Pseudosimilarity in Graphs— A Survey

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This survey reviews the results on pseudosimilarity which have been obtained in the years since it was discovered by Harary and Palmer. A number of open questions and a full bibliography are given.

1. INTRODUCTION

Graph theorists seem to have stumbled on the concept of pseudosimilarity quite by accident. If two vertices u and v in a graph G are similar, that is, there is an automorphism of G which maps one into the other, then it is clear that $G - u$ and $G - v$ are isomorphic graphs. However the converse is not true, because $G - u$ and $G - v$ can be isomorphic without u and v being similar in G. The smallest graph for which this can happen is shown in Figure 1. Nobody seems to have given this phenomenon any thought until (as reported by Harary and Palmer [17]) someone apparently found a proof of the celebrated Reconstruction Conjecture which depended on the assumption that if $G - u$ and $G - v$ are isomorphic then u and v must be similar. To Harary and Palmer goes the credit of taking what could simply have remained a curious counter-example, and turning it into a graph theoretic concept worthy of investigation. Their 1965 and 1966 papers proved the first results and set the scene for further studies. In more than twenty-five years which have passed since the Harary and Palmer papers, a number of authors have found new results and unearthed more problems. So it seems useful to attempt to present a unified look at the progress made and the open questions that emerge in the hope that this may indicate the most fruitful directions to follow in further investigations and the most appropriate tools to employ.

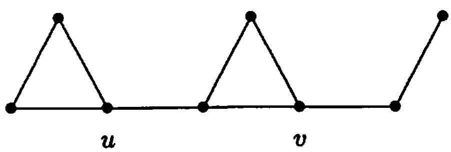

Figure 1

We shall mostly follow the graph theoretic terminology of [14] the most notable exception being that here we use the terms vertex and edge instead of point and line respectively. Unless explicitly stated otherwise, all graphs considered are finite and simple. For any graph G, the sets of vertices and of edges will be denoted by $V(G)$ and $E(G)$ respectively. The *order* of G

is $|V(G)|$. Two adjacent vertices are said to be *neighbours* and the set of neighbours of a vertex v in G is denoted by $N_G(v)$ or simply $N(v)$. As in [14], two vertices u and v of G are *similar* if there exists some automorphism α of G such that $\alpha(u) = v$; they are *removal-similar* if $G - u$ and $G - v$ are isomorphic, and they are *pseudosimilar* if they are removal-similar but not similar. Further notation and terminology will be defined as and when it is required.

2. BASIC RESULTS

Although at first it might seem unexpected that pseudosimilar vertices do exist, upon further consideration one comes to realise that, in fact, such vertices should be a natural occurrence. The simplest way to create such vertices in a graph is possibly the following. Let G be a graph and let u and v be a pair of similar vertices in G. If u and v are adjacent in G, then let $H = G - uv$, whereas if they are not let $H = G + uv$. The vertices u and v are still removal-similar in H but the addition or deletion of the edge uv could have destroyed their similarity, making them pseudosimilar. One necessary condition for this to happen is that there is no automorphism α of G such that $\alpha(u) = v$ and $\alpha(v) = u$. This sets the scene for the investigation of why pseudosimilar vertices arise. We call two sets of vertices A and B in a graph G *interchange similar* if there is an automorphism α of G such that $\alpha(A) = B$ and $\alpha(B) = A$.

Theorem 1. *Let u, v be pseudosimilar vertices in a graph G, and let $A = N(u) \cap V(G - u - v)$, $B = N(v) \cap V(G - u - v)$. Then either A and B are similar but not interchange similar in $G - u - v$ or else $G - v$ contains a vertex pseudosimilar to u.*

Proof. Let $\alpha: G - v \to G - u$ be an isomorphism. If $\alpha(u) = v$ then restricting α to $V(G) - u - v$ gives an automorphism mapping A into B. Of course, A and B cannot be interchange similar in $G - u - v$ as otherwise u and v would be similar in G.

We therefore assume that $\alpha(u) \neq v$. Let $w = \alpha^{-1}(v)$; therefore $w \neq u$. It now follows that u and w are removal similar in $G - v$, for $(G - v) - w \simeq \alpha((G - v) - w) = (G - u) - v = (G - v) - u$.

Now suppose that u and w are similar in $G - v$. Let β be an automorphism of $G - v$ with $\beta(u) = w$. Then $\beta\alpha$ is an isomorphism from $G - v$ to $G - u$ with $\beta\alpha(u) = v$. We therefore again obtain, as above, that there is an automorphism of $G - u - v$ mapping A into B.

The only alternative left is that u and w are not similar in $G - v$, giving that $G - v$ contains a vertex pseudosimilar to u. □

This theorem easily gives, as a corollary, two results about vertices which cannot be pseudosimilar in a tree; the first of these was one of the earliest results on pseudosimilarity. We first need to recall the following fact about similar vertices in a tree.

