Non-binary matroids having four non-binary elements*

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ABSTRACT. An element e of a matroid M is called *non-binary* when $M\backslash e$ and M/e are both non-binary matroids. Oxley in [6] gave a characterization of the 3-connected non-binary matroids without non-binary elements. In [4], we constructed all the 3-connected matroids having exactly 1, 2 or 3 non-binary elements. In this paper, we will give a characterization of the 3-connected matroids having exactly four non-binary elements.

1 Introduction

We start this paper by giving Tutte's definition of n-connected matroids. For more definitions and notation, the reader shall consult Oxley[7] or Welsh[13]. For a positive integer k, we say that $\{X, E\backslash X\}$ is a *k-separation* of a matroid M on E if

$$\xi(M;X) = r(X) + r(E \setminus X) - r(E) + 1 \leq k \text{ and } \min\{|X|, |E \setminus X|\} \geq k.$$

A k-separation $\{X, E \setminus X\}$ is said to be *exact* if $\xi(M;X) = k$. A matroid is said to be *k-connected* when it does not have an l-separation for every $1 \leq l < k$. Oxley[5] proved that

(1.1) If M is a 3-connected non-binary matroid and e is an element of M such that $M\backslash e$ and M/e are binary matroids, then M is isomorphic to $U_{2,4}$. □

Using the previous theorem of Oxley, it is not dificult to see that:

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$$\begin{aligned} D(M) &= \{e \in E(M) : M \backslash e \text{ is binary}\}, \\ C(M) &= \{e \in E(M) : M/e \text{ is binary}\} \text{ and} \\ N(M) &= \{e \in E(M) : M \backslash e \text{ and } M/e \text{ are not binary}\} \end{aligned}$$

is a partition of the ground set $E(M)$ of a 3-connected matroid M, when M is non-binary and non-isomorphic to $U_{2,4}$.

Observe that $N(M) = N(M^*)$ and $C(M) = D(M^*)$. An element of M is called *binary*, when it belongs to $C(M)$ or $D(M)$, and *non-binary* when it belongs to $N(M)$.

Oxley in [6] gave a characterization of the 3-connected non-binary matroids without non-binary elements, namely:

(1.2) The following two statements are equivalent for a matroid M.

(i) M is non-binary, 3-connected, and, for every element e, $M\backslash e$ or M/e is binary.

(ii) (a) M is isomorphic to $U_{2,n}$ or $U_{n-2,n}$ for some $n \geq 4$; or

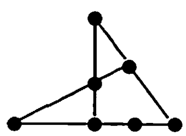
(b) both the rank and corank of M exceed two and M can be obtained from a 3-connected binary matroid by relaxing a circuit-hyperplane. □

We have constructed in [4] all the 3-connected matroids satisfying $1 \leq |N(M)| \leq 3$, namely:

(1.3) If M is a 3-connected non-binary matroid, then $N(M) = \emptyset$ or $|N(M)| \geq 3$. □

(1.4) If M is a 3-connected non-binary matroid such that $|N(M)| = 3$, then M is isomorphic to M_7 or M_7^*. □

A geometric representation for the matroid M_7^* is given below.



These results are surprising, since there are infinitely many non-binary 3-connected matroids whose elements are all binary, by **(1.2)**. The main result of this paper is a characterization of the 3-connected matroids having four non-binary elements. This result is similar to **(1.2)**, since there are infinitely many of these matroids.

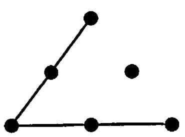
(1.5) Suppose that M is a 3-connected matroid. $|N(M)| = 4$ if and only if

(i) M is isomorphic to Q_6, M_8 or M_8^*; or

(ii) M can be obtained from a 3-connected binary matroid by relaxing a pair of circuit-hyperplanes of order 2; or

(iii) M is equal to the blowing up of a 3-connected binary matroid along a special line.

A geometric representation for the matroid Q_6 is given below. The definition of M_8 can be found in section 4.



In section 5, we define the operation of relaxing a pair of circuit-hyperplanes of order 2 and prove some properties about this construction. At the last subsection of this paper, we define a special line of a binary matroid, and describe how to construct the blowing up along this line.

I guess that is very difficult to get a similar characterization for the class of matroids having exactly m ($m \geq 5$) non-binary elements, because a matroid in this family may have a large line L such that $r(L) < r(M) - 1$. We conjecture that this class has infinitely many non-isomorphic matroids.

We became interested in this problem, as we were trying to solve a problem proposed by Oxley: find a minor-excluded characterization for the class of matroids $\mathcal{M}$, which is the union of the classes of binary and ternary matroids. A 3-connected excluded minor M for $\mathcal{M}$, which is not isomorphic to $U_{2,5}$ or $U_{3,5}$, has F_7 or F_7^* as a minor, by the Bixby-Reid-Seymour minor excluded characterization for the class of ternary matroids[2,8]. So, there are $A, B \subseteq E(M)$ such that $M \backslash B / A$ is isomorphic to F_7 or F_7^*. Observe that $B \subseteq D(M)$ and $A \subseteq C(M)$, and hence $|N(M)| \leq 7$. Using Oxley's extension of Seymour's splitter theorem, it is not difficult to prove that $|N(M)| \leq 4$, when $|E(M)| \geq 10$. So, every 3-connected excluded minor for $\mathcal{M}$, having at least 10 elements, belongs to the family of matroids described in **(1.2)** or to one of the two families described in **(1.5)**.

2 Known results

Except for theorems (2.7) and (2.8), which have been proved by Seymour, all the results presented in this section are proved in [4]. These results will be useful throughout this paper.

L is said to be a *line* of a matroid N, if L is a union of circuits of N and $r^*(N \mid L) = 2$. If $C_1, \ldots, C_n$ are the circuits of N contained in a line L of N, the partition $\{L \setminus C_1, \ldots, L \setminus C_n\}$ of L is called *canonical*. A line L is called *large* when $n \geq 4$ and *connected* when $n \geq 3$.

(2.1) Suppose that L is a large line of a 3-connected matroid M. Then
(i) $D(M) \subseteq L$ and $L \setminus \{\alpha\}$ is a circuit of M for every $\alpha \in C(M) \cap L$.
(ii) L spans $C(M)$ in M.
(iii) If L contains more than four circuits of M, then $L \cap C(M) = \emptyset$. □

(2.2) If C_1 and C_2 are circuits of a 3-connected matroid M such that $|C_1 \triangle C_2| = 2$, then $C_1 \cup C_2$ is a large line of M. □

Suppose that M is a 3-connected non-binary matroid non-isomorphic to $U_{2,4}$. For $a \neq b$, we denote by L_{ab} the following set

$$\{L \cap N(M) : L \text{ is a spanning large line of } M \text{ and } L \cap C(M) = \{a, b\}\}.$$

(2.3) Suppose that M is a 3-connected non-binary matroid non-isomorphic to $U_{2,4}$.
(i) If $b \neq c$ and $A \in L_{ab} \cap L_{ac}$, then $A \in L_{bc}$.
(ii) If $A \in L_{ab}$, $B \in L_{ac}$ and $A \triangle B = \{e, f\}$, then $\{a\} \cup A \cup B \cup D(M)$ is a large line of M, which contains the circuit $\{a, e, f\}$ of M. □

(2.4) Suppose that L is a large line of a 3-connected non-binary matroid M non-isomorphic to $U_{2,4}$. If $|L \cap C(M)| > 2$ and $|C(M)| > 3$, then $D(M) = \emptyset$ and $M \mid C(M) \simeq U_{2,|C(M)|}$. □

(2.5) Suppose that M is a 3-connected non-binary matroid non-isomorphic to $U_{2,4}$, and that every large line L of M spans $E(M)$.
(i) If $A, B \in L_{ab}$ then $|A \triangle B| \neq 2$ or $M \simeq U_{2,|E(M)|}$.
(ii) If L is a large line of M, then $|L \cap C(M)| \leq 2$ or $M \simeq U_{2,|E(M)|}$. □

Let N be a matroid having a circuit-hyperplane C. We denote by N_C the matroid obtained from N by relaxing the circuit-hyperplane C.

