Counting frequencies of configurations in Steiner Triple Systems

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1 Introduction

A Steiner triple system of order v is a pair (V, B) where V is a set of cardinality v and B is a family of 3-subsets of V having the property that every pair of elements of V occurs exactly once among the elements of B. A Steiner triple system of order v is denoted by $STS(v)$. The elements of V shall be called points and the elements of B shall be called lines.

An *n-line configuration* or *n-configuration*, $n \geq 1$, is any collection of n lines of an $STS(v)$. Two n-line configurations are considered isomorphic if there is a bijection between the points of the configurations carrying lines to lines. Points of an n-line configuration are sometimes referred to as vertices of the configuration and the degree of a vertex is the number of lines that vertex is on in the configuration.

An n-line configuration is called *constant* if for any given admissible value of v the configuration occurs the same number of times in every $STS(v)$. Otherwise the configuration is called *variable.*

In [1] the authors showed that for $n = 1, 2$, and 3, every n-line configuration is constant. For $n = 4$, they showed that five of the sixteen 4-line configurations were constant and the rest were variable. The number of occurences of each variable configuration could be written as a constant plus a multiple of the number of Pasch configurations. (See Figure 1.)

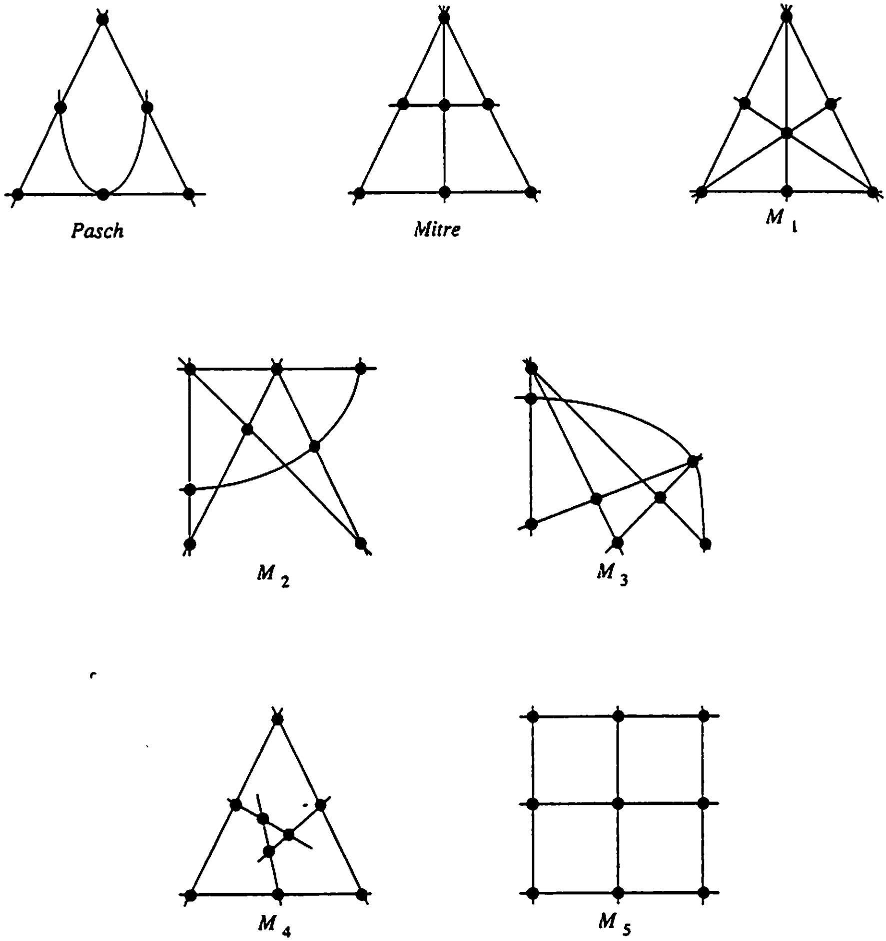

Figure 1

A generating set M for n-line configurations is a set of m-line configurations, $1 \leq m \leq n$, such that for each admissible v the number of occurences of any n-line configuration can be expressed as a linear combination of the number of occurences of the configurations in M, where the coefficients are polynomials in v. A minimal generating set will be called a basis. So, using this terminology, it has been shown in [1] that any constant configuration and the Pasch configuration together form a basis for 4-configurations.