Theorem 2. [47] *Any two similar vertices in a tree are interchange similar.*

We also recall that an *endvertex* is a vertex of degree equal to 1 and an *end-cutvertex* in a tree is a vertex having only one neighbour with degree greater than one.

Corollary 1. *Let T be a tree. Then:*
(i) [17] *Any two removal-similar endvertices in T are similar.*
(ii) [26] *Any two removal-similar end-cutvertices in T are similar.*

Proof. We shall prove (i) by induction on the number of vertices of the tree T; the proof of (ii) is analogous. Suppose u, v are pseudosimilar endvertices of T. Let x, y be the neighbours of u, v respectively. Since, by the induction hypothesis, u cannot be pseudosimilar to any endvertex in $T-v$, it follows, from Theorem 1, that x and y are similar but not interchange similar in $T-u-v$. But this contradicts Prins' Theorem. Therefore u and v cannot be pseudosimilar. □

So it seems that one way to try and obtain pseudosimilar vertices is to take a graph with two sets of similar but not interchange similar sets of vertices, and join two new vertices to them. Harary and Palmer [17], by exploiting this idea, gave the first systematic way of constructing pairs of pseudosimilar vertices. Take any graph H, and let X and Y be two sets of vertices of H such that there is no automorphism α of H with $\alpha(X) = Y$ (for example, choose X and Y with $|X| \neq |Y|$). Then take three copies H_1, H_2, H_3 of H and form G by adding two new vertices u and v joining u to X in H_1 and to Y in H_2, and v to X in H_2 and to Y in H_3. Then u and v are pseudosimilar in G. The graph in Figure 1 could have been constructed this way, with $H = K_2$, the complete graph on two vertices, $X = V(H)$ and Y containing only one vertex.

But the most general construction from which all pairs of pseudosimilar vertices can be obtained was found by Godsil and Kocay [12] who showed that such pairs can in fact always obtained by destroying some cyclic symmetry in a graph. The basic idea came from Herndon and Ellzey [20] who were studying methods of constructing "removal-cospectral" vertices and developed a method which actually produced removal-similar, and hence pseudosimilar, vertices. Take a graph H with vertices u and v and an automorphism α such that $\alpha^t(u) = v$ for some $t > 1$ and $\alpha^r(u) \neq v$ for $1 \leq r < t$. Then u and v are removal-similar in $G = H - \{\alpha(u), \ldots, \alpha^{t-1}(u)\}$; if moreover they also happen to be not similar, then we have a pair of pseudosimilar vertices. Godsil and Kocay showed that, in fact, every pair of pseudosimilar vertices can be obtained this way.

Theorem 3. [12] *Let u and v be a pair of pseudosimilar vertices in a graph G. Then G is an induced subgraph of some graph H such that H has an automorphism α with $\alpha(G-v) = G-u$ and $\alpha^t(u) = v$, and such that $V(H) - V(G) = \{x_1, \ldots, x_r\}$, where $x_i = \alpha^{t+i}(u)$ and $\alpha(x_r) = u$.*

The reason why pairs of pseudosimilar vertices occur is therefore quite well understood in terms of a sort of truncation of cyclic symmetry. The area where most of the unanswered questions lie is the situation where a graph has several pseudosimilar vertices. We can ask for two ways in which this can happen. One way requires the graph to have several pairs of pseudosimilar vertices. This case is taken up in Section 4. Alternatively one can ask for the graph to have a large set of mutually pseudosimilar vertices (the vertices in a subset S of $V(G)$ are said to be mutually pseudosimilar if any two vertices of S are removal-similar but no two are similar). This situation is discussed in Section 5. We shall first give, in the next section, those definitions and results involving graphs and groups upon which most of the material in Sections 4 and 5 depends.

3. GRAPHS AND GROUPS

Let Γ be a finite group and let Ω be a set of generators of Γ not containing the identity element. The *Cayley colour-graph* $D(\Gamma, \Omega)$ is a directed graph with vertex set Γ and arcs (x, y) for all pairs of elements of Γ with $x^{-1}y \in \Omega$; moreover, each arc (x, y) is coloured with colour $x^{-1}y$. When the set Ω also has the property that if $x \in \Omega$ then $x^{-1} \in \Omega$ we can define the *Cayley graph* $G(\Gamma, \Omega)$ to be the graph with vertex set Γ and edges $\{x, y\}$ for all pairs of elements of Γ with $x^{-1}y \in \Omega$.