(2.6) Suppose that M is a 3-connected matroid non-isomorphic to $U_{2,4}$ such that every large line spans M.
(i) If $r(M) = |D(M)| + |N(M)|$, then $N(M) = \emptyset$.
(ii) If $r(M) = |D(M)|$, then $C(M) = \emptyset$ or $N(M) = \emptyset$ and there is a binary matroid N such that $M = N_{D(M)}$.
(iii) If $r(M) = |D(M)| + 1$, then $|C(M)| \leq 1$.
(iv) If $|D(M)| < r(M) = |D(M)| + |N(M)| - 1$, then $|C(M)| \leq 1$. □

We will use the main theorem of Seymour[4]:

(2.7) Suppose that M is a 3-connected non-binary matroid. If $a, b \in E(M)$, then there is a large line L of M such that a and b belong to different sets of the canonical partition of L. □

Now we will give the definition of 2-sum of matroids. Let M_1 and M_2 be matroids such that $E(M_1) \cap E(M_2) = \{e\}$, e is not a loop or coloop of M_1 or M_2 and $|E(M_1)|, |E(M_2)| \geq 3$. The *2-sum* of M_1 and M_2 is the matroid $M_1 \Delta M_2$ on $(E(M_1) \cup E(M_2)) \setminus \{e\}$, whose circuits are all subsets S of $(E(M_1) \cup E(M_2)) \setminus \{e\}$ such that S is a circuit of M_i for some i, or $(S \cap E(M_i)) \cup \{e\}$ is a circuit of M_i for $i = 1, 2$. Seymour in [9] proved that:

(2.8) If $\{X_1, X_2\}$ is an exact 2-separation of a matroid M then there are matroids M_1 and M_2 on $X_1 \cup \{e\}$ and $X_2 \cup \{e\}$, respectively (where e is a new element), such that M is the 2-sum of M_1 and M_2. Conversely, if M is the 2-sum of M_1 and M_2 then $\{E(M_1) \setminus E(M_2), E(M_2) \setminus E(M_1)\}$ is an exact 2-separation of M, and M_1, M_2 are isomorphic to minors of M. □

(2.9) Suppose that $M = N_C$ is a 3-connected matroid. If N is not a 3-connected matroid, then $N = N_1 \Delta N_2$, where $E(N_1) = C \cup \{e\}$ and $N_2 \simeq U_{2,|E(N_2)|}$. □

3 Preliminary results

Throughout this section, we shall suppose that M is a 3-connected non-binary matroid such that $|N(M)| = 4$.

(3.1) If L is a non-spanning large line of M, then $r(L) = r(M) - 1$. Moreover, $\overline{L} = C(M) \cup D(M)$ or $\overline{L} = C(M) \cup D(M) \cup \{\alpha\}$, for some $\alpha \in N(M)$.

Proof: If $r(L) < r(M) - 1$, then $r(M^* \mid N(M)) \leq 2$. Since M is 3-connected, it follows that $M^* \mid N(M) \simeq U_{2,4}$, a contradiction since $N(M)$

spans $D(M) \cup C(M)$ by **(2.1)**. □

We say that a matroid M is a *circuit*, when $E(M)$ is its only circuit.

(3.2) Suppose that $M' = M_1 \triangle M_2$ is a connected matroid without parallel elements, $e \in E(M_1) \cap E(M_2)$ and the series class of e in M_2 is trivial. If $|N(M')| \leq 3$, M_1 is not a circuit and M_2 is non-binary, then $|E(M_1) \cap N(M')| \geq 2$ and

(i) there is an $f \in E(M_1)$ such that $\{e, f\}$ is a circuit of M_1, and $E(M_1) \backslash \{e\}$ is a circuit of M' with 3 or 4 elements; or

(ii) $E(M_1) \setminus S$ is a circuit of M with 3 elements, where S is the non-trivial series class of e in M_1.

Proof: Observe that M_1 is a binary matroid, otherwise $|N(M')| > 3$.

(i) Suppose that $a \in E(M_1) \backslash \{e, f\}$. By hypothesis, a and f are not parallel elements in M_1 and hence M_2 is a minor of both $M' \backslash a$ and M'/a. Hence, $a \in N(M')$ and $E(M_1) \setminus \{e, f\} \subseteq N(M')$. So, $|E(M_1)| \leq 5$ and the result follows since M_1 is binary and M' does not have parallel elements.

(ii) Suppose that $a \in E(M_1) \setminus S$. Observe that M_2 is a minor of both $M' \backslash a$ and M'/a. So, $E(M_1) \setminus S \subseteq N(M')$. Since $E(M_1) \setminus S \subseteq N(M')$ is a union of connected components of $M_1 \backslash e$, it follows that $E(M_1) \setminus S$ is a circuit with 3 elements of M'. If $S = \{e\}$, then there is an $f \in E(M_1)$ such that $\{e, f\}$ is a circuit of M_1 and the result follows as well. □

Let M'_e be a matroid obtained from M/e after the deletion of all but one element from every non-trivial parallel class of M/e, and let M''_e be a matroid obtained from $M \backslash e$ after the contraction of all but one element from every non-trivial series class of $M \backslash e$.

(3.3) Suppose that $\alpha \in N(M)$. If M''_α is not a 3-connected matroid, then there are $\beta, \gamma \in N(M)$ such that $\{\alpha, \beta, \gamma\}$ is a cocircuit of M and M has a non-spanning large line L such that $E(M) \setminus \overline{L} = \{\alpha, \beta, \gamma\}$. Moreover, there is $\delta \in E(M)$ such that $\{\beta, \gamma, \delta\}$ is a circuit of M, and

(i) $M''_\alpha \backslash \delta$ is a 3-connected non-binary matroid; or

(ii) $\delta \in N(M)$ and the series class of β in $M''_\alpha \backslash \delta$ is non-trivial and $M''_\alpha \backslash \delta / \beta$ is 3-connected non-binary matroid.

Proof: Suppose that M''_α is not a 3-connected matroid. Hence, there exist matroids M_1 and M_2 such that $M \backslash \alpha = M_1 \triangle M_2$, M_1 and M_2 are not circuits. Since $M \backslash \alpha$ is a non-binary matroid, we may suppose that M_2 is a non-binary matroid and the series class of $e \in E(M_1) \cap E(M_2)$ in M_2 is trivial. By **(3.2)**, this decomposition is unique, since $|E(M_1) \cap N(M)| \geq 2$, and we have two cases:

Case 1: there is a $\delta \in E(M_1)$ such that $\{e, \delta\}$ is a circuit of M_1 and

$E(M_1) \setminus \{e\}$ is a circuit of M with 3 or 4 elements.

If $\beta, \gamma \in E(M_1) \setminus \{e, \delta\}$, then β, γ are in the same series class of $M \backslash \alpha$. Hence $\{\alpha, \beta, \gamma\}$ is a cocircuit of M, $\{\beta, \gamma, \delta\}$ is a circuit of M and $|E(M_1)| = 4$. If L is a large line of M_2 which contains e, then $L \triangle \{e, \delta\}$ is a non-spanning large line of M.

Case 2: $E(M_1) \setminus S$ is a circuit of M with 3 elements, where S is the non-trivial series class of e in M_1.

Observe that there are elements $\beta, \gamma \in E(M_1) \setminus S$ belonging to the same series class of M_1. Hence $\{\alpha, \beta, \gamma\}$ is a cocircuit of M. If $\delta \in E(M_1) \setminus S \cup \{\beta, \gamma\}$, then δ and $\{\beta, \gamma\} \cap E(M''_\alpha)$, say β, are parallel elements of M''_α, and $E(M''_\alpha) \cap S$ and β are in the same series class of $M''_\alpha \backslash \delta$. If L is a large line of M_2 which contains e, then $(L \cup S \cup \{\delta\}) \setminus \{e\}$ is a non-spanning large line of M. □

As an imediate consequence of this lemma, we have the following result:

(3.4) (i) If every large line of M is spanning, then M''_α is a 3-connected matroid for every $\alpha \in N(M)$.

(ii) If M''_α is not 3-connected for some $\alpha \in N(M)$, then M has a non-spanning large line L such that $E(M) \setminus \overline{L} \subseteq N(M)$ is a triad of M, and that M''_δ is a 3-connected matroid for $\delta \in N(M) \cap \overline{L}$. □

4 A non-spanning large line

During this section, we shall suppose that M is a 3-connected non-binary matroid such that $|N(M)| = 4$. Except in **(4.4)** and **(4.5)**, we also suppose that M has a non-spanning large line.

