}{3}(2g^2-5g-3)}\right)\right| = \frac{1}{3}\left(2g^2+g-3\right)$ we obtain with $b = \frac{1}{3}\left(2g^2-5g-3\right)$ from (1) $\frac{1}{3}\left(2g+1\right)\left(2g+3\right) = 2a+\frac{d}{3}\left(2g^2-2g-3\right)$, which is equivalent to

$$(3) \qquad 4g^2+8g+3 = 6a + d\left(2g^2-2g-3\right).$$

Since for each $g \in \mathbb{N}$, the numbers $4g^2+8g+3$ and $2g^2-2g-3$ are odd, (3) shows that d must be odd. Furthermore, we assert that $d=1$. Simple computation shows that even for $g \geq 7$ and $d \geq 3$ equation (3) yields $a < 0$, which is a contradiction.

Putting $d=1$, we obtain from (3)

$$3a = g^2+5g+3.$$

From our assumption $g \equiv 0(3)$ or $g \equiv 1(3)$ follows, that $a = \frac{g^2+5g+3}{3} \in \mathbb{N}$. Finally we obtain $a_{max} = a + |V(P_{g,\frac{1}{3}(2g^2-5g-3)})| - 1 = \frac{g^2+5g+3}{3} + \frac{2}{3}g(g-1) - 1 = g(g+1)$.

Now we prove the existence of an $(\frac{g^2+5g+3}{3}, 1)$-antimagic labeling for each parachute $P_{g,\frac{1}{3}(2g^2-5g-3)}$, $g \geq 9$, $g \not\equiv 2(3)$.

Theorem 3:

$$\bigwedge_{\substack{g\in\mathbb{N}\\ g\not\equiv 2(3)\\ g\geq 9}} P_{g,\frac{1}{3}(2g^2-5g-3)} \text{ is } \left(\frac{g^2+5g+3}{3}, 1\right)\text{-antimagic}$$

Proof: To prove this theorem we distinguish the two cases a) $g \equiv 3(6)$ or $g \equiv 1(6)$ and b) $g \equiv 0(6)$ or $g \equiv 4(6)$.

a) Let $g \equiv 3(6)$, $g \geq 9$, and $b = \frac{1}{3}(2g^2 - 5g - 3)$, then an appropriate edge labeling of $P_{g,b}$ is given by the bijective mapping

$$f : \begin{cases} E(P_{g,b}) \longrightarrow \{1, 2, \ldots, |E(P_{g,b})|\} \\ h_1 \longrightarrow 1 \\ h_i \longrightarrow g + (i-2), \quad i = 2, 3, \ldots, \frac{1}{2}(g-1) - 1 \\ h_{\frac{1}{2}(g-1)+i} \longrightarrow \frac{g+3}{2} + i, \quad i = 0, 1, \ldots, , \frac{g-5}{2} \\ h_{g-2} \longrightarrow 2g - 1 \\ h_{g-1} \longrightarrow 2g - 3 \\ h_g \longrightarrow 2g - 2 \\ e \longrightarrow |E(P_{g,b})| = \frac{1}{3}\left(2g^2 + g - 3\right) \\ e' \longrightarrow y = |E(P_{g,b})| - \frac{b}{2} = \frac{1}{6}\left(2g^2 + 7g - 3\right) \\ e'_{2i+1} \longrightarrow x + i = \frac{1}{3}g(g+2) + i, \quad i = 0, 1, \ldots, \frac{g-3}{2} \\ e'_{2i} \longrightarrow 1 + i, \quad i = 1, 2, \ldots, \frac{g-1}{2} \\ e_{2i-1} \longrightarrow x - i, \quad i = 1, 2, \ldots, i_0 = \frac{g^2-4g}{3} \\ e_{2i_0+1} \longrightarrow 2g - 4 \\ e_{2i_0+1+2j} \longrightarrow (2g-4) - j, \quad j = 1, 2, \ldots, \frac{g-3}{2} \\ e_{2i} \longrightarrow |E_{g,b}| - i = \frac{1}{3}\left(2g^2 + g - 3\right) - i, \quad i = 1, 2, \ldots, \frac{b-2}{2} \end{cases}$$

Now we have to check whether the induced mapping g_f is injective and $g_f(V(P_{g,b})) = \{a = \frac{1}{3}\left(g^2 + 5g + 3\right), a+1, a+2, \ldots, a_{max} = g(g+1)\}$. As the proof of the injectivity of g_f is a matter of routine checking we only need to show that $g_f(V(P_{g,b})) = \{a, a+1, \ldots, a_{max}\}$. In order to do this we successively compute the values $g_f(v)$, $g_f(x_1)$, $g_f(v_1)$, $g_f(v_i)$, $i = 2, 3, \ldots, b$, $g_f(x_i)$, $i = g-2, g-1, g$, $g_f(x_i)$, $i = 2, 3, \ldots, \frac{1}{2}(g-3)$, and $g_f(x_{\frac{1}{2}(g-1)+i})$, $i = 0, 1, \ldots, \frac{g-5}{2}$. At first, we obtain $g_f(v) = \sum_{i=1}^{g} f(h_i) = 1 + 6g - 6 + \left(\frac{g+3}{2} + \frac{g+5}{2} + \cdots + g - 1\right) + \left(g + (g+1) + \cdots + (g + \frac{g-7}{2})\right) = g^2 + g - 1 = a_{max} - 1$. In the same way we get $g_f(x_1) = f(e) + f(e'_1) + f(h_1) = \frac{1}{3}\left(2g^2 + g - 3\right) + \frac{1}{3}g(g+2) + 1 = g(g+1) = a_{max}$. $g_f(v_1) = f(e) + f(e_1) = a_{max} - 2$; for $i = 2, 3, \ldots, i_0 = \frac{g^2-4g}{3}$, we obtain the values $g_f(v_2) =$