It is well known that Γ is the group of colour-preserving automorphisms of $D(\Gamma, \Omega)$ and that the arcs of $D(\Gamma, \Omega)$ can be replaced by appropriate "gadgets" to give a graph whose automorphism group is abstractly isomorphic to Γ. It is equally standard knowledge that $G(\Gamma, \Omega)$ is a vertex-transitive graph and that Γ is a subgroup (sometimes proper) of its automorphism group. The action of Γ on $G(\Gamma, \Omega)$ is the *left regular action* which associates to every $g \in \Gamma$ an automorphism α_g of $G(\Gamma, \Omega)$ defined by $\alpha_g(x) = gx$. If Γ is in fact the full automorphism group of $G(\Gamma, \Omega)$ then $G(\Gamma, \Omega)$ is called a *graphical regular representation (GRR) of* Γ. Except for a finite number of known groups, all finite, nonabelian groups which are not generalised dicyclic groups have GRR's. A number of authors contributed towards obtaining this result, the final steps being provided in [21] and [10]. We shall only be requiring GRRs for groups of odd order; (it follows (see [3], for example) that such groups must be nonabelian).

Theorem 4. [22] *Except for one group of order 27, all nonabelian groups of odd order have GRR's.*

Although given any group Γ there is a graph whose automorphism group is abstractly isomorphic to Γ, it is not true that every permutation group is equivalent to the group of a graph. However it is possible to obtain a graph with an automorphism group whose action on a subset of its vertices is equivalent to a given permutation group. Suppose Γ is a permutation group acting transitively on a set X. For some $x \in X$, let Γ_x be its stabiliser. Then the action of Γ on X is equivalent to the action of premultiplication by Γ on the set of left cosets of Γ_x. Construct a Cayley colour-graph $D(\Gamma, \Omega)$ and, by replacing its arcs with "gadgets" in the standard way, obtain a graph G whose automorphism group is abstractly isomorphic to Γ. Then add a new vertex to G joining it to the vertices corresponding to Γ_x; repeat this by adding a new vertex for every left coset of Γ_x. The resulting graph has an automorphism group whose action restricted to the set of new vertices is equivalent to Γ. If Γ is not transitive on X then the construction can be repeated separately for all the orbits of X. This way one basically obtains the following theorem, a short proof of which can also be found in Problem 12.21 of the book by Lovász [44].

Theorem 5. [5] *Let Γ be a permutation group acting on a set X. Then there exists a graph G such that $X \subseteq V(G)$, X is invariant under the action of AutG and the restriction of AutG to X gives a permutation group equivalent to Γ.*

The usefulness of this result in tackling certain problems on pseudosimilar vertices can be illustrated by the following solution of Kocay [33] to a problem raised by E. Farrell and B.D. McKay. Let u and v be pseudosimilar vertices in a graph G, and let H be a graph with a distinguished vertex a. Denote by $G_u(H)$ and $G_v(H)$ the graphs obtained from G and H by identifying vertex a of H with the vertices u and v, respectively. Is it possible for $G_u(H)$ and $G_v(H)$ to be isomorphic? Kocay answered this question in the affirmative in the following fashion, which is also a good illustration of how pseudosimilar vertices can be constructed by destroying some cyclic symmetry between vertices in a graph. Let a graph H with a distinguished vertex a be given and let A_4 denote the alternating group acting on $X = \{u, v, w, x\}$. Use Theorem 5 to construct a graph K such that $X \subseteq V(K)$ and the restriction of Aut$K$ to X is equivalent to A_4. Form $K_x(H)$ and let $G = K_x(H) - w$. (To ensure that the only automorphisms of G are still those arising from A_4 one might have to adjust slightly the construction of Theorem 5, for example by adding "tails" to some of the vertices to distinguish them by their degree.) Then u and v are pseudosimilar in G and $G_u(H) \simeq G_v(H)$.

4. GRAPHS WITH SEVERAL PAIRS OF PSEUDO-SIMILAR VERTICES

Kimble, Schwenk and Stockmeyer [25] were the first to consider the problem of graphs with several pseudosimilar vertices. The following result was essentially proved in their paper.

Theorem 6. [25] *There exist graphs in which every vertex has a pseudosimilar mate.*

Proof. Let Γ be a group of odd order and let H be a GRR of Γ (as we have noted above, Γ must be nonabelian and, by Theorem 4, such Γ and H do exist). We note that H is a regular graph and that the stabiliser of any vertex is just the identity element of Γ. Therefore, if r is any vertex of H, then $G = H - r$ has the identity automorphism group.

Now, let v be any vertex in G. There is an automorphism α of H mapping r to v. The vertices $\alpha^{-1}(r)$ and $v = \alpha(r)$ are distinct, because otherwise α would contain a cycle of length 2, which is impossible since Γ has odd order. Since α^{-1} maps $\{v, r\}$ onto $\{r, \alpha^{-1}(r)\}$, it follows that $G - v = H - r - v \simeq H - \alpha^{-1}(r) - r = G - \alpha^{-1}(r)$, that is, $v = \alpha(r)$ and $\alpha^{-1}(r)$ are removal-similar in G. But G has the identity automorphism group, therefore v and $\alpha^{-1}(r)$ are pseudosimilar. □

Kimble, Schwenk and Stockmeyer also give neat constructions as concrete examples of the above theorem. For the sake of illustration we give one simple example from their constructions. We describe this in terms of Cayley graphs, which is easily seen to be equivalent to the formulation given by Kimble, Schwenk and Stockmeyer.