An Erdös configuration of order n is an n-line configuration on $n+2$ points which contains no subconfiguration of m lines on $m+2$ points, $1 < m < n$. For example, the Pasch configuration is an Erdös configuration of order 4. It was conjectured in [1] that any constant configuration, e.g., the "star", n lines all intersecting in a common point, together with all Erdös configurations of order m, $1 < m \leq n$, form a basis for the n-line configurations. From [1], the conjecture is true for $n = 1, 2, 3$, and 4.

In this paper we prove that an arbitrary constant configuration together with all m-line configurations, $m \leq n$, having all vertices of degree ≥ 2, form a generating set for the n-line configurations. When $n = 5$ this collection is the same as the collection of the conjecture above. This means that when $n = 5$ the generating set is also a basis, so the conjecture is true.

We also show that, for $n = 6$, our generating set forms a basis. This basis has 8 elements (see Figure 1) which refutes the conjecture in [1]. The above statement and additional supporting evidence leads us to:

Conjecture 1. A constant n-configuration together with all m-line configurations, $m \leq n$, having all vertices of degree at least 2, form a basis for n-configurations.

In addition, we show that the five n-configurations in Figure 2 (which are all possible n-configurations that can be obtained from a star on $n-1$ lines by adding a line) are constant configurations. We strongly believe the following conjecture is true.

Conjecture 2. For any $n \geq 3$ there are exactly 5 constant configurations.

2 The Generating Set.

For a configuration C we denote by $|C|$ its number of occurences in a Steiner triple system. If L is a line of C then by $C-L$ we understand a configuration obtained from C by removing the line L and all isolated vertices, if any.

The next theorem is the main result of this paper.

Theorem 1. *Any constant n-configuration together with all m-line configurations, $m \leq n$, having all vertices of degree at least 2, form a generating set for the n-line configurations.*

Proof: Let STS denote a fixed Steiner triple system on v points.

We prove the theorem by double induction on n and s, where s is the number of vertices in an n-configuration. The statement is trivial for $n = 1$. Suppose $n > 1$ and the statement is true for all $n-1$ configurations. Let C be an n-configuration on s vertices. If all vertices of C are of degree 2 we are done as in this case C belongs to our generating set. So suppose C contains a vertex of degree 1. Further, assume that the theorem is valid for all n-configurations on $s-1$ vertices. We show that $|C|$ satisfies the relation

$$p(v)|C'| = \alpha|C| + \alpha_1|C_1| + \cdots + \alpha_m|C_m| \qquad (1)$$

where C' is an $(n-1)$-configuration, $C_1, \cdots, C_m$ are n-configurations on $s-1$ vertices, $p(v)$ is a polynomial in v, and $\alpha, \alpha_1, \cdots, \alpha_m$ are absolute constants depending only on the structure of C and not on our choice of STS. When s is the smallest value, i.e., C is an n-configuration with the smallest number of vertices among all n-configurations, then the right hand side of (1) is just the term $\alpha|C|$. C is either a generator or can be expressed in terms of C'. Obviously, proving (1) finishes the proof of this theorem. Let $L = \{u, v, w\}$ be a line of C such that at least one of the vertices, u, v, w is of degree 1 in C. Let $C' = C - L$. We distinguish three cases:

I. All three vertices, u, v and w are of degree 1 in C. Denote by $C_1, \cdots, C_m$ all non-isomorphic n-line configurations on $s - 1$ vertices which can be obtained from C' by adding a new (pendant) line $L^* = \{x, y, z\}$, where x is a vertex from C' and y, z are new vertices (and thus will be of degree 1 in $C' \cup L^*$). Now consider all C' configurations of STS, i.e. all occurences of C' in STS. To each of them we add a line $M = \{u, v, w\}$ for all possible choices of vertices u, v, which are not in C'. In this way we obtain $\binom{v-(s-3)}{2}|C'|$ n-line configurations of STS, because C' is on $s - 3$ vertices. Not all of them are necessarily distinct but any of them is isomorphic to one of $C, C_1, \cdots, C_m$. Therefore

$$\binom{v-s+3}{2}|C'| = \alpha|C| + \alpha_1|C_1| + \cdots + \alpha_m|C_m|.$$

To prove (1) in this case it suffices to show that $\alpha, \alpha_1, \cdots, \alpha_m$ are absolute constants. For a configuration D we set $S_D = \{D - M$, where M is a line of $D\}$. Let b_D be the number of configurations in S_D which are isomorphic to C'. Then any C_i configuration, $i = 1, \cdots, m$, and any C configuration of STS has been obtained by the procedure described above from b_{C_i} different C'-configurations and from b_C different C' configurations respectively. Hence, for $1 \le i \le m, \alpha_i = b_{C_i}$, and $\alpha = 3b_C$ as by three different choices of the pair of vertices u, v we get the same configuration C from a fixed configuration C'.

II. Precisely two vertices of the line $L = \{u, v, w\}$, say u and v, are of degree one in C. Let k be the number of vertices of $C' = C - L$ such that adding a pendant line to C' at any of them results in a configuration isomorphic to C. Clearly, the vertex w is one (possibly the only one) of them. Let K be the set of these vertices. Further, let $C_1, \cdots, C_m$ be all non-isomorphic n-configurations on $s - 1$ vertices obtained from C' by adding a line of the form $\{w', x, y\}$, where $w' \in K, x$ is a vertex of C' and y does not belong to C'. This means that w' and x are not on a line in C'. Consider all C'-configurations of STS. To each of them add a line $\{w', x, y\}$, $w' \in K, y$ is not in C' for all possible choices of w' and y. Clearly, this yields $k(v - (s - 2))|C'|$, not necessarily distinct, n-configurations, as C' is on $s - 2$ vertices. Any of them is isomorphic to one of $C, C_1, \cdots, C_m$. Therefore, $k(v - (s - 2))|C'| = \alpha|C| + \alpha_1|C_1| + \cdots + \alpha_m|C_m|$. Let b_D be defined as in case I. Then $\alpha_i = b_{C_i}$ for $1 \le i \le m$ and $\alpha = 2b_C$ as by two different choices of the vertex y we obtain the same C-configuration from a given C'-configuration.

III. Exactly one vertex, say u, of the line $L = \{u, v, w\}$ is of degree 1 in C. Let $K = \{\{v, w\} : v, w$ are vertices of $C'\}$ be the set of all pairs of vertices such that $C' \cup \{v, w, x\}$ is isomorphic to C. Set $k = |K|$. Let $C_1, \cdots, C_m$ be all non-isomorphic n-configuration on $s-1$ vertices of the form $C' \cup \{v, w, z\}$, where $\{v, w\} \in K$ and z is in C'. To any C'-configuration of STS we take k n-configurations of the type $C' \cup \{v, w, x\}$ where $\{v, w\} \in K$. Any of them is isomorphic to one of $C, C_1, \cdots, C_m$ and we have

$$k|C'| = \alpha|C| + \alpha_1|C_1| + \cdots + \alpha_m|C_m|$$

where $\alpha_i = b_{C_i}, i = 1, \cdots, m, \alpha = b_C$. The proof is complete.

Remark 1. Since we showed how to determine all coefficients in (1) one could use (1) for deriving, for any n-configuration C, a formula expressing the number of occurences of C in terms of the numbers of occurences of the elements from the generating set.