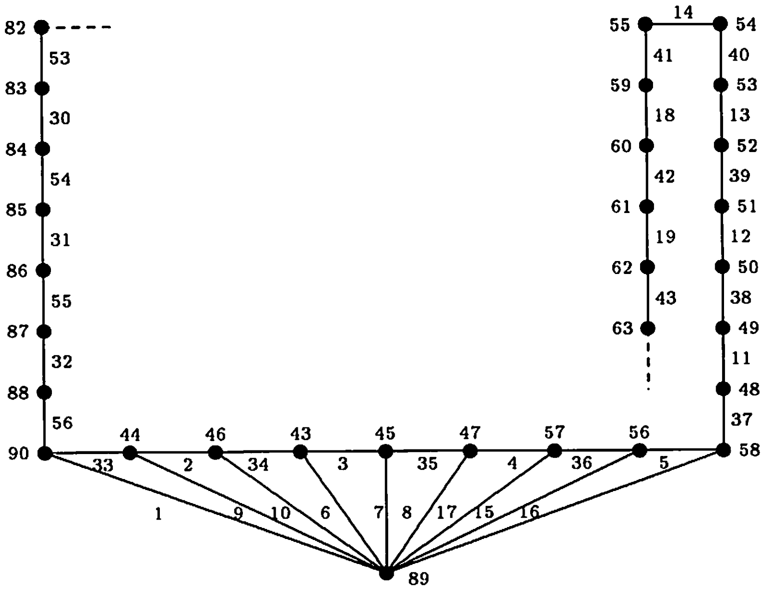

Figure 6: $P_{9,38}$ is $(43,1)$-antimagic

$a_{max}-3$, $g_f(v_3) = a_{max}-4, \dots, g_f(v_{2i_0}) = f(e_{2i_0})+f(e_{2i_0-1}) = \frac{g^2+11g-3}{3}$. For $i = i_0+1, i_0+2, \dots, b-1$ the corresponding values are $g_f(v_{2i_0+1}) = f(e_{2i_0+1})+f(e_{2i_0}) = 2g-4+\frac{g^2+5g-3}{3} = \frac{g^2+11g-15}{3}$, $g_f(v_{2i_0+2}) = \frac{g^2+11g-18}{3}$, $\dots$, $g_f(v_{b-1}) = f(e_{b-1})+f(e_{b-2}) = \frac{3g-5}{2}+\frac{2g^2+7g-3}{6}+1 = \frac{g^2+8g-6}{3}$. The last value $g_f(v_b) = \frac{g^2+8g-9}{3}$. Now it remains to compute the values $g_f(x_i)$, $i = 2, 3, \dots, g$. In case of $i = g$, $g-1$ and $g-2$ we obtain $g_f(x_g) = f(e') + f(h_g) + f(e'_{g-1}) = \frac{1}{6}(2g^2+7g-3)+2g-2+\frac{g+1}{2} = \frac{g^2+11g-6}{3}$, $g_f(x_{g-1}) = f(e'_{g-2}) + f(e'_{g-1}) + f(h_{g-1}) = \frac{2g^2+7g-9}{6}+\frac{g+1}{2}+2g-3 = \frac{g^2+11g-12}{3}$, and $g_f(x_{g-2}) = f(e'_{g-3}) + f(e'_{g-2}) + f(h_{g-2}) = \frac{g-1}{2}+\frac{2g^2+7g-9}{6}+2g-1 = \frac{g^2+11g-9}{3}$.

It is a matter of routine checking to see that the still missing $g-4$ integers $a = \frac{1}{3}(g^2+5g+3), a+1, \dots, a+g-5 = g_f(v_b)-1$ appear as values of the induced mapping g_f at the $g-4$ vertices $x_2, x_3, \dots, x_{g-3}$.

Figure 6 shows a $(43,1)$-antimagic labeling of $P_{9,38}$.