(4.1) Suppose that $C(M) \cup D(M)$ is a large line of M and $|C(M)| = 3$. If $\alpha \in N(M)$, then $M \mid (C(M) \cup N(M) \setminus \{\alpha\}) \simeq M(K_4)$.

Proof: If $\beta \in N(M) \cap \overline{C(M) \cup D(M)}$, then $M \mid C(M) \cup D(M) \cup \{\beta\}$ contains a large line L of M such that $\beta \in L$. Hence $L = D(M) \cup \{a, b, \beta\}$ for some $a, b \in C(M)$. So, $\{a, b, \beta\}$ is a circuit of M and $\{a, b, \beta\} \cup C(M)$ is a large line, by **(2.2)**, which does not contain $D(M)$, a contradiction. Hence $C(M) \cup D(M) = \overline{C(M) \cup D(M)}$ and $N(M)$ is a cocircuit of M. By **(3.4)**, M''_α is a 3-connected matroid.

Choose $\beta \in N(M) \setminus \{\alpha\}$. Let S be the series class of β in $M \backslash \alpha$. Observe that $S \cap N(M) = \{\beta\}$, since $N(M)$ is a cocircuit of M. As M''_α is a 3-connected matroid, it follows that $M \backslash \{\alpha\} \cup S$ is connected. So, there is a large line L of $M \backslash \{\alpha\} \cup S$ such that $N(M) \setminus \{\alpha, \beta\} = \{\gamma, \delta\} \subseteq L$. Hence $L = D(M) \cup \{a, b, \gamma, \delta\}$ for some $a, b \in C(M)$. The canonical partition of L is $\{\{a\}, \{b\}, \{\gamma, \delta\}, D(M)\}$ because $D(M) \cup \{a, b\}$ is a ciruit of M, by **(2.1)**.

Observe that $\{a, b, \gamma, \delta\} \cup C(M)$ is a line of M which contains 3 circuits, since $D(M) \neq \emptyset$. So, $\{\gamma, \delta, c_{\gamma\delta}\}$ is a circuit of M for some $c_{\gamma\delta} \in C(M)$. Similarly, there are $c_{\beta\delta}, c_{\beta\gamma} \in C(M)$ such that $\{\beta, \delta, c_{\beta\delta}\}$ and $\{\beta, \gamma, c_{\beta\gamma}\}$ are circuits of M.

If $c_{\beta\delta} = c_{\beta\gamma}$, then $M \mid \{\beta, \gamma, \delta, c_{\beta\delta}\} \simeq U_{2,4}$, a contradiction. Hence $C(M) = \{c_{\gamma\delta}, c_{\beta\delta}, c_{\beta\gamma}\}$ and the result follows. □

(4.2) If $C(M) \cup N(M)$ is a large line of M and $|C(M)| = 3$, then $M \mid C(M) \cup N(M) \simeq F_7$.

Proof: If $\alpha \in N(M)$, then, by **(4.1)**,

$$M \mid (C(M) \cup N(M) \setminus \{\alpha\}) \simeq M(K_4),$$

and its matrix representation over $GF(2)$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

As $M \mid C(M) \cup N(M)$ is a binary matroid and α is not in parallel with any element of $C(M) \cup N(M) \setminus \{\alpha\}$, it follows that $M \mid C(M) \cup N(M)$ is isomorphic to F_7. □

(4.3) If $C(M) \cup N(M)$ is a large line of M and $|C(M)| = 3$, then $M \simeq M_8$. Moreover, every large line of M^* is spanning.

By **(4.1)** and **(4.2)**, this happens when $N(M)$ contains different elements α and β such that both M''_α and M''_β are 3-connected.

Proof: By **(4.2)**, $M \mid C(M) \cup N(M) \simeq F_7$. Observe that $r(D(M)) = |D(M)|$, $r(M) = |D(M)| + 2$ and $r(C(M) \cup N(M)) = 3$. Hence

$$\xi(M; D(M)) = r(D(M)) + r(C(M) \cup N(M)) - r(M) + 1 = 2$$

and $\{D(M), C(M) \cup N(M)\}$ is a 2-separation of M, unless $|D(M)| = 1$. As M is 3-connected, it follows that $D(M) = \{d\}$.

Claim 1: $A \cup \{d\}$ is not a circuit of M, for every $A \subseteq N(M)$ such that $|A| = 2$.

Suppose that $A \cup \{d\}$ is a circuit of M. As $M \mid C(M) \cup N(M) \simeq F_7$, there is a unique element of $C(M)$, which will be denoted by c_A, such that $A \cup \{c_A\}$ is a circuit of M. So, $M \mid (A \cup \{c_A, d\}) \simeq U_{2,4}$ and $A \cup \{c_A, d\}$ is a spanning set of M, a contradiction.

Claim 2: If $A \subseteq N(M)$ has 2 elements, then $A \cup \{d\} \cup C(M) \setminus \{c_A\}$ is a large line of M whose canonical partition is $\{\{a\}, C(M) \setminus \{a, c_A\}, \{d\}, A\}$, for $a \in C(M)$.

Observe that $M \mid C(M) \cup D(M) \cup A$ is a conected matroid, since $M \mid C(M) \cup A$ is connected and d is not a coloop in $M \mid C(M) \cup D(M) \cup A$, otherwise $(N(M) \setminus A) \cup \{d\}$ is a cocircuit of M having just one element in common with the line $C(M) \cup D(M)$. So, there is a circuit in $M \mid C(M) \cup D(M) \cup A$ which contains $A \cup \{d\}$. Hence $A \cup \{a, d\}$ is a circuit of M for some $a \in C(M)$. Observe that $(A \cup \{a, d\}) \cup (\{d\} \cup C(M) \setminus \{c_A\})$ is a large line of M. Its canonical partition is $\{\{a\}, C(M) \setminus \{a, c_A\}, \{d\}, A\}$.

Claim 3: If $\alpha \in N(M)$, then $(N(M) \setminus \{\alpha\}) \cup \{d\}$ is a circuit of M.

Suppose that A and B are subsets of $N(M)$ having 2 elements such that $A \cup B = N(M) \setminus \{\alpha\}$. So, there is $a \in C(M) \setminus \{c_A, c_B\}$ and $(A \cup \{a, d\}) \cup (B \cup \{a, d\})$ is a large line of M. Hence $A \cup B \cup \{d\}$ is a circuit of M.

So, we have just found all circuits of M, and we say that a matroid having those circuits is isomorphic to M_8.

A non-spanning large line of M^* must be equal to $C(M) \cup D(M)$, since $N(M)$ is a circuit of M. This cannot happen because a large line in M^* has at least $r(M^*) + 1 = 6$ elements, by **(3.1)**. □

(4.4) If $|D(M)| = 1$ and $|C(M)| = 2$, then every large of M is spanning.

Proof: Suppose that M has a non-spanning large line L. $L = D(M) \cup C(M) \cup \{\alpha\}$ for some $\alpha \in N(M)$, since $|L \cap N(M)| \leq 1$. Observe that $M \mid L \simeq U_{2,4}$ and $C(M) \cup \{\alpha\}$ is a circuit of M. For each subset of A of $N(M) \setminus \{\alpha\}$ with 2 elements, there is a circuit C of $M \backslash D(M)$ such that $A \subseteq C$ since $M \backslash D(M)$ is connected. As $N(M) \setminus \{\alpha\}$ is a cocircuit of M, it follows that $C \neq N(M) \setminus \{\alpha\}$ since M is 3-connected, and hence $C \cap (C(M) \cup \{\alpha\}) \neq \emptyset$. Observe that $N(M) \setminus \{\alpha\} \not\subseteq C$, otherwise $M^* \mid (D(M) \cup N(M) \setminus \{\alpha\}) \simeq U_{2,4}$, since $N(M) \setminus \{\alpha\}$ is a cocircuit of M. So, we may suppose that $|C \cap (C(M) \cup \{\alpha\})| = 1$, otherwise we can replace C by $C \triangle (C(M) \cup \{\alpha\})$, which is a circuit of $M \backslash D(M)$ since $M \backslash D(M)$ is binary. This circuit is denoted by C_A.