Let Γ be the metacyclic group of order 21 given by the presentation

$$\Gamma = \langle \alpha, \beta | \beta^7 = \alpha^3 = 1, \beta\alpha = \alpha\beta^2 \rangle.$$

Let H be the Cayley graph $G(\Gamma, \Omega)$ where

$$\Omega = \{\alpha, \alpha^{-1}, \beta, \beta^{-1}, \alpha\beta, \alpha^{-1}\beta^3, \alpha\beta^5, \alpha^{-1}\beta\}.$$

All we have to show in order to prove that H is a GRR of Γ is that the only automorphism of H which fixes 1 is the identity. We do this by employing the method used in [19]. Let σ be an automorphism of H that fixes 1. Therefore σ, when restricted to the subgraph H' of H induced by the neighbours of 1, is an automorphism. (The graph H' is shown in Figure 2.)

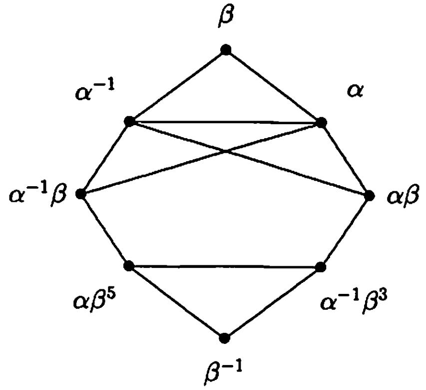

Figure 2

Hence β and β^{-1} are also fixed by σ. But H is vertex-transitive, therefore it follows that if σ fixes some vertex γ, then it also fixes $\gamma\beta$; therefore σ fixes $\beta^0, \beta^1, \ldots, \beta^6$. From H' one can also deduce that σ either fixes α and α^{-1} or it reverses them. But α is adjacent to β^3 and α^{-1} is not, so σ must fix α and α^{-1}. Hence σ fixes H, as required.

5. LARGE SETS OF MUTUALLY PSEUDO-SIMILAR VERTICES

It is perhaps under this heading that most of the interesting open questions on pseudosimilarity arise. This problem has been investigated by a number of authors [13, 24, 25, 31, 34, 40, 42, 43]. It is clear that a graph G cannot have all its vertices mutually pseudosimilar. Otherwise G would be regular and a regular graph cannot have pseudosimilar vertices because if α is an isomorphism from $G-u$ to $G-v$, then α must map the neighbours of u into the neighbours of v, and therefore it can be extended to an automorphism of G mapping u into v. Therefore the first question which arises is to determine the largest size which a set of mutually pseudosimilar vertices in a graph of order n can have. This seems to be a very difficult problem, and below we shall consider a more restricted version of this question. (This problem has been solved for trees: in [26] it is shown that it is not possible to have three or more mutually pseudosimilar vertices in a tree.) Another problem is to obtain necessary conditions for the existence of sets of mutually pseudosimilar vertices analogous to Theorem 3 for pairs of such vertices. The progress registered on this question is also discussed below.

The first to construct graphs with large sets of mutually pseudosimilar vertices were, independently, Krishnamoorthy and Parthasarathy [40] and Kimble, Schwenk and Stockmeyer [25]. The latter construction is particularly simple to describe. The transitive tournament on k vertices, T_k, is the tournament with vertex set $\{1, \ldots, k\}$ and in which vertex i dominates vertex j iff $i < j$. Clearly, the vertices of T_k are all mutually pseudosimilar. The problem is therefore to transform T_k into an undirected graph while preserving the pseudosimilarity of its vertices. The usual replacement of arcs by "gadgets" will give a graph whose automorphism group is still the identity. However, in order to make the vertices of T_k removal-similar in the resulting graph, we have to take into consideration the fact that now removing such a vertex will leave behind it "tails" and "heads", and these must be compensated for. We must therefore first add to the tournament T_k appropriate "semi-arcs" and then replace these by the corresponding "semi-gadgets". This process is illustrated for T_4 in Figure 3.

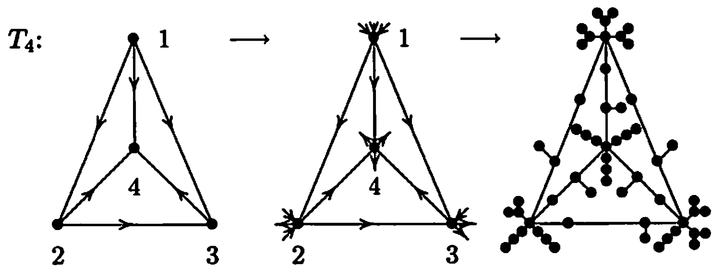

Figure 3

This method therefore constructs a sequence of graphs G_k having k mutually pseudosimilar vertices and order $O(k^2)$. Krishnamoorthy and Parthasarathy [40] constructed a sequence of graphs G_k having 2^k mutually pseudosimilar endvertices and order $O(3^k)$. A slightly more general construction of Lauri and Marino [42] and Lauri [43], which we now describe, produces a denser "packing" of mutually pseudosimilar vertices in a graph.