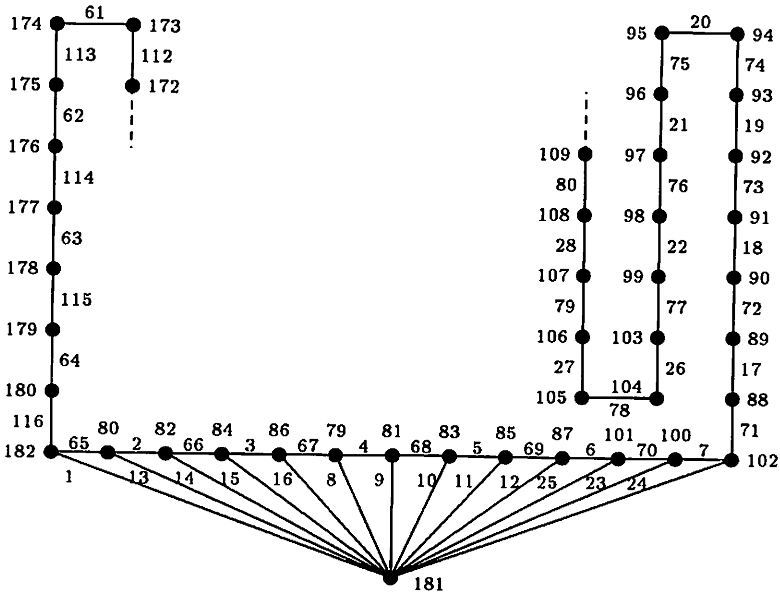

Figure 7: $P_{13,90}$ is $(79,1)$-antimagic

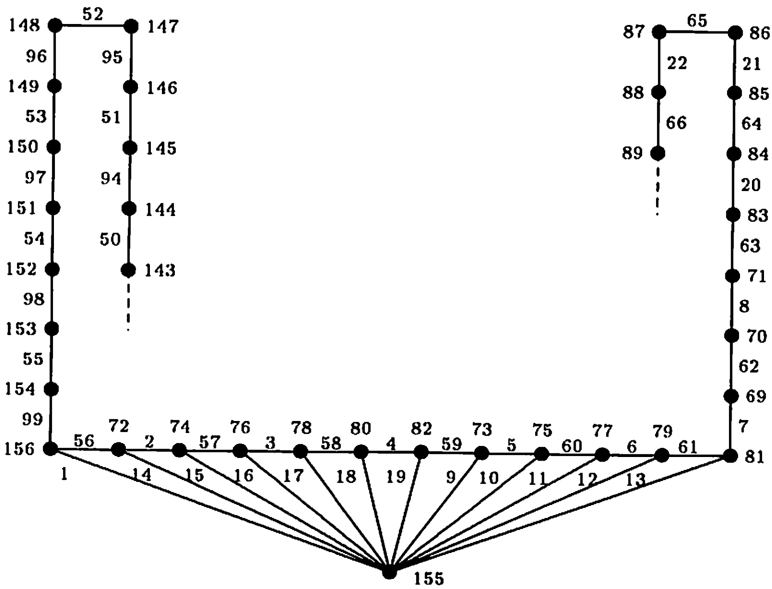

Figure 8: $P_{12,75}$ is $(69,1)$-antimagic

Hence it is shown that $g_f(V(P_{g,b})) = \{a, a+1, \ldots, a_{max}\}$ and f is a $(\frac{1}{3}(g^2+5g+3), 1)$-antimagic labeling of the parachute $P_{g,\frac{1}{3}(2g^2-5g-3)}$.

Figure 7 shows an $(79,1)$-antimagic labeling of $P_{13,90}$ which corresponds to $g \equiv 1(6)$. In this case, too, the edge labeling f defined above yields an $\left(\frac{g^2+5g+3}{3}, 1\right)$-antimagic labeling for $P_{g,\frac{1}{3}(2g^2-5g-3)}$, $g \equiv 1(6)$, $g \geq 7$. Hence part a) is completely proved.

Now we deal with case b), $g \equiv 0(6)$ or $g \equiv 4(6)$. Firstly let $g \equiv 0(6)$. Figure 8 shows an $(69,1)$-antimagic labeling of $P_{12,75}$. This gives us the motivation to define the edge labeling f for $P_{g,\frac{1}{3}(2g^2-5g-3)}$ in the following way.

$$f: \begin{cases} E_{g,\frac{1}{3}(2g^2-5g-3)} \longrightarrow \{1,2,\ldots,|E_{g,\frac{1}{3}(2g^2-5g-3)}|\} \\ h_1 \longrightarrow 1 \\ h_i \longrightarrow g+i \quad \text{for} \quad i = 2,3,\ldots,\frac{g}{2}+1 \\ h_i \longrightarrow i+1 \quad \text{for} \quad i = \frac{g}{2}+2, \frac{g}{2}+3, \ldots, \frac{g}{2}+\frac{g}{2} = g \\ e \longrightarrow |E(P_{g,b})| = \frac{1}{3}(2g^2+g-3) \\ e' \longrightarrow \frac{g+2}{2} \\ e'_{2i+1} \longrightarrow \frac{1}{3}g(g+2)+i, \quad i = 0,1,\ldots,\frac{g}{2}-1 = \frac{g-2}{2} \\ e'_{2i} \longrightarrow 1+i, \quad i = 1,2,\ldots,\frac{g}{2}-1 = \frac{g-2}{2} \\ e_{2i} \longrightarrow |E_{g,b}| - i = \frac{1}{3}(2g^2+g-3) - i, \quad i = 1,2,\ldots,\frac{b-1}{2} \\ e_{2i-1} \longrightarrow \frac{1}{3}g(g+2) - i, \quad i = 1,2,\ldots,\frac{b-3}{2} \\ e_{b-2} \longrightarrow \frac{g+4}{2} \end{cases}$$

In order to show that $g_f(V_{g,b}) = \{a, a+1, \ldots, a_{max}\}$ we successively compute the values $g_f(v)$, $g_f(x_1)$, $g_f(v_1)$, $g_f(v_i)$, $i = 2,3,\ldots,b$, $g_f(x_i)$, $i = 2,3,\ldots,\frac{g}{2}+1$, $g_f(x_i)$, $i = \frac{g}{2}+2,\ldots,g$. At first, we obtain $g_f(v) = \sum_{i=1}^{g} f(h_i) = 1 + (g+2+g+3+\cdots+g+\frac{g}{2}+1) + ((\frac{g}{2}+3)+(\frac{g}{2}+4)+\cdots+(g+1)) = \frac{8g^2+8g-8}{8} = a_{max} - 1$. Similarly $g_f(x_1) = f(e) + f(e'_1) + f(h_1) = \frac{1}{3}(2g^2+g-3) + \frac{1}{3}g(g+2) + 1 = g(g+1) = a_{max}$ and $g_f(v_1) = f(e) + f(e_1) = a_{max} - 2$. If $i = 2,3,\ldots,b-3$ then $g_f(v_i) = f(e_i) + f(e_{i-1})$ gives the set $\{g_f(v_2), g_f(v_3), \ldots, g_f(v_{b-3})\} = \{a_{max}-3, a_{max}-4, \ldots, \frac{g^2+8g+9}{3} =$