Observe that C_A is unique and $|C_A \cap C_B| = 1$, when $A \neq B$, otherwise $M \backslash D(M)$ has a circuit with cardinality 2. So, $M \backslash D(M) \simeq M(K_4)$. We arrive at a contradiction, since $D(M) \cup C(M)$ and A are parallel class of M/α, where $\alpha \in C_A$, and M'_α is binary. Hence, every large line of M is spanning. □

(4.5) If $|C(M)| = 3$ and $\alpha \in N(M)$, then $D(M) \cup C(M) \cup \{\alpha\}$ cannot be a non-spanning large line of M

Proof: Suppose that $D(M) \cup C(M) \cup \{\alpha\}$ is a non-spanning large line of M. Then $N(M) \setminus \{\alpha\} = \{\beta, \gamma, \delta\}$ is a cocircuit of M. By **(3.4)**, M''_α is 3-connected. So, there is a large line L of $M \backslash \alpha$ such that $\beta, \gamma \in L$.

If $\delta \in L$, then $L = D(M) \cup \{\beta, \gamma, \delta, c_1, c_2\}$ for $c_1, c_2 \in C(M)$. Observe that $\{c_1\}$ and $\{c_2\}$ belong to the canonical partition of L. As $\{\beta, \gamma, \delta\}$ is a cocircuit of M, it follows that $\{\beta, \gamma, \delta\} \subseteq A$, where A belongs to the canonical partition of L, since L contains only four circuits by **(2.1)**. Hence $L \backslash A = (D(M) \backslash A) \cup \{c_1, c_2\}$ is a circuit of M and $D(M) \cup (C(M) \backslash \{c_3\}) \cup \{\alpha\}$ is also a circuit of M for $c_3 \in C(M) \setminus \{c_1, c_2\}$, a contradiction. So, $\delta \notin L$ and $L = D(M) \cup C(M) \cup \{\beta, \gamma\}$.

Similarly $D(M) \cup C(M) \cup \{\beta, \delta\}$ and $D(M) \cup C(M) \cup \{\gamma, \delta\}$ are large lines of M. We arrive at a contradiction, since $(D(M) \cup \{\beta, \delta\, c_1, c_2\}) \cup (D(M) \cup \{\beta, \gamma\, c_1, c_2\})$ is also a large line of M, by **(2.2)**. □

5 All large lines span M

During this section, we shall suppose that M is a 3-connected non-binary matroid such that every large line of M is a spanning set of M and $|N(M)| = 4$.

5.1 Case 1: $C(M) \neq \emptyset$

By **(2.6)**,

$$|D(M)| < r(M) < |D(M)| + |N(M)| = |D(M)| + 4.$$

By **(2.6)**, $|C(M)| = 1$, when $r(M) \neq |D(M)| + 2$.

(5.1) $r(M) \neq |D(M)| + 3$.

Proof: If $r(M) = |D(M)| + 3$, then $|C(M)| = 1$. So,

$$r(M^*) = |E(M)| - r(M) = |N(M)| + |D(M)| + |C(M)| - (|D(M)| + 3) = 2,$$

a contradiction. □

(5.2) $r(M) \neq |D(M)| + 1$.

Proof: If $r(M) = |D(M)| + 1$, then $|C(M)| = 1$, say $C(M) = \{c\}$. Observe that M^* has a non-spanning large line L^* by **(5.1)**. Since $|L^*| = r(M^*) + 1 = 5$, $C(M) \subset L^*$ and $|L^* \cap N(M)| \leq 1$, it follows that $|L^* \cap D(M)| = 3$ and hence $|D(M)| = 3$ by **(2.4)**. So, $L^* = C(M) \cup D(M) \cup \{\alpha\}$, for some $\alpha \in N(M)$, and we arrive at a contradiction by **(4.5)**. □

So, we have proved that:

(5.3) $r(M) = |D(M)| + 2$. □

(5.4) $L_{ab} = \{A, N(M) \setminus A\}$, for some $A \subset N(M)$ such that $|A| = 2$.

Proof: By **(3.4)**, M''_α is a 3-connected matroid for every $\alpha \in N(M)$. By **(2.7)**, there is a large line L_α of M such that $a, b \in L_\alpha$ and $\alpha \notin L_\alpha$. $|L_\alpha \cap C(M)| = 2$, otherwise $M \simeq U_{2,|E(M)|}$ by **(2.5)**. Hence, there is an $A_\alpha \in L_{ab}$ such that $\alpha \notin A_\alpha$. $A_\alpha = A_\beta$ or $A_\alpha \cap A_\beta = \emptyset$, otherwise $M \simeq U_{2,|E(M)|}$ by **(2.5)**. So, $L_{ab} = \{A, N(M) \setminus A\}$ for some $A \subseteq N(M)$. □

(5.5) Suppose that $T \subset N(M)$ and $|C(M)| \geq 2$. If $|T| = 3$, then T is not a circuit of M. Moreover, if L^* is a non-spanning large line of M^*, then $L^* = C(M) \cup D(M)$ and $|D(M)| = 3$.

Proof: Suppose that $T = \{\alpha, \beta, \gamma\}$ is a circuit of M and $L_{ab} = \{\{\alpha, \beta\}, \{\gamma, \delta\}\}$ for some $a, b \in C(M)$. Observe that $L = (D(M) \cup \{\alpha, \beta, a\}) \cup \{\alpha, \beta, \gamma\}$ is a large line of M, otherwise $D(M) \cup \{\gamma, a\}$ is a circuit of M. Since T is a circuit of L, the canonical partition of L is $\{\{\alpha\}, \{\beta\}, \{\gamma\}, D(M) \cup \{a\}\}$. By **(2.1)**, $D(M) = \emptyset$ and $r(M) = r(L) = 2$, a contradiction. □

(5.6) Suppose that $|C(M)| \geq 2$. If $a, b_1, b_2, b_3 \in C(M)$, then there are i and j $(1 \leq i < j \leq 3)$ such that $L_{ab_i} = L_{ab_j}$.

Proof: Suppose that $|A_i \cap A_j| = 1$, for every $1 \leq i < j \leq 3$, where $\alpha \in A_k \in L_{ab_k}$, for some fixed $\alpha \in N(M)$. If $A_k = \{\alpha, \alpha_k\}$, then by **(2.3)**, $\{a, \alpha_1, \alpha_2\}$ and $\{a, \alpha_1, \alpha_3\}$ are circuits of M. So $L = \{a, \alpha_1, \alpha_2, \alpha_3\}$ is a large line of M, and hence $r(M) = r(L) = 2$, a contradiction. □

(5.7) If $|C(M)| \geq 4$, then $L_{ab} = L_{cd}$ for every $a, b, c, d \in C(M)$.

Proof: Suppose that $L_{ab} \neq L_{bc}$. By **(2.3)**, $L_{ac} \neq L_{bc}$ and $L_{ab} \neq L_{ac}$. Without loss of generality, we can suppose that $L_{ac} = L_{ad}$, by **(5.6)**. Observe that $L_{ac} = L_{ad} = L_{dc}$, by **(2.3)**. We arrive at a contradiction, since L_{bd} cannot be equal to L_{ab} or L_{bc}. □

(5.8) Suppose that $C(M) = \{a, b, c\}, N(M) = \{\alpha, \beta, \gamma, \delta\}$ and that $\{\alpha, \beta\} \in L_{ab}, \{\alpha, \gamma\} \in L_{ac}, \{\alpha, \delta\} \in L_{bc}$. $M \mid C(M) \cup N(M)$ is is isomorphic to F_7 and its cycle space is generated by $N(M)$, $\{a, \alpha, \delta\}$, $\{b, \alpha, \gamma\}$ and $\{c, \alpha, \beta\}$.

Proof: Observe that $D(M) \neq \emptyset$, otherwise M has a large line with 4 elements. So, $M \mid C(M) \cup N(M)$ is binary. By **(2.3)**, it follows that $\{a, \alpha, \delta\}$, $\{b, \alpha, \gamma\}$ and $\{c, \alpha, \beta\}$ are circuits of M. Since $\{a, \beta, \gamma\}$ is also a circuit of $M \mid C(M) \cup N(M)$, $N(M) = \{a, \alpha, \delta\} \triangle \{a, \beta, \gamma\}$ is a circuit of M. □

(5.9) Suppose that $L_{ab} = L_{cd}$ for every $a, b, c, d \in C(M)$. For each $\alpha \in N(M)$, there is an $a \in C(M)$ such that $\{a\} \cup (N(M) \setminus \{\alpha\}) \cup D(M)$ is a large line of M.