Let G' be a graph containing r endvertices, all of which are mutually pseudosimilar. Let G be the graph obtained from G' by removing all its endvertices, and let R be the set of neighbours of the endvertices of G' (note that since no two endvertices are similar, no two can share a common neighbour, therefore $|R| = r$). Let X be the set of all those vertices of G which are similar to some vertex in R under the action of Aut$G$. We now construct a sequence of graphs G_t, $t = 1, 2, \ldots$, containing r^t mutually pseudosimilar endvertices. Let $G_1 = G'$ and let H_1 be G_1 less one of its endvertices. Having constructed G_t, let H_t be G_t less one of its pseudosimilar endvertices. Then, G_{t+1} is obtained by attaching a copy of G_t to each vertex in R and a copy of H_t to each of the other vertices in $X - R$. (By attaching a copy of G_t (or H_t) to a vertex v of G we mean joining v to every vertex of G_t (or H_t) which is not an endvertex.)

Each graph G_t so obtained has r^t mutually pseudosimilar endvertices and $O(|X|^t)$ vertices. Therefore if $k = r^t$ is the number of pseudosimilar endvertices, then the total number of vertices in G_t is $O(k^{\frac{\log|X|}{\log|R|}})$.

Such a sequence of graphs is constructed in [43] with $|X| = 2|R| = 8$, therefore giving a sequence G_t with $k = 4^t$ pseudosimilar endvertices and order $O(k^{3/2})$. These constructions suggest the problem of obtaining a sequence of graphs having k mutually pseudosimilar (end)vertices and order $O(k^{1+\epsilon})$, with ϵ as small as possible.

The crucial step in this construction is finding the starting graph G', that is, one with endvertices all of which are mutually pseudosimilar. One way to do this is the following. Suppose Γ is a group of permutations acting on some set X such that, for some $R \subset X$, the following two conditions hold: (i) the setwise stabiliser $\Gamma_{\{R\}}$ of R is the identity and, (ii) for any two $(|R|-1)$-subsets A, B of R, there is a permutation α in Γ such that $\alpha(A) = B$. Then, by employing Theorem 5, one can construct a graph G with minimum degree at least 2 and $X \subseteq V(G)$ and whose automorphism group has the same action as Γ on X. Therefore if we attach one endvertex to each vertex of $R \subset V(G)$ we obtain the starting graph G' all of whose endvertices are mutually pseudosimilar. Hence such starting graphs can be constructed if permutation groups satisfying conditions (i) and (ii) are found.

In [43] is described a simplified version of a construction of such permutation groups due to Cameron [6]. This construction produces groups with the required properties if $|X| = O(|R|^{2|R|})$, therefore the question still remains of finding groups of substantially smaller degree having these properties.

However, by the previous discussion, these groups do make possible the construction, for any r, of a graph G containing r endvertices all of which are mutually pseudosimilar. Such a graph can be transformed, as follows, into one containing a set of mutually pseudosimilar vertices which are not endvertices. Let Δ be the maximum degree of G and let $p = \max\{r, \Delta\}$. Identify the endvertices of G with distinct vertices of the complete graph K_p. In the resulting graph, the vertices which were endvertices in G are still mutually pseudosimilar. This brute-force construction does not, of course, solve the problem of obtaining as dense a packing of mutually pseudosimilar vertices as possible.

Godsil and Kocay [13] used Cayley graphs and the action of a group on the cosets of a subgroup to construct graphs with three mutually pseudosimilar vertices. Let Γ again be the metacyclic group of order 21 with the presentation given above. Let the set of generators Ω this time be $\{\alpha, \alpha^{-1}, \beta, \beta^{-1}\}$. Let G be the Cayley graph $G(\Gamma, \Omega)$, and let S be the subgroup $\{1, \beta\alpha, \beta^3\alpha^2\} = \langle\beta\alpha\rangle$. Let $\gamma = \beta^3\alpha^2$. Then the cosets $S, \beta S, \beta^2 S$ and $\beta^3 S$ are distinct and $\gamma S = \beta^{-1}\gamma\beta^2 S = S$. Therefore, remembering that β and γ are automorphisms (by premultiplication) of G, one can see that, if H is obtained from G by joining new vertices u, v, w to the vertices of $S, \beta S$ and $\beta^2 S$, respectively, then $H - u, H - v$, and $H - w$ are isomorphic. Further calculation shows that u, v and w are actually pseudosimilar, not similar. Godsil and Kocay give further examples using this method and Kocay [31] modified this construction to give the smallest known graph (of order 17) containing three mutually pseudosimilar vertices.

Kocay, Niesink and Zarnke [36] exploit further the equivalence of the action of a permutation group Γ on a set X with the action of Γ on the set of left cosets of a stabiliser. They systematically search for groups Γ with a subgroup K such that the action of Γ on the cosets of K can be used to construct graphs with $k > 2$ mutually pseudosimilar vertices.