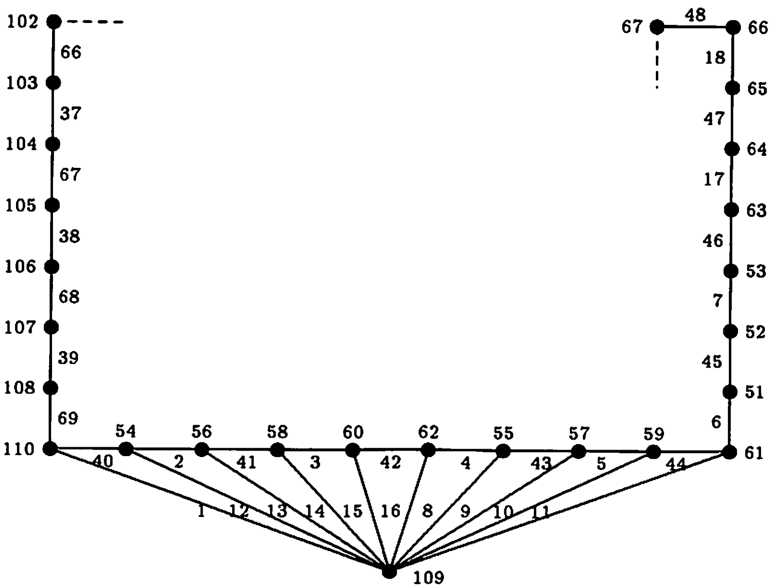

Figure 9: $P_{10,49}$ is $(51,1)$-antimagic

$a+g+2\}$. Furthermore, it is easy to see that $g_f(v_{b-2}) = a+2$, $g_f(v_{b-1}) = a+1$ and $g_f(v_b) = a$. It is a matter of routine checking to show that $\{g_f(x_i) \mid i = 2, 3, \ldots, g\} = \{a+3, a+4, \ldots, a+g, a+g+1\}$. This proves $g_f(V_{g,b}) = \{a, a+1, \ldots, a_{max}\}$.

Now let $g \equiv 4(6)$. Figure 9 shows an $(51,1)$-antimagic labeling of $P_{10,49}$. It turns out that in the case $g \equiv 4(6)$ the appropriate edge labeling f for $P_{g,\frac{1}{3}(2g^2-5g-3)}$ is exactly the same as described in the former case $g \equiv 0(6)$. Hence b) and consequently Theorem 3 is completely proved.

4 The Existence of a Minimum Integer b_{min}

Because of the infiniteness of $\Gamma_1(P)$, $\Gamma_2(P)$, $\Gamma'(P) \subseteq \Gamma(P)$ we know that $\Gamma(P)$ is an infinite set. This fact arises the question whether it is possible to determine the set $\Gamma(P)$ explicitely. Since the complete answer to this question is very extensive we have to restrict ourselves to dealing with a partial solution by showing the existence of a minimum t_{min} or b_{min} for every $g \geq 26$ such that for each divisor t' smaller than t_{min} the corresponding parachute $P_{g,b'}$ is not (a,d)-antimagic. In order to do this it is useful to introduce some auxiliary concepts. At first we consider the two quadratic auxiliary functions $r(b)$ and $s(b)$ defined in the following way

$$r : \begin{cases} \mathbb{R} \longrightarrow \mathbb{R} \\ b \longrightarrow 3b^2 + b\left(3 + 9g - g^2\right) + 7g^2 + 4g - g^3 \end{cases}$$

and

$$s : \begin{cases} \mathbb{R} \longrightarrow \mathbb{R} \\ b \longrightarrow 4b^2 + b\left(4 + 11g - g^2\right) + 8g^2 + 5g - g^3 \end{cases}$$

where $g \geq 4$ is an arbitrary integer. While r has the two zeros $b_1(r) = \frac{1}{6}\left(g^2 - 9g - 3 - \sqrt{D_r}\right) < 0$ and $b_2(r) = \frac{1}{6}\left(g^2 - 9g - 3 + \sqrt{D_r}\right) > 0$, discriminant $D_r = g^4 - 6g^3 - 9g^2 + 6g + 9 > 0$ for every $g \geq 8$, the other quadratic function s has got the two zeros $b_1(s) = \frac{1}{8}\left(g^2 - 11g - 4 - \sqrt{D_s}\right) < 0$ and $b_2(s) = \frac{1}{8}\left(g^2 - 11g - 4 + \sqrt{D_s}\right) > 0$, discriminant $D_s = g^4 - 6g^3 - 15g^2 + 8g + 16 > 0$ for every $g \geq 15$. For every $g \geq 16$ there exists a nonnegative real number $\overline{b}_2(r) = \frac{1}{6}\left(2g^2 - 13g\right)$ such that $\overline{b}_2(r) < b_2(r)$ and $r(b) < 0$ for every b in the open interval $\left(b_1(r), \overline{b}_2(r)\right)$. In the same way one can show the existence of a real number $\overline{b}_2(s)$ with $0 \leq \overline{b}_2(s) = \frac{2g^2 - 15g}{8} < b_2(r)$ for every $g \geq 19$ with the property $s(b) < 0$ for every b in the open interval $\left(b_1(s), \overline{b}_2(s)\right)$. Comparing the two positive zeros $b_2(r)$ and $b_2(s)$ of r and s it turns out that the inequality $b_2(s) < b_2(r)$ holds for $g \geq 15$. Similarly, the inequality $b_1(s) < b_1(r)$ is true.