Proof: Choose $\beta, \gamma \in N(M) \setminus \{\alpha\}$ belonging to different elements of L_{ab}. Since M''_α is a 3-connected matroid by **(3.4)**, there is a large line L of $M \backslash \alpha$ which contains β and γ by **(2.7)**. Observe that $|L \cap C(M)| = 1$, otherwise $L \cap C(M) = \{a, b\}$ for some $a, b \in C(M)$ and $\{\beta, \gamma\} \in L_{ab}$. So, $L = \{a\} \cup (N(M) \setminus \{\alpha\}) \cup D(M)$ for some $a \in C(M)$. □

(5.10) Suppose that $L_{ab} = L_{cd}$ for every $a, b, c, d \in C(M)$. If $H = D(M) \cup A$ or $H = D(M) \cup (N(M) \setminus A)$, where $A \in L_{ab}$, then $H \cup \{a\}$ is a circuit of M for every $a \notin H$.

Proof: The result follows by the definition of L_{ab}, **(5.9)**, **(2.7)** and **(2.1)**. □

Suppose that N is a 3-connected binary matroid having a pair of circuit-hyperplanes C_1 and C_2 such that $|C_1 \triangle C_2| > 2$. Let N_{C_1,C_2} be the matroid obtained from N by relaxing both C_1 and C_2. When we relax both circuit-hyperplanes, we get a matroid because C_2 is a circuit-hyperplane of N_{C_1}. We say that N_{C_1,C_2} is obtained from N by *relaxing a pair of circuit-hyperplanes of order* $|C_1 \setminus C_2|$. This construction works for several circuit-hyperplanes and a result similar to the next lemma holds in this case.

(5.11) N_{C_1,C_2} is a 3-connected matroid such that

(i) $N(N_{C_1,C_2}) = C_1 \triangle C_2$;

(ii) $D(N_{C_1,C_2}) = C_1 \cap C_2$;

(iii) $C(N_{C_1,C_2}) = E(N) \setminus (C_1 \cup C_2)$.

Proof: Observe that $N_{C_1,C_2} \setminus \{e\}$ is binary when $e \in C_1 \cap C_2$, since every large line of N_{C_1,C_2} contains C_1 or C_2. Moreover, $N_{C_1,C_2} \setminus \{e\}$ is not binary when $e \notin C_1 \cap C_2$. The result follows since $(N_{C_1,C_2})^* = N^*_{E(M)\setminus C_1, E(M)\setminus C_2}$. □

When $|C_1 \triangle C_2| = 4$, $D(N_{C_1,C_2}) \neq \emptyset$, otherwise $|C_1| = |C_2| = 2$ and N is not a 3-connected matroid.

(5.12) $|C(M)| = 1$, or $|C(M)| = 3$ and $L_{ab} \neq L_{ac}$ when $b \neq c$, or $M = K_{C_1,C_2}$ for some 3-connected binary matroid K having a pair of circuit-hyperplanes C_1 and C_2 such that $|C_1 \triangle C_2| = 4$.

Proof: Suppose that $|C(M)| \geq 2$. If $L_{ab} \neq L_{ac}$ for some $a, b, c \in C(M)$, then $|C(M)| = 3$ by **(5.8)**, and the result follows. We may suppose that $L_{ab} = L_{cd}$ for every $a, b, c, d \in C(M)$.

By **(5.10)**, M is obtained from a matroid N by relaxing the circuit-hyperplane $D(M) \cup A$, where $A \in L_{ab}$. If N is not 3-connected, then by **(2.9)**, $M = N_1 \triangle N_2$, where N_1 is a matroid on $D(M) \cup A \cup \{e\}$, N_2 is on $(E(M) \setminus (D(M) \cup A)) \cup \{e\}$ and $N_2 \simeq U_{2,|E(N_2)|}$. Hence $|E(N_2)| \leq 4$, otherwise M contains a large line with four elements. So, $|N(M)| + |C(M)| - 2 \leq 3$ or $|C(M)| \leq 1$, a contradiction.

If N is 3-connected, then by **(5.10)** N is obtained from a matroid K by relaxing the circuit-hyperplane $D(M) \cup (N(M) \setminus A)$. If K is 3-connected and non-binary, there is a large line L of M such that $a, b \in L$ and $L \neq \{a, b\} \cup D(M) \cup A$ and $L \neq \{a, b\} \cup D(M) \cup (N(M) \setminus A)$, a contradiction. So, K is not 3-connected or K is binary. If K is not 3-connected, then $|C(M)| \leq 1$ as before, and we arrive at a contradiction. Hence K is 3-connected and binary, and $M = K_{D(M) \cup A, D(M) \cup (N(M) \setminus A)}$. □

Now we will prove the main result of this subsection:

(5.13) If M is a 3-connected matroid such that every large line of M is spanning and $|N(M)| = 4$, then:

(i) M is isomorphic to Q_6 or M_8^*; or

(ii) $M = N_{C_1, C_2}$ for some 3-connected binary matroid having a pair of circuit-hyperplanes C_1 and C_2 such that $|C_1 \triangle C_2| = 4$.

Proof: By **(5.12)**, we can suppose that $|C(M)| = 1$ or $|C(M)| = 3$ otherwise the result follows.

When every large line of M^* is spanning, then $|D(M)| = 1$ or $|D(M)| = 3$, otherwise the result follows: we remind the reader that

$$(N_{C_1, C_2})^* = N^*_{E(M) \setminus C_1, E(M) \setminus C_2}.$$

So, we have three cases:

Case 1: $|C(M)| = |D(M)| = 3$.

By **(5.8)**, if $A \subset N(M)$ and $|A| = 2$, then there is an $a \in C(M)$ such that $A \cup \{a\}$ is a circuit of M. Using this result for M^*, then for each $B \subset N(M)$ and $|B| = 2$, there is a $b \in D(M)$ such that $B \cup \{b\}$ is a cocircuit of M. We arrive at a contradicition, since we can choose B such that $|A \cap B| = 1$ and M has a circuit and a cocircuit having one element in common.

Case 2: $|C(M)| = 3$ and $|D(M)| = 1$.

If $A \in L_{ab}$, then $D(M) \cup A \cup \{a, b\}$ is a large line of M. So, $\{a, b, d\}$ or $\{a, b, \alpha\}$ is a circuit of M, where $D(M) = \{d\}$ and $\alpha \in A$. Since $\{a, b, c\}$ is a circuit of M by **(5.8)**, it follows that M has a large line L such that $|L| = 4$ and hence $r(M) = r(L) = 2$, a contradiction.

Case 3: $|C(M)| = |D(M)| = 1$.

Since $r(M) \neq 2$, it follows that $E(M) \setminus \{\gamma\}$ is a large line of M for every $\gamma \in N(M)$ and $(N(M) \setminus \{\gamma\}) \cup \{d\}$ is a circuit of M for $d \in D(M)$, by **(2.1)**. Hence $\{d\} \cup N(M)$ is a large line of M and $N(M)$ is a circuit of M. For $a \in C(M)$, $\{a\} \cup N(M)$ is a non-large line and $\{a, \gamma, \delta\}$ and $\{a, \alpha, \beta\}$ are circuits of M, where $N(M) = \{\alpha, \beta, \gamma, \delta\}$. Observe that the canonical partition of the large line $E(M) \setminus \{\alpha\}$ is $\{\{a\}, \{d, \beta\}, \{\gamma\}, \{\delta\}\}$. Hence $\{a, d, \beta, \gamma\}$ and $\{a, d, \beta, \delta\}$ are circuits of M. Similarly, $\{a, d, \alpha, \gamma\}$, $\{a, d, \alpha, \delta\}$, $\{a, d, \gamma, \alpha\}$, $\{a, d, \delta, \alpha\}$ and $\{a, d, \delta, \beta\}$ are circuits of M. Observe that this matroid is isomorphic to Q_6.

When M^* has a non-spanning large line, we have two cases:

Case 4: $|C(M)| = 3$.

By **(5.5)**, it follows that $C(M) \cup D(M)$ is a large line of M^*, and by **(4.3)** $M^* \simeq M_8$, a contradiction since $|E(M)| = 10$.

Case 5: $|C(M)| = 1$.

