Graphs with Maximum Edge-Integrity

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ABSTRACT. The edge-integrity of a graph G is given by the minimum of $|S|+m(G-S)$ taken over all $S \subseteq E(G)$, where $m(G-S)$ denotes the maximum order of a component of $G-S$. An honest graph is one with maximum edge-integrity (viz. its order). In this paper lower and upper bounds on the edge-integrity of a graph with given order and diameter are investigated. For example, it is shown that the diameter of an honest graph on n vertices is at most $\sqrt{8n}-3$, and this is sharp. Also, a lower bound for the edge-integrity of a graph in terms of its eigenvalues is established. This is used to show that for d sufficiently large almost all d-regular graphs are honest.

1 Introduction

In this paper we consider finite undirected graphs without loops or multiple edges. The edge-integrity of a graph attempts to measure the disruption caused by the removal of edges from the graph. The order of a component or graph is the number of its vertices, and we let $m(H)$ denote the maximum order of a component of graph H. Barefoot, Entringer and Swart [6] defined the edge-integrity of a graph G with edge set $E(G)$ by

$$I'(G) = \min_{S \subseteq E(G)} (|S| + m(G - S)).$$

Any set S of edges which realizes this value is called an *I'-set*; one of minimum cardinality is called a *minimum I'-set*. We say that a graph is *honest* if its edge-integrity is equal to its order. Of course, the empty set shows that $I'(G) \leq m(G)$ in general.

Barefoot, Entringer and Swart [6] proposed this parameter as a measure of how hard it is to disrupt thoroughly a network by edge failures. Properties of edge-integrity were also investigated by Bagga et al., and by others. See for example [2, 4, 8, 10]. There is also a survey [3].

In this paper we investigate the range of values that the edge-integrity may take given the order and diameter of a graph. One of the first results in this direction was given by Bagga et al. who showed that graphs with diameter 2 are honest:

Proposition 1. [2]. *If G has n vertices and diameter 2 then $I'(G) = n$.*

For graphs with diameter 3 we prove a sharp lower bound of $3n^{2/3}/2 - O(n^{1/3})$ on the edge-integrity of such graphs. But, for graphs with higher diameter there is no better lower bound than that was observed by Barefoot et al.:

Proposition 2. [6]. *If G has n vertices and is connected then $I'(G) \geq \lceil 2\sqrt{n} \rceil - 1$.*

They showed that the path P_n on n vertices has $I'(P_n) = \lceil 2\sqrt{n} \rceil - 1$. We show that there are graphs with diameter 4 (and indeed with radius 2) with the same edge-integrity. At the other extreme, we show the diameter of an honest graph on n vertices is at most $\sqrt{8n} - 3$, and this is sharp.

In the final section we establish a link between the edge-integrity of a graph and its eigenvalues. As a consequence we show that, for d sufficiently large, almost all d-regular graphs are honest.

2 Minimum Edge-Integrity and Diameter

We know already by Proposition 1 that a graph with diameter 2 is honest. Our first result gives a tight lower bound on the edge-integrity of a graph of diameter 3:

Theorem 1. *Let graph G have n vertices and diameter 3. Then*

$$I'(G) \geq 3n^{2/3}/2 - n^{1/3}/2 - O(1).$$

Proof: Let S be an I'-set of G. If every vertex of G is incident with an edge of S then $|S| \geq n/2$ and we are done. Otherwise, let $H_1, H_2, \ldots, H_t$

be the components of $G - S$ which contain a vertex that is not incident to an edge of S. Then, as the diameter of G is at most 3, S contains an edge between H_i and H_j for $1 \leq i < j \leq t$. Let $H_1, H_2, \ldots, H_t$ have a total of r vertices. Then $|S| \geq \binom{t}{2} + (n-r)/2$. Also $m(G-S) \geq r/t$.

Thus $I'(G)$ is at least the minimum of

$$\frac{r}{t} + \binom{t}{2} + (n-r)/2,$$

taken over $1 \leq t < r \leq n$. For $t = 1$ the minimum of the expression is $(n+1)/2$. For $t \geq 2$ the minimum can be determined by using calculus (and a computer): it is attained at $r^* = n$ and $t^* \approx n^{1/3}$, and has the above value. □

There are graphs of diameter 3 which have edge-integrity that matches the lower bound. For example, for t even let G_t be the graph formed by taking t disjoint cliques, each with $t^2 - t/2$ vertices, and adding one edge between every pair of cliques. The graph G_t has $n_t = t^3 - t^2/2$ vertices and edge-integrity $i_t = 3t^2/2 - t$. The limit of $i_t - (3n_t^{2/3}/2 - n_t^{1/3}/2)$ as t goes to infinity is $-1/24$.