The graphs of r and s are parabolas whose extremal points have the b-coordinate $b_v(r) = \frac{g^2 - 9g - 3}{3}$ and $b_v(s) = \frac{1}{4}\left(g^2 - 11g - 4\right)$, respectively (Figure 10).

In order to show the relationship between the two quadratic functions r, s on the one side and the parachutes $P_{g,b}$, $g \geq 3$, $b \geq 1$, on the other side, we assume $P_{g,b}$ is either $(a_1, 1)$- or $(a_2, 2)$-antimagic (compare Proposition 3 below). Putting $d = 1$ or $d = 2$ in (1) we obtain the two equations

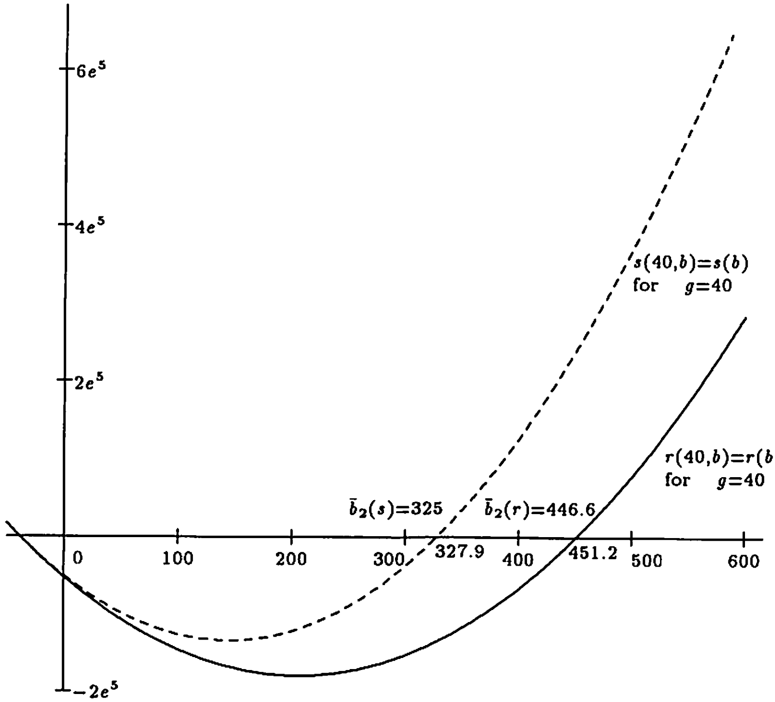

Figure 10

$$(4) \qquad 2a_1(g+b+1) + (g+b+1)(g+b) = 2(2g+b)(2g+b+1)$$

$$(5) \qquad a_2(g+b+1) + (g+b+1)(g+b) = (2g+b)(2g+b+1).$$

These two equations imply the desired relationship given as

$$(6) \qquad a_{max,1} - \frac{g(g+1)}{2} = \frac{r(b)}{2(g+b+1)} \quad \text{and}$$

$$(7) \qquad a_{max,2} - \frac{g(g+1)}{2} = \frac{s(b)}{2(g+b+1)}$$

such that we have as an immediate consequence

Proposition 4:

(a) $$\bigwedge_{g\geq 4}\ \bigwedge_{b\geq 1}\ [P_{g,b}\ \text{ is } (a_1,1)\text{-antimagic } \longrightarrow r(b)\geq 0]$$

(b) $$\bigwedge_{g\geq 4}\ \bigwedge_{b\geq 1}\ [P_{g,b}\ \text{ is } (a_2,2)\text{-antimagic } \longrightarrow s(b)\geq 0]$$

The proof is an immediate consequence of the inequality

$$a_{max,i}\geq \frac{g(g+1)}{2},\quad i=1,2.$$

Looking at Proposition 4 one immediately notices that it only deals with the cases $d=1$ and $d=2$. The following proposition shows that we are allowed to restrict ourselves to considering the values $d=1$ and $d=2$ for we begin by establishing Proposition 5, which asserts that a parachute $P_{g,b}$ can only be $(a_1,1)$- or $(a_2,2)$-antimagic — a_1,a_2 are suitable integers ≥ 3 — for sufficiently large $g\in\mathbb{N}$.

Proposition 5:

$$\bigwedge_{\substack{g,b\in\mathbb{N}\\ g\geq 24}}\ \bigwedge_{\substack{a,d\in\mathbb{N}\\ a\geq 3}}\ [P_{g,b}\ \text{ is } (a,d)\text{-antimagic} \longrightarrow d=1\vee d=2]$$

Proof: The proof is divided into several steps. Assume $g\geq 45$ and let $P_{g,b}$ denote an (a,d)-antimagic parachute. From the Diophantine equation (1) follows

$$\text{(8)}\qquad a = \frac{1}{2(g+b+1)}\left(g^2(8-d)+gb(8-2d)+g(4-d)+\right.$$
$$\left.+b^2(2-d)+b(2-d)\right).$$

From (8) we see that d is necessarily ≤ 7 (for $d\geq 8$ the integer a is negative). In the next step we give an estimate of the quotient $\frac{g}{b}$. By using