Remark 2. It is possible to define the notion of generating set for n-configurations in Steiner systems with block size $k > 3$. In this case Theorem 1 would read that the set consisting of a constant n-configuration and all m-configurations, $m \leq n$, with the property that each line of them contains at least 3 vertices of degree at least 2, is a generating set for the n-configuration. The proof of the theorem would require only trivial modifications.

Theorem 2. *Any constant configuration, together with Pasch and Mitre configurations (see Figure 1), form a basis for the 5-configurations.*

Proof: It is easy to check that for $m \leq 5$ there are only two configurations, Pasch and Mitre, with all vertices of degree at least two. Thus, the three configurations in the statement of the theorem form a generating set for the 5-configurations. The minimality of the set is proven in [1], so it is a basis.

Theorem 3. *The seven configurations listed in Figure 1 together with any one constant 6-configuration form a basis for the 6-configurations.*

Proof: First of all we need to show that the set given above is the set of all m configurations, $m \leq 6$, with all vertices of degree at least 2. As mentioned in the proof of Theorem 2, for $m \leq 5$, there are exactly two such configurations, the Pasch and Mitre configurations. For 6-configurations with the property we get that none of them could have a vertex of degree 4, so all vertices are of degree either 2 or 3. A trivial calculation shows that for the number s of vertices we have $7 \leq s \leq 9$. If $s = 7$ then the configuration is M_1, a Fano plane without one line. For $s = 8$, we have that the configuration has exactly two vertices of degree 3. If these are on the same line we get configuration M_2 from Figure 1, if they are not, the configuration M_3. Finally, for $s = 9$, all the vertices are of degree 2. If there is a triangle it has to be M_4, if there is no triangle in it we get M_5. To show that this generating set is also a basis, it is sufficient to take the eight STS on 19 vertices given in the Appendix and find out the number of occurences of all 8 configurations in all of them. The numbers are listed in matrix A where each row corresponds to one STS and each column corresponds to a particular configuration. As the rank of A is 8 the set forms a basis.

$$A = \begin{bmatrix} star & Pasch & Mitre & M_1 & M_2 & M_3 & M_4 & M_5 \\ 1596 & 18 & 38 & 0 & 161 & 44 & 346 & 73 \\ 1596 & 19 & 38 & 1 & 181 & 27 & 325 & 91 \\ 1596 & 9 & 38 & 0 & 211 & 28 & 324 & 70 \\ 1596 & 17 & 23 & 1 & 210 & 45 & 305 & 60 \\ 1596 & 12 & 42 & 0 & 187 & 30 & 339 & 80 \\ 1596 & 12 & 29 & 2 & 220 & 45 & 325 & 69 \\ 1596 & 14 & 31 & 0 & 209 & 29 & 305 & 69 \\ 1596 & 24 & 52 & 4 & 129 & 44 & 428 & 75 \end{bmatrix}$$

Remark 3. The numbers in A were obtained by computer. Although the calculation is not a complicated one, we used two different programs. The first one took all 4-,5-, and 6-configurations and checked which of them were from our list. The second program, a fast one, made use of the structure of the configurations.

Let us point out that for orders $v = 3, 7, 9$ and 13, as there are less than 3 isomorphism classes of Steiner triple systems, the sets given above do not form bases. More interestingly, although there are 80 $STS(15)$s, a computation shows that a linear basis for 6-configurations among $STS(15)$s contains only 6 members.

3 Constant configurations

It is not difficult to see the n-star is a constant configuration. Here we show that configurations obtained from a star by adding an additional line are also constant. As we stated in the introduction we believe that there is no other constant configuration. We denote the n-line star by A_n; there are four other possible n-configurations derived by adding one line to A_{n-1}, and we shall denote them by B_n, C_n, D_n, E_n and define them as in Figure 2.