If $C(M) \cup D(M)$ is a large line of M^*, then $M^* \simeq M_8$ as before. So, we may suppose that $C(M) \cup A \cup \{\alpha\}$ is a large line of M^*, where $\alpha \in N(M)$ and $A \subseteq D(M)$, $|A| = 2$, otherwise $|D(M)| = 3$ and we arrive at a contradiction by **(4.5)**. As $A \cup C(M) \cup \{\alpha\}$ spans $D(M)$, it follows that $M^* \mid D(M) \cup C(M) \cup \{\alpha\}$ has rank 2. If $|D(M)| \geq 3$, then $D(M) \cup \{\alpha\}$ is a large line of M^*, a contradiction. So $|D(M)| = 2$. By **(4.4)**, every large line of M^* is spanning, a contradiction. □

5.2 Case 2: $C(M) = \emptyset$

Suppose that L^* is a large line of M^*. If L^* is a non-spanning set of M^*, then $D(M) \subseteq \overline{L^*}$ by **(2.1)** and $T = E(M) \setminus \overline{L^*} \subseteq N(M)$ is a circuit of M. As M is 3-connected, it follows that $|T| \geq 3$. Hence $|L^* \cap N(M)| \leq 1$ and $|L^* \cap D(M)| \geq 3$. By **(2.4)**, $|D(M)| = 3$ and $r(M) = 3$. Since M^* is 3-connected, $E(M) \setminus \{a\}$ is a line of M^* for every $a \in E(M)$, a contradiction.

So, we may suppose that every large line of M^* is spanning. Observe that $C(M^*) = D(M) = \emptyset$, since $C(M) = D(M^*) \neq \emptyset$ for every matroid M^* listed in **(5.13)**. Hence $|E(M)| = 4$, a contradiction. We have proved that:

(5.14) If M is a 3-connected matroid such that every large line of M is spanning and $|N(M)| = 4$, then $C(M) \neq \emptyset$. □

6 M and M^* have non-spanning large lines

Throughout this section, we shall suppose that M is a 3-connected matroid such that $|N(M)| = 4$, M and M^* have non-spanning large lines $L = D(M) \cup A$ and $L^* = C(M) \cup A^*$ respectively, where $D(M) \cap A = \emptyset$ and

$C(M) \cap A^* = \emptyset$. So,

$$|L| = r(M) + 1 \text{ and } |L^*| = r(M^*) + 1$$

and hence

$$|A| + |A^*| = 2 + |N(M)| = 6.$$

Observe that $|A|, |A^*| \leq 4$ since $|A \cap N(M)| \leq 1$ and $|A^* \cap N(M)| \leq 1$.

(6.1) $|A| = |A^*| = 3$.

Proof: Without loss of generality, we may suppose that $|A| = 4$. Observe that $|A \cap C(M)| \geq 3$ and by **(2.4)**, $C(M) \subseteq L$ and $|C(M)| = 3$. So, $L = D(M) \cup C(M) \cup \{\alpha\}$ for some $\alpha \in N(M)$. We arrive at a contradiction by **(4.5)**. □

(6.2) $A \cap N(M) \neq \emptyset$ and $A^* \cap N(M) \neq \emptyset$.

Proof: Without loss of generality, we may suppose that $A \cap N(M) = \emptyset$. So, $C(M) \cup D(M)$ is a large line of M. By **(4.3)** $M \simeq M_8$, which is impossible since every large line of M_8^* is spanning. □

So, there is a circuit T and a cocircut T^* of M with $|T| = |T^*| = 3$ and $T, T^* \subset N(M)$. We may suppose that

$$T \cap T^* = \{\gamma, \delta\}, T \setminus T^* = \{\alpha\} \text{ and } T^* \setminus T = \{\beta\}.$$

(6.3) The large lines of M are the following:

(i) $D(M) \cup \{\alpha, a, b\}$ for every $a, b \in C(M)$ (its canonical partition is $\{\{a\}, \{b\}, A, (D(M) \setminus A) \cup \{\alpha\}\}$, where $A \cup \{a, b\}$ is the circuit of $M \mid D(M) \cup \{a, b\}$);

(ii) $D(M) \cup \{\gamma, \delta, a, b\}$ for every $a, b \in C(M)$ (its canonical partition is $\{\{a\}, \{b\}, A, (D(M) \setminus A) \cup \{\gamma, \delta\}\}$, where $A \cup \{a, b\}$ is the circuit of $M \mid D(M) \cup \{a, b\}$);

(iii) $D(M) \cup N(M)$ (its canonical partition is $\{\{\alpha\}, \{\gamma\}, \{\delta\}, D(M) \cup \{\beta\}\}$);

(iv) $D(M) \cup \{\beta, \gamma, \delta, a\}$ for every $a \in C(M)$ (its canonical partition is $\{\{a\}, \{\beta\}, A, (D(M) \setminus A) \cup \{\gamma, \delta\}\}$, where $A \cup \{\beta, \gamma, a\}$ is the circuit of $M \mid D(M) \cup \{\beta, \delta, a\}$);

(v) $D(M) \cup \{\alpha, \beta, \delta, a\}$ for every $a \in C(M)$, when $\{\beta, \delta, a\}$ is not a circuit of M (its canonical partition is $\{\{a\}, \{\beta, \delta\}, A, (D(M) \setminus A) \cup \{\alpha\}\}$, where $A \cup \{\beta, \delta, a\}$ is the circuit of $M \mid D(M) \cup \{\beta, \delta, a\}$);

(vi) $D(M) \cup \{\alpha, \beta, \gamma, a\}$ for every $a \in C(M)$, when $\{\beta, \gamma, a\}$ is not a circuit of M (its canonical partition is $\{\{a\}, \{\beta, \gamma\}, A, (D(M) \setminus A) \cup \{\alpha\}\}$, where $A \cup \{\beta, \gamma, a\}$ is the circuit of $M \mid D(M) \cup \{\beta, \gamma, a\}$).

Proof: Let N be the matroid $(M \backslash \beta) \backslash \{\gamma, \delta\}$. Observe that γ and δ are in series in $M \backslash \beta$, and if we contracted γ in $M \backslash \beta$, then δ became parallel with α. Since $M \backslash \beta$ is non-binary and connected, it follows that N is also non-binary and connected.

If $c \in C(M)$, then there is a large line L' of N such that $c \in L'$. If $\alpha \notin L'$, then $|C(M)| = 3$ and $L = C(M) \cup D(M)$. Let $C(M) = \{a, b, c\}$. So, the canonical partition of L' is $\{\{a\}, \{b\}, \{c\}, D(M)\}$. As $L = D(M) \cup A$ and $\alpha \in A$, it follows that $|A \cap C(M)| = 2$, say $a, b \in A$. Hence the canonical partition of L is $\{\{a\}, \{b\}, \{\alpha\}, D(M)\}$, since $L' \setminus \{c\} = D(M) \cup \{a, b\}$ is a circuit of M contained in L. So, $C(M)$ and $\{a, b, \alpha\}$ are circuits of M and $C(M) \cup \{\alpha\}$ is a large line of M by **(2.2)**, a contradiction because $D(M) \neq \emptyset$ and $D(M)$ is contained in every large line of M.

Hence $\alpha \in L'$ and $D(M) \cup \{\alpha, c\}$ is a circuit of M for every $c \in C(M)$. Hence $D(M) \cup \{\alpha, a, b\}$ is a large line of M for every $a, b \in C(M)$. As $\{\alpha, \delta, \gamma\}$ is a circuit and $\{\gamma, \delta\}$ a cocircuit of $M \backslash \beta$, it follows that $D(M) \cup \{\gamma, \delta, a, b\}$ is a large line of M for every $a, b \in C(M)$.

If $D(M) \cup \{\beta, \delta, a, b\}$ is a large line of M, for $a, b \in C(M)$, then, by **(2.2)**, $(D(M) \cup \{\beta, \delta, a\}) \cup (D(M) \cup \{\gamma, \delta, a\})$ is a large line of M with partition $\{\{\beta\}, \{\gamma\}, \{a\}, D(M) \cup \{\delta\}\}$ and $\{\beta, \gamma, a\}$ is a circuit of M. Similarly $\{\beta, \gamma, b\}$ is a circuit of M. By **(2.2)**, $\{\beta, \gamma, a\} \cup \{\beta, \gamma, b\}$ is a large line of M and $D(M) = \emptyset$, a contradiction. So, $D(M) \cup \{\beta, \delta, a, b\}$ is not a large line of M. Similarly $D(M) \cup \{\beta, \gamma, a, b\}$ is not a large line of M.