Obtaining results analogous to Theorem 3 for sets containing $k > 2$ mutually pseudosimilar vertices is much more difficult than for $k = 2$. In the previous example of Godsil and Kocay with three mutually pseudosimilar vertices, if the vertices of the Cayley graph G are partitioned into a complete family of cosets of S and if, for every coset, a new vertex is added, joining it to all the elements in the coset, then the three vertices u, v and w do become similar. Thus, for such graphs a result similar to Theorem 3 exists, that is, the graph H can be extended so as to make u, v and w similar. Kocay has obtained the following result.

Theorem 7. [34] *Let G be a graph with a set $U = \{u_0, u_1, \ldots, u_{k-1}\}$ of k mutually pseudosimilar vertices. Let $p_i : X - u_i \rightarrow X - u_0$ be isomorphisms, for $i = 1, 2, \ldots, k-1$. Then G can be extended to a graph H, and each p_i can be extended to a permutation p_i^* of $V(H)$ such that:*

(i) G is an induced subgraph of H;
(ii) each p_i^* is an automorphism of H;
(iii) the vertices of U are all similar in H;
(iv) the vertices of $V(H) - V(G)$ are all in the same orbit as U under the action of $AutH$.

Theorem 7 is an analogue of Theorem 3, with an important difference — the graph H in Theorem 3 is finite whereas the graph H in Theorem 7 is infinite. There is therefore still the problem of obtaining a full analogue of Theorem 3 with H finite. The main stumbling block towards finding conditions like those of Theorem 3 for sets of three or more pseudosimilar vertices is that with only one pair of pseudosimilar vertices there is only one permutation p_1^*. Therefore, forcing p_1^* to have finite order in the course of the proof of Theorem 3, it is easy to determine the group which is generated, namely the cyclic group. But, with more than one generator, unless one is willing to settle for the free group generated by $\{p_1^*, p_2^*, \ldots, p_{k-1}^*\}$, it is no longer possible to say in general what the structure of the group one is working with will be.

Godsil and Kocay [13] managed to some extent in going round this problem for $k = 3$ by considering as a special case the situation where there are no edges between a certain set of vertices containing the three mutually pseudosimilar vertices. The algorithm they give for extending the graph with three mutually pseudosimilar vertices into one in which the vertices are similar is essentially the Todd-Coxeter algorithm for the enumeration of cosets of a subgroup of a group.

It is interesting to note, in this connection, that in all known examples, the subgraph induced by mutually pseudosimilar vertices is either the complete graph or the null graph, and this raises the question of whether this must necessarily always be the case.

6. OTHER RESULTS

Pseudosimilar edges

Pseudosimilar edges can be defined in an analogous way to pseudosimilar vertices. Of course, the constructions of the previous section giving sets of mutually pseudosimilar endvertices also construct mutually pseudosimilar edges. Kimble [24] looked for a result like Theorem 6 for pseudosimilar edges, and he managed to construct a sequence of graphs in which the proportion of edges which have a pseudosimilar mate tends to 1. The following is a much simpler construction. Let C_1 and C_2 be two directed cycles, each on n vertices, n odd. Let G_n be constructed as follows: Join every vertex of C_1 to every vertex of C_2 and replace each arc of C_1 by Gadget 1 and each arc of C_2 by Gadget 2, as shown in Figure 4; delete one of the edges joining a vertex of C_1 to a vertex of C_2.

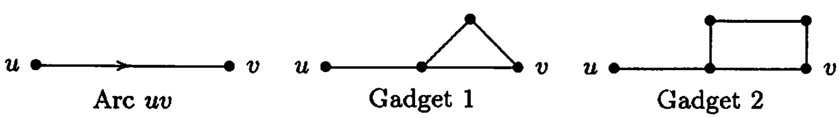

Figure 4

The resulting graph G_n has the identity automorphism group, and each edge joining a pair of vertices from C_1 and C_2 has a corresponding edge to which it is removal-similar. Therefore G has $n^2 - 1$ edges which have pseudosimilar mates, and a total of $n^2 + 9n - 1$ edges. Therefore the proportion of edges which have a pseudosimilar mate tends to 1 as n tends to ∞.

The question of whether there exist graphs in which every edge has a pseudosimilar mate remains, however, open. Such a graph K would be a line graph $L(G)$ of a graph G such as those constructed in Section 4.