If the diameter is 4, however, then it turns out that the edge-integrity can be as small as what connectivity guarantees (recall Proposition 2). For example, construct graph H_s as follows. Take s disjoint cliques, each with s vertices, and designate one vertex in each clique; then add $s-1$ edges between the designated vertices to form a star. The resulting graph H_s has radius 2 and the same edge-integrity as the path on s^2 vertices, viz. $2s - 1$. The graph H_4 is illustrated in Figure 1.

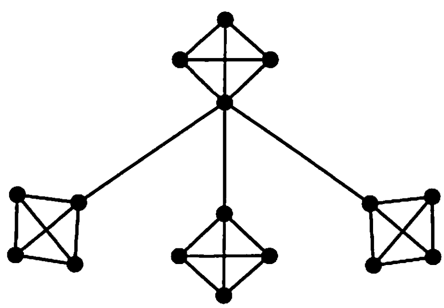

Figure 1.
The graph H_4 has radius two and minimum edge-integrity

3 Maximum Edge-Integrity and Diameter

For graphs with large edge-integrity and large diameter consider the following graphs. Given a sequence $a_0, a_1, \ldots, a_d$ of positive integers, we define the "leveled" graph $G[a_1, a_1, \ldots, a_d]$ as follows: take disjoint cliques $A_0, A_1, \ldots, A_d$ where A_i has a_i vertices ($i = 0, 1, \ldots, d$), and add all edges between A_i and A_{i+1} for $i = 0, 1, \ldots, d-1$. Figure 2 shows $G[1, 1, 2, 2, 2, 1, 1]$. Of course, $G[a_0, a_1, \ldots, a_d]$ has diameter d.

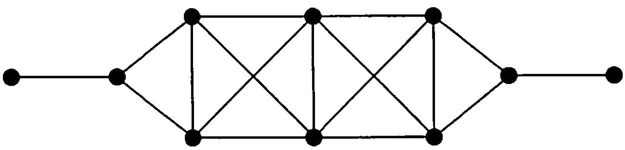

Figure 2. An honest leveled graph of diameter 6

The following lemma aids in the calculation of the edge-integrity of leveled graphs:

Lemma 2. *Let $G = G[a_0, a_1, \ldots, a_d]$ be a leveled graph, and let S be a minimum I'-set of G. Then*

(a) the removal of S does not split any of the A_i, and

(b) if the removal of S separates A_i and A_{i+1} then $i \in \{1, 2, \ldots, d-2\}$ and $a_i \leq a_{i+2}$ and $a_{i+1} \leq a_{i-1}$.

Proof:

(a) Suppose the removal of S splits some A_i. Let i be the smallest such index. Let H_1 and H_2 be two components of $G - S$ that contain vertices of A_i. Let $B_r = \{j\colon A_j \cap H_r \neq \emptyset\}$ for $r = 1, 2$. Two cases arise:

(1) *One of the B_r is a subset of the other.* Say $B_1 \subseteq B_2$. Then combine H_1 and H_2; that is, expunge from S and add to $G - S$ all the edges that join vertices of H_1 and H_2. The number of edges expunged from S is at least $|H_1|$, while the increase in the maximum component order is at most $|H_1|$. Thus S was not a minimum I'-set, a contradiction.

(2) *Neither of the B_r is a subset of the other.* Say $i-1 \in B_1 - B_2$. Let $T = H_1 \cap A_{i-1}$, $U = H_1 \cap A_i$, $V = H_2 \cap A_i$, and $W = H_2 \cap A_{i+1}$. If $|W| \leq |T|$ then add to $G - S$ the edges between V and $T \cup U$ and remove the edges between V and W. Effectively this transfers

V from H_2 to H_1, while saving at least $|V|$ edges. Thus S was not a minimum I'-set.

If $|W| > |T|$ then remove from $G - S$ the edges between U and T, splitting H_1 into two pieces, and add all edges between the piece of H_1 containing U and H_2. It can be checked that the net decrease in the number of edges removed is again at least the increase in the maximum component order. Thus S was not a minimum I'-set, a contradiction.