By **(3.4)**, M'_β is a 3-connected matroid. So, there are large lines $L'_{a,\delta}$ and $L'_{a,\gamma}$ of M'_β such that $\{a, \delta\} \subseteq L'_{a,\delta}$ and $\{a, \gamma\} \subseteq L'_{a,\gamma}$ for $a \in C(M)$. Hence there are large lines $L_{a,\delta}$ and $L_{a,\gamma}$ of M such that $\{a, \delta, \beta\} \subseteq L_{a,\delta}$ and $\{a, \gamma, \beta\} \subseteq L_{a,\gamma}$ (β belongs to these lines because they have to be spanning lines of M).

Case 1: $L_{a,\delta} = L_{a,\gamma}$ for some $a \in C(M)$.

Hence $L_{a,\delta} = D(M) \cup \{a, \beta, \gamma, \delta\}$ and $D(M) \cup \{\beta, \gamma, \delta\}$ is a circuit of M. As $D(M) \cup \{b, \gamma, \delta\}$ is a circuit of M for every $b \in C(M)$, it follows that $(D(M) \cup \{b, \gamma, \delta\}) \cup (D(M) \cup \{\beta, \gamma, \delta\})$ is a large line for every $b \in C(M)$.

If $(D(M) \cup \{\beta, \gamma, \delta\}) \cup (\{\alpha, \gamma, \delta\})$ is not a large line, then $D(M) \cup \{\beta, \alpha\}$ is a circuit of M, a contradiction since $D(M) \cup \{\beta, \alpha\} \cap T^* = \{\beta\}$. So, $D(M) \cup \{\alpha, \beta, \gamma, \delta\}$ is a large line of M. $D(M) \cup \{\alpha, \beta, \gamma, a\} = (D(M) \cup \{\alpha, \beta, \gamma\}) \cup (D(M) \cup \{\alpha, a\})$ is a large line of M, for $a \in C(M)$, unless $\{\beta, \gamma, a\}$ is a circuit of M. Similarly, $D(M) \cup \{\alpha, \beta, \delta, a\}$ is a large line of M, for $a \in C(M)$, unless $\{\beta, \delta, a\}$ is a circuit of M.

Case 2: $L_{a,\delta} \neq L_{a,\gamma}$ for every $a \in C(M)$.

So, $L_{a,\delta} = D(M) \cup \{a, \alpha, \beta, \delta\}$ and $L_{a,\gamma} = D(M) \cup \{a, \alpha, \beta, \gamma\}$. Observe that $(D(M) \cup \{\alpha, \beta, \delta\}) \cup (D(M) \cup \{\alpha, \beta, \gamma\})$ is also a large line of M by **(2.2)**. Note that $(D(M) \cup \{\beta, \gamma, \delta\}) \cup (D(M) \cup \{\gamma, \delta, a\})$ is a large line of M for every $a \in C(M)$.

Those are all the large lines of M, since a large line L of M contains $D(M)$. □

6.1 Reduction to a binary matroid

Observe that every large line of M contains $D(M) \cup \{\alpha\}$ or $D(M) \cup \{\gamma, \delta\}$. We shall prove that the family $\mathcal{F}$ whose sets are

(i) $D(M) \cup \{\alpha\}$, $D(M) \cup \{\gamma, \delta\}$; and

(ii) all the circuits of M which do not contain $D(M) \cup \{\alpha\}$ or $D(M) \cup \{\gamma, \delta\}$;

is the family of circuits of a binary matroid N.

Suppose that $\mathcal{G}$ is a family of subsets of a finite set. We say that L is a *line* of $\mathcal{G}$, when L is a minimal set which does not belong to $\mathcal{G}$ and is the union of sets belonging to $\mathcal{G}$. The set

$$\mathcal{S}(L) = \{L \setminus C : C \subseteq L \text{ and } C \in \mathcal{G}\}$$

is called the *associated set* of L. When $\mathcal{G}$ is the family of circuits of a matroid M, the lines of $\mathcal{G}$ are precisely the lines of M and its associated set is its canonical partition in M.

(6.4) $\mathcal{S}(L)$ is a partition of L having cardinality 2 or 3, for every line L of $\mathcal{F}$.

Proof: The result is true when L does not contain $D(M) \cup \{\alpha\}$ and $D(M) \cup \{\gamma, \delta\}$. We remind the reader that every large line of M contains $D(M) \cup \{\alpha\}$ or $D(M) \cup \{\gamma, \delta\}$.

For the other lines, we have the following:

(i) $D(M) \cup \{\alpha, \gamma, \delta\}$: its canonical partition is $\{\{\alpha\}, \{\gamma, \delta\}, D(M)\}$;

(ii) $D(M) \cup \{\alpha, a, b\}$ $(a, b \in C(M))$: its canonical partition is $\{\{a, b\}, A, (D(M) \setminus A) \cup \{\alpha\}\}$, where $A \cup \{a, b\}$ is the circuit of $M \mid D(M) \cup \{a, b\}$;

(iii) $D(M) \cup \{\gamma, \delta, a, b\}$ $(a, b \in C(M))$: its canonical partition is $\{\{a, b\}, A, (D(M) \setminus A) \cup \{\gamma, \delta\}\}$, where $A \cup \{a, b\}$ is the circuit of $M \mid D(M) \cup \{a, b\}$;

(iv) $D(M) \cup \{\beta, \gamma, \delta, a\}$ ($a \in C(M)$): its canonical partition is $\{\{a, \beta\}, A, (D(M) \setminus A) \cup \{\gamma, \delta\}\}$, where $A \cup \{\beta, \gamma, a\}$ is the circuit of $M \mid D(M) \cup \{\beta, \delta, a\}$;

(v) $D(M) \cup \{\alpha, \beta, \delta, a\}$ ($a \in C(M)$ and $\{\beta, \delta, a\}$ is not a circuit of M): its canonical partition is $\{\{\beta, \delta, a\}, A, (D(M) \setminus A) \cup \{\alpha\}\}$, where $A \cup \{\beta, \delta, a\}$ is the circuit of $M \mid D(M) \cup \{\beta, \delta, a\}$;

(vi) $D(M) \cup \{\alpha, \beta, \gamma, a\}$ ($a \in C(M)$ and $\{\beta, \gamma, a\}$ is not a circuit of M): its canonical partition is $\{\{\beta, \gamma, a\}, A, (D(M) \setminus A) \cup \{\alpha\}\}$, where $A \cup \{\beta, \gamma, a\}$ is the circuit of $M \mid D(M) \cup \{\beta, \gamma, a\}$;

(vii) $D(M) \cup \{\alpha, \beta, \delta, a\}$ ($a \in C(M)$ and $\{\beta, \delta, a\}$ is a circuit of M): its canonical partition is $\{\{\beta, \delta, a\}, D(M) \cup \{\alpha\}\}$;

(viii) $D(M) \cup \{\alpha, \beta, \gamma, a\}$ ($a \in C(M)$ and $\{\beta, \gamma, a\}$ is a circuit of M): its canonical partition is $\{\{\beta, \gamma, a\}, D(M) \cup \{\alpha\}\}$.

So, the result follows. □

As a consequence of this result, we have the following:

(6.5) $\mathcal{F}$ is the set of circuits of a binary matroid, which is denoted by $\mathcal{N}(M)$.

Proof: Suppose that C_1 and C_2 are diferent sets belonging to $\mathcal{F}$. Choose an element e in $C_1 \cap C_2$. Let L be a line of $\mathcal{F}$ such that $L \subseteq C_1 \cup C_2$. If $e \notin L$, then $L \subseteq (C_1 \cup C_2) \setminus \{e\}$ and $(C_1 \cup C_2) \setminus \{e\}$ contains an element of $\mathcal{F}$. If $e \in L$, then $L \setminus \{e\} \subseteq (C_1 \cup C_2) \setminus \{e\}$ and $(C_1 \cup C_2) \setminus \{e\}$ contains an element of $\mathcal{F}$ by **(6.5)**. So, $\mathcal{F}$ is the set of circuits of a matroid. This matroid is binary because it does not have a large line. □

We say that $\mathcal{N}(M)$ was obtained from M by the *implosion* of $N(M)$.

We remind the reader that the set of lines of a matroid N is the set of circuits of another matroid, which is denoted by $\mathcal{L}(N)$.