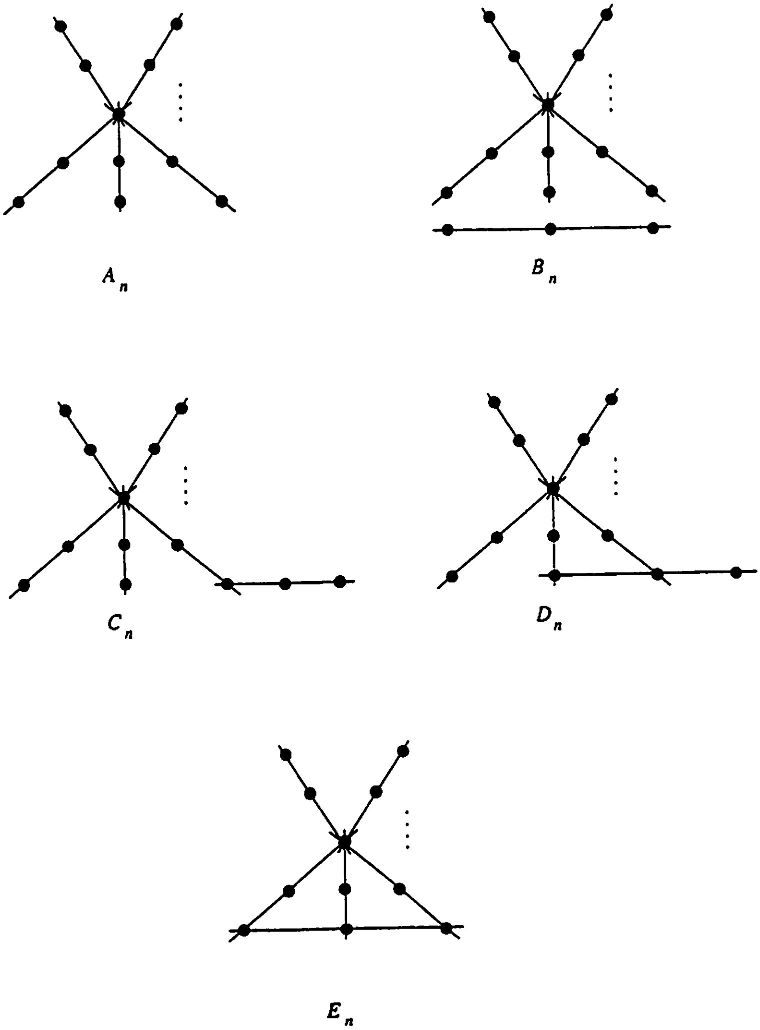

Figure 2

Theorem 4. *Configurations A_n, B_n, C_n, D_n and E_n are constant configurations for any admissible value of n.*

Proof: Let STS be a fixed Steiner triple system on v points. We shall prove the statement by induction on n. It is shown in [1] that A_n, B_n, C_n, D_n and E_n are constant configurations for any admissible value $n \leq 4$. We could have used (1) to obtain the desired result. However, the following argument gives a recurrent formula in terms of the given configuration alone. Let F_n be an n-configuration isomorphic to one of A_n, B_n, C_n, D_n or E_n, and let F_{n-1} be an $(n-1)$-configuration of the same type as F_n, $n \geq 5$. Denote by z the center of the star of F_{n-1}. For each F_{n-1} configuration of STS there are $\frac{v-s-k}{2}$ lines L of STS such that $F_{n-1} \cup L$ forms an F_n configuration (i.e. each of these lines contains z and is a pendant line), where s is the total number of vertices of F_{n-1} and k is the number of vertices of F_{n-1} which are not collinear with z. On the other hand, any F_n configuration is obtained by this procedure from α different configurations F_{n-1}, where α is the number of pendant lines of F_n containing z. Thus

$$\frac{v-s-k}{2}|F_{n-1}| = \alpha|F_n| \tag{2}$$

Hence F_n is a constant configuration.