(b) Now suppose S separates A_i and A_{i+1}. If $i = 0$ then reinsert into $G - S$ the edges between A_i and A_{i+1}. If $a_{i-1} < a_{i+1}$ then reinsert into $G - S$ the edges between A_i and A_{i+1} and remove (if necessary) the edges between A_i and A_{i-1}. In both cases, the saving in edges removed is at least $|A_i|$, while the increase in maximum component order is at most $|A_i|$. So S was not a minimum I'-set, a contradiction.

A similar argument holds if $i = d - 1$ or $a_{i+2} < a_i$.

□

Corollary 3. *Let $G[a_0, a_1, \ldots, a_d]$ be a leveled graph for which there exists an r such that for $0 \leq i \leq r-2$ it holds that $a_i < a_{i+2}$, and for $r \leq i \leq d-2$ it holds that $a_i > a_{i+2}$. Then G is honest.*

Proof: By the above lemma a minimum I'-set of the graph is empty. □

For example the leveled graph $G[1, 1, 2, 2, 2, 1, 1]$ of Figure 2 is honest (use $r = 3$). We will show that the cheapest way to satisfy the hypothesis of the corollary gives the honest graph of diameter d with minimum order. We will need the following lemma:

Lemma 4. *Let m be a given integer, and consider the following problem:*

Minimize $\sum_{i=0}^{m} b_i$ such that $b_j b_{j+1} \geq \sum_{i=0}^{j} b_i$ for $j = 0, 1, \ldots, m-1$,

where the b_i are positive integers. Then the minimum is $\lfloor (m+2)^2/4 \rfloor$, and the unique best sequence of the b_i is given by the first $m+1$ entries of $\mathcal{B} = 1, 1, 2, 2, 3, 3, 4, 4 \ldots$

Proof: Let $\beta_j = \sum_{i=0}^{j} b_i$. Rearranged, the constraint says that $b_j(b_{j+1} - 1) \geq \beta_{j-1}$. So by calculus it follows that $b_j + (b_{j+1} - 1) \geq \lceil 2\sqrt{\beta_{j-1}} \rceil$. Thus $\beta_{j+1} = b_{j+1} + b_j + \beta_{j-1} \geq 1 + \lceil 2\sqrt{\beta_{j-1}} \rceil + \beta_{j-1}$. Since $\beta_1 \geq 1$ and $\beta_2 \geq 2$, by induction it then follows that $\beta_j \geq \lfloor (j+2)^2/4 \rfloor$. (We omit the straight-forward calculation.) The characterization of when equality occurs is also proved by induction. □

Theorem 5. *The minimum number of vertices in an honest graph of diameter d is $\lceil (d+2)(d+4)/8 \rceil$.*

Proof: Let G be an honest graph with n vertices and diameter d. Let v be a vertex such that there is a vertex at distance d from v. Let A_i denote the set of vertices at distance i from v, and let $a_i = |A_i|$. Further, let r be the largest index such that $\sum_{i=1}^{r} a_i \leq n/2$.

By considering the removal of all edges between A_j and A_{j+1}, it follows that necessarily

$$a_j a_{j+1} \geq \sum_{i=0}^{j} a_i \qquad \text{for } j = 0, \ldots, r,$$

and

$$a_j a_{j+1} \geq \sum_{i=j+1}^{d} a_i \quad \text{for } j = r+1, \ldots, d-1.$$

By the above lemma it follows that $\sum_{i=0}^{r} a_i \geq \lfloor (r+2)^2/4 \rfloor$, and $\sum_{i=r+1}^{d} a_i \geq \lfloor (d-r+1)^2/4 \rfloor$. Since the two sets $\cup_{i=0}^{r} A_i$ and $\cup_{i=r+1}^{d} A_i$ are disjoint, it follows that

$$8n \geq 2(r+2)^2 + 2(d-r+1)^2 - 8\varepsilon_{dr},$$

where $\varepsilon_{dr} = 0$ if r is even and d is odd, $\varepsilon_{dr} = 1/2$ if d and r both odd, and $\varepsilon_{dr} = 1/4$ otherwise.