Observe that $L = D(M) \cup \{\alpha, \gamma, \delta\}$ is a line of $\mathcal{N}(M)$ and that $\mathcal{L}(\mathcal{N}(M))/L \simeq U_{1,|E(M) \setminus L|}$. So, L is a circuit-hyperplane of $\mathcal{L}(\mathcal{N}(M))$ and $\mathcal{L}(M)$ is obtained from $\mathcal{L}(\mathcal{N}(M))$ relaxing L.

(6.6) $\mathcal{N}(M)$ is a 3-connected matroid.

Proof: Suppose that $N = \mathcal{N}(M)$ is not a 3-connected matroid. So, there is a $k \in \{1, 2\}$ and a subset X of $E(M)$ such that

$$r_N(X) + r_N(E(M) \setminus X) - r_N(E(M)) + 1 = k$$

and $\min\{|X|, |E(M) \setminus X|\} \geq k$. Since $r_N(X) = r_M(X)$ for every $X \neq D(M) \cup \{\alpha\}, D(M) \cup \{\gamma, \delta\}$ and M is 3-connected, it follows that $k = 2$

and $X = D(M) \cup \{\alpha\}$ or $X = D(M) \cup \{\gamma, \delta\}$. By **(2.8)**, we can split M as the 2-sum of matroids N_1 and N_2 such that $E(N_1) = D(M) \cup \{\alpha, e\}$ or $E(N_1) = D(M) \cup \{\gamma, \delta, e\}$, where e is a new element.

Case 1: $E(N_1) = D(M) \cup \{\gamma, \delta, e\}$

Observe that $\{e, \alpha\}$ is a circuit of N_2 and that $A \cup \{e\}$ is is a circuit of N_2 for every $A \subseteq E(M)$ such that $|A| = 2$ and $A \cap (D(M) \cup \{\alpha, \gamma, \delta, e\}) = \emptyset$. Hence $N_2 \backslash \{e\} \simeq U_{2,|E(N_2)|-1}$, since $N_2 \backslash \{e\}$ does not have parallel elements. So, $M \mid C(M) \cup \{\alpha, \beta\} \simeq U_{2,|C(M)|+2}$. Since every large line of M contains $D(M)$, it follows that $|C(M)| + 2 \leq 3$, a contradiction.

Case 2: $E(N_1) = D(M) \cup \{\alpha, e\}$

As $\{\alpha, \gamma, \delta\}$ is a circuit of N, it follows that $\{e, \alpha\}$ is a circuit of N_1. So, $|D(M)| = 1$, say $D(M) = \{d\}$, otherwise $\{D(M), E(M) \setminus D(M)\}$ is also a 2-separation of N. Since $\{d, \alpha, a, b\}$ is a large line of M for every $a, b \in C(M)$, it follows that $\{\alpha, a, b\}$ is a circuit of M and hence $M \mid C(M) \cup \{\alpha\} \simeq U_{2,|E(M)|+1}$. Hence $|C(M)| = 2$ and $|E(M)| = 7$. We arrive at a contradiction by **(4.4)**, since M has a non-spanning large line. □

6.2 A characterization

In this section, we will define the inverse operation of the implosion. Let H be a 3-connected binary matroid having a line $L = D \cup \{\alpha', \gamma', \delta'\}$, $D \neq \emptyset$. L is said to be a *special line* of H, when its canonical partition is $\{D, \{\alpha'\}, \{\gamma', \delta'\}\}$ and

(i) H has a triad $\{\beta', \gamma', \delta'\}$ for $\beta' \notin D$;

(ii) $C = E(H) \setminus D \cup \{\alpha', \beta', \gamma', \delta'\}$ has at least 2 elements;

(iii) C, $\{\gamma', \delta'\}$ and $\{\beta'\}$ are the parallel class of $H/D \cup \{\alpha'\}$, and $\{a, \beta', \gamma'\}$ is a circuit of $H/D \cup \{\alpha'\}$ for $a \in C$;

(iv) $C \cup \{\beta'\}$ is a parallel class and $\{\alpha'\}$ is a loop of $H/D \cup \{\gamma', \delta'\}$.

Let $\mathcal{G}$ be the family whose elements are:

(i) all circuits of H except $D \cup \{\alpha'\}$ and $D \cup \{\gamma', \delta'\}$;

(ii) $D \cup \{\alpha', a\}$ and $D \cup \{\gamma', \delta', a\}$ for every $a \in C$;

(iii) $D \cup \{\beta', \gamma', \delta'\}$, $D \cup \{\alpha', \beta', \gamma'\}$ and $D \cup \{\alpha', \beta', \delta'\}$.

(6.7) $\mathcal{S}(L)$ is a partition of L, for every line L of $\mathcal{G}$.

Proof: The result is true when L does not contain $D \cup \{\alpha'\}$ and $D \cup \{\gamma', \delta'\}$. When L contains one of these sets, the result follows similarly to **(6.4)**. □

Similarly to **(6.5)**, $\mathcal{G}$ is the set of circuits of a matroid, which is denoted by $H_{D,\alpha',\gamma'\delta'}$ and is called the *blowing up* of H along the special line $D \cup \{\alpha', \gamma', \delta'\}$. Observe that $\mathcal{N}(M)$ is a 3-connected binary matroid by **(6.6)**. Moreover, $D(M) \cup \{\alpha, \gamma, \delta\}$ is a special line of $\mathcal{N}(M)$ and $(\mathcal{N}(M))_{D(M),\alpha,\gamma\delta} = M$.

(6.8) If H is a binary matroid having a special line as described at the beginning of this section and $|E(H)| > 7$, then

(i) $D(H_{D,\alpha',\gamma'\delta'}) = D$;

(ii) $C(H_{D,\alpha',\gamma'\delta'}) = C$;

(iii) $N(H_{D,\alpha',\gamma'\delta'}) = \{\alpha', \beta', \gamma', \delta'\}$.

Proof: Observe that every large line of $H_{D,\alpha',\gamma'\delta'}$ contains D. So, $D \subseteq D(H_{D,\alpha',\gamma'\delta'})$. As $H/a = H_{D,\alpha',\gamma'\delta'}/a$ for every $a \in C$, it follows that $C \subseteq C(H_{D,\alpha',\gamma'\delta'})$. Since $D \cup \{\alpha', \beta', \gamma', \delta'\}$ is a large line of $H_{D,\alpha',\gamma'\delta'}$, it follows that $D \cup \{\alpha', \gamma', \delta'\}$ is a large line of $H_{D,\alpha',\gamma'\delta'}/\beta'$. Observe that $D \cup \{a, b, \alpha'\}$ is a large line of $H_{D,\alpha',\gamma'\delta'} \backslash \beta'$ when $a, b \in C$. Hence $\beta' \in N(H_{D,\alpha',\gamma'\delta'})$. By **(1.3)** and **(1.4)**, $|N(H_{D,\alpha',\gamma'\delta'})| = 4$ and the result follows. □

Now we can prove the main result of this section:

(6.9) Suppose that M is a 3-connected matroid such that $|N(M)| = 4$. If both M and M^* have a non-spanning large line, then M is equal to the blowing up of a 3-connected binary matroid along a special line.

Proof: Observe that $|E(M)| > 7$ by **(4.4)**. The result follows from **(6.5)**, **(6.6)**, **(6.7)**, **(6.8)** and

$$(\mathcal{N}(M))_{D(M),\alpha,\gamma\delta} = M. \square$$

7 Proof of the main result

Now we prove the main result of this paper:

Proof of (1.5): We have two cases to consider:

Case 1: Every large line of M or M^* is spanning.

The result follows from **(5.14)** and **(5.13)**.

Case 2: Both M and M^* have non-spanning large lines.

The result follows from **(6.9)**. □

As a consequence of **(1.5)**, we have:

(7.1) There are infinitely many 3-connected matroids having exactly four non-binary elements. □

We define $nb(m)$ as the number of non-isomorphic 3-connected non-binary matroids such that $|N(M)| = m$. Oxley in [6] proved that $nb(0)$ is infinite. We proved in [4] that $nb(1) = nb(2) = 0$ and $nb(3) = 2$. In this paper we showed that $nb(4)$ is again infinite. We conjecture that $nb(m)$ is infinite, when $m \geq 4$. For m even, the conjecture follows from **(5.11)**.

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