Infinite graphs

Most work done on pseudosimilarity is concerned only with finite graphs. Godsil and Kocay [12] do mention constructions of pseudosimilar pairs of vertices in infinite graphs and they actually give Theorem 3 in a way which is applicable to the infinite case (the main amendment to Theorem 3 in order to allow for the possibility that G be infinite is basically that, if x is any vertex in $V(H) - V(G)$, then x can be equal to $\alpha^k(u)$ or $\alpha^{-k}(u)$ for some integer $k \geq 1$). They also mention the following problem (see also [11]) which brings out the sharp differences between pseudosimilarity in finite and infinite graphs: Is there a connected locally finite graph G such that the subgraphs $G-u, u \in V(G)$, are all isomorphic but G is not regular (and hence G contains pseudosimilar vertices)? As they observed, while the answer to the question is, of course, no in the finite case, on the other hand, for infinite graphs, if the condition of connectedness or local finiteness is dropped, then the answer is yes. Thomassen [54] solved the problem proving the following stronger statement.

Theorem 8 [54] *Let G be a locally finite infinite graph without isolated vertices. Then G is vertex-transitive iff all its vertex-deleted subgraphs are isomorphic.*

Vertex (edge) orbits and vertex-deleted (edge-deleted) subgraphs

Some authors [1, 2, 8, 55] take the following point of view of pseudosimilarity : If a graph G has k vertex (edge) orbits, that is, similarity classes under the action of Aut$G$, then the number of isomorphism classes of vertex-deleted (edge-deleted) subgraphs of G is at most k; if this number is less than k, then G has pseudosimilar vertices (edges). One natural question to ask in this context is: For a given value of k, do there exist graphs having k vertex (edge) orbits and fewer than k isomorphism classes of vertex-deleted (edge-deleted) subgraphs? For example, our earlier remark that if all vertex-deleted subgraphs of G are isomorphic then all its vertices are similar is equivalent to saying that the answer to the above question for $k = 2$ is no for vertex-deleted subgraphs.

The above-mentioned authors have studied this question for edge-deleted subgraphs, and the main result here is that the answer is no for $k = 3$, that is, if a graph has pseudosimilar edges, then it has at least four edge orbits, or equivalently, if a graph has exactly two isomorphism classes of edge-deleted subgraphs then it does not have pseudosimilar edges [1]. The answer to the question (for edge-deleted subgraphs) is yes for $k \geq 6$ but it is still unresolved for $k = 4, 5$. It is also awaiting investigation generally for vertex-deleted subgraphs.

Directed graphs

Pseudosimilar vertices and edges can analogously be defined for digraphs. Some of the questions posed and the results obtained for graphs, like Theorem 3, for example, also carry over to digraphs. In general, it would seem that it is easier to create pseudosimilar vertices or edges in digraphs, and in fact, most of the above constructions do start from digraphs. Jean [23] has shown that the only tournaments with all vertices mutually pseudosimilar are the transitive tournaments. Stockmeyer's tournaments [50, 51] in which every vertex has a pseudosimilar mate are important in the construction of counterexamples to the Digraph Reconstruction Conjecture.

Vertex switching

A *vertex-switching* G^S of a graph G is obtained by deleting from G all edges with exactly one end in the set of vertices S, and then adding to G all edges of the complement of G with exactly one end in S. If $S = \{u\}$ then G^S is denoted by G^u. Two vertices u and v are said to be *vertex-switching pseudosimilar* (or *VS-pseudosimilar*) if $G^u \simeq G^v$ but u and v are not similar in G. Ellingham [9] has obtained a result analogous to Godsil and Kocay's Theorem 3 above for graphs (finite or infinite) with pairs of VS-pseudosimilar vertices .

Pseudosimilarity and reconstruction

The discovery of the concept of pseudosimilarity as a flaw in a purported proof of the Reconstruction Conjecture has of course drawn upon it suspicion as a possible reason for the eventual falsity of the conjecture. There is however little concrete evidence for this, the most notable being Stockmeyer's counterexamples to the digraph reconstruction conjecture [50, 51]. The basis for these counterexamples is a family of tournaments with some remarkable properties, amongst which the fact that, in each tournament, every vertex has a pseudosimilar mate (see also [35]). The counterexamples are constructed by combining together pairs of such tournaments. Although, as we have seen in Section 4, graphs in which every vertex has a pseudosimilar mate do exist, there does not seem to be any analogous way of using them as building blocks for nonreconstructible graphs [52].

The complementary point of view would be to exploit the absence of pseudosimilarity in order to prove reconstructibility. For example, Thomassen's Theorem 8 implies that infinite graphs whose vertex-deleted subgraphs are all isomorphic is reconstructible, and Anderson, Ding and Vestergaard [2] have shown that a graph which has exactly two isomorphism classes of edge-deleted subgraphs is reconstructible from its two nonisomorphic subgraphs. Also, the fact that endvertices and end-cutvertices in a tree cannot be pseudosimilar was an important component in the proof of restricted versions of the reconstruction conjecture for trees [15, 46].