If d is even, then the lower bound for $8n$ is minimized at $r = d/2$ or $r = d/2 - 1$. It follows that $8n \geq d^2+6d+8$, as required. If d is odd and r is even, then the lower bound for $8n$ is minimized at $r = (d-1)/2$ where it has value $8n \geq d^2+6d+9$. If d and r are both odd then we use the above lemma to observe that $\sum_{i=0}^{r-1} a_i \geq \lfloor (r+1)^2/4 \rfloor$ and $\sum_{i=r}^{d} a_i \geq \lfloor (d-r+2)^2/4 \rfloor$, and thus that

$$8n \geq 2(r+1)^2 + 2(d-r+2)^2 \geq d^2 + 6d + 9.$$

To obtain a best sequence of the a_i we put together two almost equalized initial segments of $\mathcal{B}$ with the second one reversed: If d is even the two initial segments differ in length by 1; if $d \equiv 1 \pmod 4$ then the two initial segments have the same length; and if $d \equiv 3 \pmod 4$ then the two initial segments differ in length by 2. For example, for $d = 6$ the best $\{a_i\}$ is $1,1,2,2,2,1,1$. For $d = 7$ it is $1,1,2,2,3,2,1,1$. The associated leveled graphs are honest by Corollary 3. □

We believe that "honest leveled graphs with tails" have nearly maximum edge-integrity for their order and diameter. Let

$$G_d^t = G[\underbrace{1,1,\ldots,1}_{t},\underbrace{1,1,2,2,3,\ldots,3,2,2,1,1}_{d},\underbrace{1,1,\ldots,1}_{t}],$$

where the middle portion gives the honest graph G_d of minimum order n_d described in Theorem 5. Then G_d^t has diameter $D = 2t + d$ and order $N = n_d + 2t$. By Lemma 2, it follows that the subgraph G_d remains virtually intact after the removal of a minimum i'-set S (except maybe losing one vertex at both ends). So $m(G_d^t - S) \geq n_d - 2$. Calculations then show that

$$I'(G_d^t) = \begin{cases} n_d, & n_d \geq t+3; \\ \approx n_d - 2 + (2t+2)/(n_d - 2), & \sqrt{2t} \leq n_d \leq t+3; \\ \approx I'(P_{D+1}), & n_d \leq \sqrt{2t}. \end{cases}$$

Specifically, we conjecture:

Conjecture 1: *For $n_d \geq t + 3$, the graph G_d^t has the maximum edge-integrity for a graph of its order N and diameter D.*

That edge-integrity is approximately $(N - D) + \sqrt{8(N-D)} + 1$, and the range of appropriate D is roughly $\sqrt{8N} - 3 \leq D \leq 2N/3$.

4 Edge-Integrity and Eigenvalues

In this section we derive a simple lower bound on the edge-integrity of a graph in terms of its eigenvalues.

Let G be a graph and A a subset of the vertices. Then the *edge boundary* $b(A)$ of A is the number of edges of G with exactly one end in A. We let $b(m)$ denote the minimum of $b(A)$ taken over sets A of m vertices. Isoperimetric inequalities give lower bounds for edge-integrity, as was observed in [7]:

Proposition 3. [7]. *Let G be a graph on n vertices and let $f(x)$ be a real convex function such that $f(m) \leq b(m)$ for all $m \in \{1, 2, \ldots, n\}$. Then*

$$I'(G) \geq \min_{x \geq 0} x + \frac{n}{2s} f(x).$$

Alon and Milman [1] established a link between the edge boundary and eigenvalues. Let L denote the Laplacian matrix $D - A$ of the graph, where A is the adjacency matrix of the graph, and D a diagonal matrix with the degrees of the vertices on the diagonal. Then the eigenvalues of L are real and nonnegative. Let λ_1 denote the second smallest eigenvalue. (The smallest is 0.) Alon and Milman showed:

Proposition 4. [1]. *For a graph on n vertices it holds that $b(m) \geq \lambda_1 m(1 - m/n)$.*

A corollary of the above two results is:

Theorem 6. *For a graph G on n vertices whose Laplacian has second smallest eigenvalue λ_1,*

$$I'(G) \geq n \cdot \min(1, \lambda_1/2).$$

Proof: By the above two propositions,

$$I'(G) \geq \min_x x + n\frac{\lambda_1 x(1 - x/n)}{2x},$$

where the minimum is taken over real $x \in [0, n]$. The minimum is attained either at $x = n$, where it has value n, or at $x = 0$, where it has value $n\lambda_1/2$. □

Corollary 7. *If $\lambda_1 \geq 2$ then G is honest.*

For example, all the hypercubes have $\lambda_1 = 2$ and are thus honest. This was first shown by Bagga et al. [2].

In [11] the third author showed that there are only finitely many cubic graphs which are honest. In contrast, when d is sufficiently large, almost every d-regular graph is honest. For, Friedman [9] showed that for a random d-regular graph almost surely $\lambda_1 \geq d - 2\sqrt{d-1} - O(\log d)$. Hence for d sufficiently large, by Corollary 7 the graph is almost surely honest.

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