$a_{max} = a + (g+b)\,d$ and $a_{max} \geq \frac{1}{2}\,g(g+1)$ such that

$$(9) \qquad \frac{g+b}{2}\,d \geq \frac{1}{2(g+b+1)}\,\left(g^3 + g^2b - 6g^2 - 7gb - 3g - 2b - 2b^2\right)$$

we find

$$(10) \quad \frac{b}{g} \geq \frac{1}{18}\,(2g - 27).$$

Observing (1) and $a \geq 3$ we finally obtain

$$(11) \quad d \;\leq\; \frac{1}{(g+b)(g+b+1)}\,\left(8g^2 + 8gb - 2g + 2b^2 - 4b - 6\right)$$

$$\leq\; \frac{8 + 8\,\frac{b}{g} + 2\,\frac{b^2}{g^2}}{1 + 2\,\frac{b}{g} + \frac{b^2}{g^2}} < 3 \quad \text{for every } g \geq 45.$$

In the case of $24 \leq g \leq 44$ we have to check the 21 values $g = 24, 25, \ldots, 44$. It turns out that every (a,d)-antimagic parachute $P_{g,b}$, $24 \leq g \leq 44$, is either $(a_1, 1)$- or $(a_2, 2)$-antimagic. Since this proof is a matter of routine checking we omit details. This completes the proof of Proposition 5.

The great advantage of Proposition 5 is the fact that $d = 1$ and $d = 2$ which simplifies the following investigations decisively.

Now we turn towards the most essential auxiliary concept of a g-minimum integer b_{min} for every $g \geq 3$.

Definition 1:

Let $g \geq 3$ be an arbitrary integer. An integer $b_{min} \geq 1$ is said to be g-minimum iff there exist two positive integers $a \geq 3$, $d \geq 1$ such that the following two conditions are satisfied

(12) $\quad g, b_{min}, a, d \quad$ fulfill the Diophantine equation (1) and

$$a_{max} = a + (g+b)\,d \geq \frac{1}{2}\,g(g+1) \quad \text{and}$$

$$(13) \qquad \bigwedge_{b \in \mathbb{N}} \quad [b < b_{min} \longrightarrow P_{g,b} \notin \Gamma(P)].$$

In order to show that for every $g \geq 3$, with exception of finitely many integers there is an integer b_{min} we have to consider the set $\mathbb{N}$ modulo 4. Then it holds

Theorem 4:

$$\text{(a)} \qquad \bigwedge_{\substack{g \equiv 0(4) \\ g \geq 16}} \quad b_{min} = \frac{g^2 - 5g - 4}{4}$$

$$\text{(b)} \qquad \bigwedge_{\substack{g \equiv 1(4) \\ g \geq 17}} \quad b_{min} = \frac{g^2 - 5g - 4}{4}$$

Proof: Ad (a). Let $g \geq 16$ be an element of $\mathbb{N}$ divisible by 4. Putting $b_{min} = \frac{g^2-5g-4}{4}$ the Diophantine equation (1) is satisfied iff $a = 2g + 4$ and $d = 2$. Then condition (12) is fulfilled such that because of $a_{max} = \frac{g^2+3g+4}{2} \geq \frac{g^2+g}{2}$. Putting $b_{min} = \frac{g^2-5g-4}{4}$ we obtain $s(b_{min}) > 0$. Now we will show that condition (13) is also satisfied. The corresponding divisor t_{min} of b_{min} satisfies the equation $\frac{2g(g-1)}{t_{min}} = 8$. Assume b is an integer $< b_{min}$. Then we know that the corresponding divisor $t \mid 2g(g-1)$ satisfies the inequality $\frac{2g(g-1)}{t} \geq 9$. If t_0 denotes the biggest divisor of $2g(g-1)$ with $t_0 < t_{min}$ we know that the corresponding $b_0 = t_0 - g - 1 = \frac{2g^2-11g-9}{9}$. Comparing b_0 and $\overline{b}_2(s) = \frac{2g^2-15g}{8}$ we recognize that $b_0 \leq \overline{b}_2(s)$ for every $g \geq 24$. This implies $s(b) < 0$ for every $b \leq b_0$. Due to Proposition 4 $P_{g,b} \notin \Gamma(P)$. For $g = 16$ and $g = 20$ we consider $b' = \frac{2g(g-1)}{10} - g - 1 = \frac{g^2-6g-5}{5}$ instead of b_0 and compute the values $s(b') = -480$ for $g = 16$ and $s(b') = -2280$ for $g = 20$. These two facts complete the proof of (a).

Ad (b). Because of the similarity of the proof we can omit details. It is worth to be mentioned that $s(b') \notin \mathbb{N}$ for $g = 17$ and $s(b') = -302$ for $g = 21$.

Theorem 5:

$$\bigwedge_{\substack{g \equiv 10(12) \\ g \geq 34}} b_{min} = \frac{g^2 - 4g - 3}{3}, \qquad \bigwedge_{\substack{g \equiv 7(12) \\ g \geq 31}} b_{min} = \frac{g^2 - 4g - 3}{3}$$

$$\bigwedge_{\substack{g \equiv 6(12) \\ g \geq 30}} b_{min} = \frac{g^2 - 4g - 3}{3}, \qquad \bigwedge_{\substack{g \equiv 3(12) \\ g \geq 27}} b_{min} = \frac{g^2 - 4g - 3}{3}$$

$$\bigwedge_{\substack{g \equiv 26(60) \\ g \geq 26}} b_{min} = \frac{2g^2 - 7g - 5}{5}, \qquad \bigwedge_{\substack{g \equiv 35(60) \\ g \geq 35}} b_{min} = \frac{2g^2 - 7g - 5}{5}$$