However, the most suggestive results in this context are the facts that trees are reconstructible from their endvertex-deleted subgraphs [18] and from their end-cutvertex-deleted subgraphs [37], and that endvertices and end-cutvertices in trees cannot be pseudosimilar. These results have lead Krasikov [37] to ask whether it is possible to show that a graph G with a "sufficiently large" set S of non-pseudosimilar vertices is reconstructible from its subgraphs $G-v, v \in S$. A good place to start investigating this sort of question could be trees — it might be revealing if one could prove, say, that trees are reconstructible from endvertex-deleted subtrees using mainly the similarity properties of endvertices in trees. In this respect, Krasikov's [38] generalisation of Corollary 1 is worth mentioning: Let T be a tree $a, b \in V(T)$, and A, B two non-isomorphic rooted trees and let $T_{a,b}(A, B)$ denote the tree obtained by identifying the roots of A and B with a and b respectively. If $T_{a,b}(A, B) \simeq T_{a,b}(B, A)$, then a and b are similar in T.

7. UNSOLVED PROBLEMS

To recapitulate this survey we gather here the main open problems which, in our opinion, the work discussed above seems to point to. Perhaps the most basic question, and certainly one of the most difficult, is the following.

Problem 1. In a graph G of order n, what is the largest possible size of a set S of mutually pseudosimilar vertices?

This question can perhaps be made more tractable by asking about the order of magnitude of $|S|$ with respect to $|V(G)|$, and also by restricting attention to endvertices.

Problem 2. Find an increasing sequence of integers k_t and a corresponding sequence of graphs G_{k_t} such that each G_{k_t} has k_t mutually pseudosimilar endvertices and order $O(k_t^{1+\epsilon})$, with ϵ as small as possible. Is it possible for the sequence of graphs to have order $O(k_t)$?

As we have seen, constructing a graph with r endvertices, all of which are mutually pseudosimilar, is equivalent to finding a group of permutations Γ acting on a set X such that, for some $R \subset X$, $|R| = r$, the setwise stabiliser of R is equal to its pointwise stabiliser and any two $(r-1)$-subsets of R are similar under the action of Γ. Such a group can be constructed with $|X| = O(r^{2r})$, and the following question naturally arises.

Problem 3. Find a permutation group of substantially smaller degree having the above properties.

While graphs in which any vertex has a pseudosimilar mate do exist, the question is still open for edges.

Problem 4. Do there exist graphs in which every edge has a pseudosimilar mate? Equivalently, do there exist line graphs in which every vertex has a pseudosimilar mate?

Most of these questions for graphs have natural analogues for digraphs. It seems reasonable to expect that, in this case, better bounds can be obtained. For example, we have already noted that all vertices in a transitive tournament are mutually pseudosimilar.

Problem 5. Repeat Problems 1, 2, 4 for digraphs.

Godsil and Kocay have shown that a finite graph G with $k = 2$ pseudosimilar vertices u_1, u_2 is an induced subgraph of a finite graph H in which the two vertices are similar and such that H has an automorphism α_1 such that $\alpha_1(G - u_1) = G - u_2$. Kocay has extended this result for $k > 2$ mutually pseudosimilar vertices, but in this case he was not able to show that H is finite. This therefore leaves open the next question whose answer we believe to be no.

Problem 6. Does there exist a graph G with $k > 2$ mutually pseudosimilar vertices $u_1, \ldots, u_k$ which is not the induced subgraph of a finite graph H in which $u_1, \ldots, u_k$ are similar and which has $k-1$ automorphisms $\alpha_1, \ldots, \alpha_{k-1}$ such that $\alpha_i(G - u_1) = G - u_{i+1}$?

The question of whether or not the subgraph induced by a set of mutually pseudosimilar vertices in necessarily the complete graph or the null graph is related to the previous problem but is also interesting in its own right.

Problem 7. Does there exist a graph G with a set S of $k > 2$ mutually pseudosimilar vertices such that the subgraph of G induced by S is not the complete graph or the null graph?

By taking line graphs and using the corresponding result for vertices, it follows that a graph cannot have all of its edges mutually pseudosimilar. It is also known that if a graph has exactly two isomorphism classes of edge-deleted subgraphs then it cannot have pseudosimilar edges. Resolving the following question would be the next step in this line of results.

Problem 8. For $k = 3$ and 4, can there be pseudosimilar edges in a graph having exactly k isomorphism classes of edge-deleted subgraphs? Investigate the analogous question for vertices, that is, for what values of $k \geq 2$ can there be pseudosimilar vertices in a graph having exactly k isomorphism classes of vertex-deleted subgraphs?

Finally, although pseudosimilarity owes its origin to the Reconstruction Problem, there does not seem to be much evidence of a direct relationship between the two problems. It is therefore not easy to suggest a concrete question relating the two concepts. We single out the following problem because it is suggested by existing results, its resolution seems relatively attainable and it seems to be one obvious place to start looking for relationships between pseudosimilarity and reconstruction.

Problem 9. Obtain a short proof that a tree is reconstructible from its endvertex-deleted subtrees by exploiting the similarity properties of endvertices in trees.

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Note added in proof: Lauri and Scapellato have found that a class of Cayley graphs constructed by Alspach and Xu gives graphs in which every edge has a pseudosimilar mate. This solves Problem 4.

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