Remark 4. Using (2) we get

$$|A_n| = \frac{v(v-1)(v-3)\cdots(v-2n+1)}{2^n n!}, \qquad n \geq 2,$$
$$|B_n| = \frac{v(v-1)(v-3)(v-7)(v-9)\cdots(v-2n-3)}{3 \cdot 2^n (n-1)!}, n \geq 3,$$
$$|C_n| = \frac{v(v-1)(v-3)(v-7)(v-9)\cdots(v-2n-1)}{2^{n-1}(n-2)!}, n \geq 4,$$
$$|D_n| = \frac{v(v-1)(v-3)(v-7)(v-9)\cdots(v-2n+1)}{2^{n-3}(n-3)!}, n \geq 4,$$
$$|E_n| = \frac{v(v-1)(v-3)(v-7)(v-9)\cdots(v-2n+3)}{3 \cdot 2^{n-3}(n-4)!}, n \geq 5.$$

Acknowledgement

The authors wish to thank Jeff Dinitz and Peter Gibbons for supplying some randomly generated Steiner triple systems of order 19.

References

[1] M.J. Grannell, T.S. Griggs, E. Mendelsohn, A small basis for four-line configurations in Steiner Triple Systems, to appear in *J. of Combinatorial Designs.*

Appendix

Eight Steiner triple systems of order 19

0 1 16	0 2 18	0 3 7	0 4 8	0 5 14	0 6 10
0 9 15	0 11 12	0 13 17	1 2 13	1 3 11	1 4 12
1 5 10	1 6 15	1 7 14	1 8 18	1 9 17	2 3 4
2 5 6	2 7 17	2 8 15	2 9 12	2 10 16	2 11 14
3 5 8	3 6 9	3 10 13	3 12 14	3 15 16	3 17 18
4 5 15	4 6 17	4 7 13	4 9 16	4 10 11	4 14 18
5 7 16	5 9 11	5 12 17	5 13 18	6 7 11	6 8 14
6 12 18	6 13 16	7 8 9	7 10 12	7 15 18	8 10 17
8 11 13	8 12 16	9 10 18	9 13 14	10 14 15	11 15 17
11 16 18	12 13 15	14 16 17			

0 1 15	0 2 9	0 3 4	0 5 16	0 6 12	0 7 18
0 8 10	0 11 17	0 13 14	1 2 10	1 3 18	1 4 14
1 5 13	1 6 17	1 7 9	1 8 16	1 11 12	2 3 11
2 4 7	2 5 17	2 6 13	2 8 14	2 12 16	2 15 18
3 5 7	3 6 8	3 9 15	3 10 17	3 12 14	3 13 16
4 5 8	4 6 11	4 9 12	4 10 15	4 13 18	4 16 17
5 6 10	5 9 11	5 12 15	5 14 18	6 7 14	6 9 18
6 15 16	7 8 15	7 10 12	7 11 16	7 13 17	8 9 17
8 11 18	8 12 13	9 10 13	9 14 16	10 11 14	10 16 18
11 13 15	12 17 18	14 15 17			

0 1 9	0 2 14	0 3 17	0 4 15	0 5 16	0 6 12
0 7 18	0 8 13	0 10 11	1 2 15	1 3 11	1 4 16
1 5 13	1 6 17	1 7 14	1 8 18	1 10 12	2 3 10
2 4 11	2 5 18	2 6 16	2 7 17	2 8 12	2 9 13
3 4 8	3 5 15	3 6 18	3 7 16	3 9 12	3 13 14
4 5 10	4 6 13	4 7 12	4 9 14	4 17 18	5 6 7
5 8 14	5 9 17	5 11 12	6 8 11	6 9 15	6 10 14
7 8 10	7 9 11	7 13 15	8 9 16	8 15 17	9 10 18
10 13 17	10 15 16	11 13 18	11 14 15	11 16 17	12 13 16
12 14 17	12 15 18	14 16 18			