$$\bigwedge_{\substack{g \equiv 50(60) \\ g \geq 50}} b_{min} = \frac{2g^2 - 7g - 5}{5}, \qquad \bigwedge_{\substack{g \equiv 11(60) \\ g \geq 71}} b_{min} = \frac{2g^2 - 7g - 5}{5}$$

$$\bigwedge_{\substack{g \equiv 38(60) \\ g \geq 38}} b_{min} = \frac{g^2 - 3g - 2}{2}, \qquad \bigwedge_{\substack{g \equiv 14(60) \\ g \geq 74}} b_{min} = \frac{g^2 - 3g - 2}{2}$$

$$\bigwedge_{\substack{g \equiv 2(60) \\ g \geq 62}} b_{min} = \frac{g^2 - 3g - 2}{2}, \qquad \bigwedge_{\substack{g \equiv 47(60) \\ g \geq 47}} b_{min} = \frac{g^2 - 3g - 2}{2}$$

$$\bigwedge_{\substack{g \equiv 59(60) \\ g \geq 59}} b_{min} = \frac{g^2-3g-2}{2}, \qquad \bigwedge_{\substack{g \equiv 23(60) \\ g \geq 83}} b_{min} = \frac{g^2-3g-2}{2}$$

Proof: Theorem 5 settles the cases $g \equiv 2$ modulo 4 and $g \equiv 3$ modulo 4 for we have $\overline{3}_4 = \overline{3}_{12} \cup \overline{7}_{12} \cup \overline{11}_{60} \cup \overline{23}_{60} \cup \overline{35}_{60} \cup \overline{47}_{60} \cup \overline{59}_{60}$ and $\overline{2}_4 = \overline{6}_{12} \cup \overline{10}_{12} \cup \overline{2}_{60} \cup \overline{14}_{60} \cup \overline{26}_{60} \cup \overline{38}_{60} \cup \overline{50}_{60}$ where $\overline{x}_m$, $x \in \mathbb{N}_0$, $m \in \mathbb{N}$, denotes the residue class modulo m of the integer x. Now we come to the proof of the first statement. Assume g is an element in $\overline{10}_{12}$ such that $g = 12k+10$, $k \geq 2$. In order to show that $b_{min} = \frac{1}{3}(g^2-4g-3)$ is g-minimum we have to verify that b_{min} satisfies Definition 1. Putting $b_{min} = \frac{1}{3}(g^2-4g-3)$ the Diophantine equation (1) becomes

$$(14) \quad 6a + \left(g^2 - g - 3\right) d = 2g^2 + 10g + 12$$

implying $d = 2$ and $a = 2g+3$. Since $a_{max} = a+(g+b)\,d = \frac{1}{3}(2g^2+4g+3) > \frac{1}{2}(g^2+g)$ the integer $\frac{1}{3}(g^2-4g-3)$ fulfills condition (12). The proof of condition (13) is much more complicated and much trickier. If we denote the corresponding divisor of $2g(g-1)$ of b_{min} by $t_{min} = \frac{1}{3}g(g-1)$ we have that the quotient $\frac{2g(g-1)}{t_{min}} = 6$. Assume that b is a positive integer such that the corresponding divisor $t = g+b+1$ of $2g(g-1)$ has the property that the quotient $\frac{2g(g-1)}{t}$ is a positive integer ≥ 7. If t_0 denotes the divisor with $\frac{2g(g-1)}{t} = 7$ the corresponding b_0 has the value $b_0 = \frac{1}{7}(2g^2-9g-7)$ such that (1) becomes

$$(15) \quad 14a + \left(2g^2 - 2g - 7\right) d = 4g^2 + 24g + 35$$

implying $d = 1$ and $a = \frac{1}{7}(g^2+13g+21)$. If we compare b_0 and $\overline{b}_2(r) = \frac{1}{6}(2g^2-13g)$ we immediately recognize the inequality $b_0 \leq \overline{b}_2(r)$ for every $g \geq 34$. Because of Proposition 4 the inequality $r(b) < 0$ is true for every $b \leq b_0$ such that every parachute $P_{g,b}$ is not $(a,1)$-antimagic for every $b \leq b_0 = \frac{1}{7}(2g^2-9g-7)$. Therefore, due to Proposition 4, it remains to show that every parachute $P_{g,b}$ cannot be $(a,2)$-antimagic for every $b \leq b_0$. Since the number $b = \frac{g^2-5g-4}{4} \notin \mathbb{N}$ for every $g \equiv 10(12)$, $g \geq 34$, it suffices to consider the divisor $t = \frac{2g(g-1)}{9}$. The corresponding $b = \frac{1}{9}(2g^2-11g-9)$. Since $b \leq \overline{b}_2(s) = \frac{1}{8}(2g^2-15g)$ we obtain $s(b') < 0$ for every $b' \leq b_0 \leq$

$\overline{b}_2(s)$ such that each parachute $P_{g,b}$, $g \geq 34$, is not $(a,2)$-antimagic. This completes the proof of first statement.

Similarly, we can carry out the proofs of the next three statements such that further details are omitted.