0 1 18	0 2 10	0 3 6	0 4 17	0 5 11	0 7 15
0 8 14	0 9 13	0 12 16	1 2 9	1 3 4	1 5 7
1 6 16	1 8 12	1 10 15	1 11 17	1 13 14	2 3 11
2 4 14	2 5 15	2 6 13	2 7 18	2 8 16	2 12 17
3 5 8	3 7 17	3 9 12	3 10 13	3 14 15	3 16 18
4 5 18	4 6 7	4 8 11	4 9 15	4 10 16	4 12 13
5 6 14	5 9 16	5 10 12	5 13 17	6 8 9	6 10 11
6 12 18	6 15 17	7 8 13	7 9 10	7 11 16	7 12 14
8 10 17	8 15 18	9 11 14	9 17 18	10 14 18	11 12 15
11 13 18	13 15 16	14 16 17			

0 1 8	0 2 10	0 3 18	0 4 7	0 5 11	0 6 15
0 9 16	0 12 13	0 14 17	1 2 7	1 3 16	1 4 10
1 5 15	1 6 17	1 9 18	1 11 13	1 12 14	2 3 9
2 4 11	2 5 14	2 6 18	2 8 15	2 12 17	2 13 16
3 4 5	3 6 12	3 7 15	3 8 14	3 10 11	3 13 17
4 6 13	4 8 12	4 9 15	4 14 18	4 16 17	5 6 8
5 7 17	5 9 10	5 12 16	5 13 18	6 7 9	6 10 14
6 11 16	7 8 13	7 10 12	7 11 14	7 16 18	8 9 17
8 10 16	8 11 18	9 11 12	9 13 14	10 13 15	10 17 18
11 15 17	12 15 18	14 15 16			

0 1 6	0 2 12	0 3 13	0 4 16	0 5 18	0 7 11
0 8 15	0 9 10	0 14 17	1 2 18	1 3 14	1 4 10
1 5 9	1 7 16	1 8 11	1 12 17	1 13 15	2 3 15
2 4 5	2 6 13	2 7 10	2 8 14	2 9 17	2 11 16
3 4 8	3 5 12	3 6 16	3 7 9	3 10 17	3 11 18
4 6 7	4 9 18	4 11 12	4 13 14	4 15 17	5 6 8
5 7 17	5 10 13	5 11 15	5 14 16	6 9 15	6 10 12
6 11 17	6 14 18	7 8 12	7 13 18	7 14 15	8 9 16
8 10 18	8 13 17	9 11 13	9 12 14	10 11 14	10 15 16
12 13 16	12 15 18	16 17 18			

0 1 3	0 2 7	0 4 8	0 5 14	0 6 12	0 9 15
0 10 11	0 13 17	0 16 18	1 2 15	1 4 7	1 5 16
1 6 13	1 8 9	1 10 18	1 11 17	1 12 14	2 3 11
2 4 9	2 5 12	2 6 16	2 8 13	2 10 17	2 14 18
3 4 5	3 6 9	3 7 18	3 8 15	3 10 13	3 12 16
3 14 17	4 6 17	4 10 14	4 11 12	4 13 16	4 15 18
5 6 18	5 7 9	5 8 17	5 10 15	5 11 13	6 7 10
6 8 14	6 11 15	7 8 12	7 11 16	7 13 14	7 15 17
8 10 16	8 11 18	9 10 12	9 11 14	9 13 18	9 16 17
12 13 15	12 17 18	14 15 16			

0 1 16	0 2 17	0 3 5	0 4 14	0 6 8	0 7 9
0 10 15	0 11 12	0 13 18	1 2 13	1 3 4	1 5 18
1 6 14	1 7 17	1 8 12	1 9 10	1 11 15	2 3 8
2 4 6	2 5 15	2 7 11	2 9 12	2 10 14	2 16 18
3 6 11	3 7 18	3 9 15	3 10 12	3 13 16	3 14 17
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5 6 16	5 8 9	5 10 13	5 11 14	5 12 17	6 7 13
6 9 18	6 10 17	6 12 15	7 8 10	7 12 16	7 14 15
8 13 15	8 14 16	8 17 18	9 11 16	9 13 14	10 11 18
11 13 17	12 14 18	15 16 17			