In the case of the 5. statement, assume g is an arbitrary integer ≥ 26 from $\overline{26}_{60}$ such that $g = 60k + 26$, $k \geq 0$. In order to prove that $b_{min} = \frac{1}{5}(2g^2 - 7g - 5)$ satisfies Definition 1 we first show that b_{min} fulfills condition (12) in the following way. Putting $b_{min} = \frac{1}{5}(2g^2 - 7g - 5)$ (1) becomes

$$(16) \quad 10a + \left(2g^2 - 2g - 5\right) d = 4g^2 + 16g + 15$$

implying $d = 1$ and $a = \frac{1}{5}(g^2 + 9g + 10) \in \mathbb{N}$. Since $a_{max} = a + b + g = \frac{1}{5}(3g^2 + 7g + 5) > \frac{g^2+g}{2}$, (12) is satisfied by b_{min}. A short sketch of the proof of condition (13) is given, because the whole proof is very extensive. Since the corresponding divisor $t_{min} = \frac{2g(g-1)}{5}$ of b_{min} gives the quotient $\frac{2g(g-1)}{t_{min}} = 5$ assume b_0 is a positive integer such that the corresponding divisor t_0 satisfies the equation $\frac{2g(g-1)}{t_0} = 7$ (the quotient cannot be equal to 6). Then $b_0 = \frac{1}{7}\left(2g^2 - 9g - 7\right) \leq \overline{b}_2(r)$ and we can quote the result of the preceding part of the proof of first statement where it is shown that the parachute $P_{g,b}$ is not $(a,1)$-antimagic for every $b' \leq b_0$. Therefore, due to Proposition 5, it remains to show that every parachute $P_{g,b}$ cannot be $(a,2)$-antimagic for every $b \leq b_0$. Assume $b = \frac{1}{9}\left(2g^2 - 11g - 9\right)$ ($\frac{2g(g-1)}{t} = 8$ is not possible). Then according to the proof of the first statement we know that $b \leq \overline{b}_2(s)$ for every $g \geq 26$ such that $P_{g,b}$, cannot be $(a,2)$-antimagic for every $b' \leq b \leq \overline{b}_2(s)$. This completes the proof of our statement.

For the sake of shortness we omit the remaining cases. Their proofs are similar to the two proofs given above.

If $\Gamma(g)$, $g \geq 3$, denotes the set $\Gamma(g) = \{P_{g,b} \in \Gamma(P) \mid b_{min} \leq b \leq b_{max}\}$ then we can ask for the cardinality of $\Gamma(g)$ for any $g \geq 3$. The answer is

Corollary 1:

$$\text{(a)} \qquad \bigwedge_{\substack{g \in \mathbb{N} \\ g \geq 26}} \left[g \equiv 0(4) \vee g \equiv 1(4) \longrightarrow |\Gamma(g)| \leq 8\right]$$

(b) $\bigwedge_{\substack{g \in \mathbb{N} \\ g \geq 27}} \bigwedge_{m \in \{3,6,7,10\}} [g \equiv m(12) \longrightarrow |\Gamma(g)| \leq 6]$

(c) $\bigwedge_{\substack{g \in \mathbb{N} \\ g \geq 26}} \bigwedge_{m \in \{11,26,35,50\}} [g \equiv m(60) \longrightarrow |\Gamma(g)| \leq 5]$

(d) $\bigwedge_{\substack{g \in \mathbb{N} \\ g \geq 38}} \bigwedge_{m \in \{2,14,38,47\}} [g \equiv m(60) \longrightarrow |\Gamma(g)| \leq 4]$

Proof: Due to Theorem 5 we know that b_{min} has four different values, namely $b_{min} = \frac{1}{4}(g^2-5g-4)$, $b'_{min} = \frac{1}{3}(g^2-4g-3)$, $b''_{min} = \frac{1}{5}(2g^2-7g-5)$ and $\bar{b}_{min} = \frac{1}{2}(g^2-3g-2)$, $g \geq 26$. The corresponding divisors of $2g(g-1)$ are $t_{min} = \frac{g}{4}(g-1)$, $t'_{min} = \frac{g}{3}(g-1)$, $t''_{min} = \frac{2g}{5}(g-1)$ and $\bar{t}_{min} = \frac{g}{2}(g-1)$, respectively. Assume $P_{g,b} \in \Gamma(g)$, $g \geq 26$. Then it is necessary that the corresponding divisor t of b satisfies one of the four inequalities $t_{min} \leq t \leq t_{max} = 2g(g-1)$, $t'_{min} \leq t \leq t_{max}$, $t''_{min} \leq t \leq t_{max}$ or $\bar{t}_{min} \leq t \leq t_{max}$ and has the property that the quotient $t_{max} = 2g(g-1)/t$ is an integer. This fact proves the four statements (a)–(d) such that Corollary 1 is completely proved.

Theorem 5 and Corollary 1 give the possibility to determine the set $\Gamma(P)$ of all (a,d)-antimagic parachutes. The first could be done in Theorems 1, 2, 3 by determining $\Gamma_1(P)$, $\Gamma_2(P)$ and $\Gamma'(P)$.

References

[1] M. Bača and I. Holländer. Prime-magic labelings of $K_{n,n}$. *Journal of the Franklin Institute*, 6(327):923–926, 1990.

[2] R. Bodendiek und G. Walther. On (a,d)-antimagic parachutes. To appear in *Ars Combinatoria*.

[3] R. Bodendiek und G. Walther. Numerierungen von Graphen. In: K. Wagner und R. Bodendiek, *Graphentheorie III*. BI-Verlag Mannheim, 1993.

[4] N. Hartsfield and G. Ringel. *Pearls in Graph Theory*. Academic Press, Boston – San Diego – New York – London, 1990.

[5] R. Schnabel und U. Spengler. Exakte Graphen. In: K. Wagner und R. Bodendiek, *Graphentheorie III*. BI-Verlag Mannheim, 1993.

[6] K. Wagner und R. Bodendiek. *Graphentheorie III*. BI-Verlag Mannheim, 1993.