On the Maximum Number of Chords in a Cycle of a Graph

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ABSTRACT. We give an exponential lower bound for the maximum number of chords in a cycle of a graph G in terms of the minimum degree of G and the girth of G. We also give regular graphs having no small cycles where the maximum number of chords possible in any cycle of the graph is approximately the fourth power of our lower bound. An immediate consequence is a recent result of Ali and Staton.

All graphs we consider in this note are finite and have neither loops nor multiple edges. For a graph G, $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. We denote the set of vertices in G adjacent to a vertex v by $N_G(v)$. The degree $d_G(v)$ of v is $|N_G(v)|$ and the minimum degree $\delta(G)$ of G is the least degree of a vertex of G.

We call $v_1v_2\cdots v_{m+1}$ a (v_1, v_{m+1})-walk in G of length m provided $v_1, \ldots, v_{m+1} \in V(G)$ and $v_1v_2, \ldots, v_mv_{m+1} \in E(G)$. The walk is closed provided $v_1 = v_{m+1}$ and the closed walk is a cycle provided the vertices $v_1, \ldots, v_m$ are distinct. Note that every closed walk of positive length contains a cycle. An edge of G, not belonging to a cycle C of G, that joins two vertices of C is called a chord of C. We denote the length of a shortest cycle in G, should one exist, by girth(G). The distance $d_G(u, v)$ between vertices u, v of G is the length of a shortest (u, v)-walk (equivalently, path), should one exist, in G. We denote the largest integer at most x by $\lfloor x \rfloor$. In general we use the notation and terminology of [2].

Recently Ali and Staton [1] proved the following result.

Theorem. *Any graph G with $\delta(G) = \delta \geq 2$ contains a cycle with at least $\delta(\delta - 2)/2$ chords. Moreover, if G contains no 3-cycles nor 5-cycles, it contains a cycle with at least $\delta(\delta - 2)$ chords.*

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In this note we give an exponential lower bound for the maximum number of chords in a cycle of a graph G in terms of the minimum degree of G and the girth of G. An immediate consequence is the above result of Ali and Staton. Our result is best possible for certain values of the girth of G. We also give regular graphs having no small cycles where the maximum number of chords possible in any cycle of the graph is approximately the fourth power of our lower bound.

For a path $P = v_1v_2\cdots v_m$, let

$$T(P) = \{P - v_{j-1}v_j + v_1v_j : v_1v_j \in E(G) \text{ with } j \geq 3\},$$
$$X(P) = \{v_j \in V(P) : v_1v_j \in E(G)\}.$$

Note that $Q \in T(P)$ is a path with the same length as P. For a collection $\mathcal{P}$ of paths in G, let

$$T(\mathcal{P}) = \cup\{T(P) : P \in \mathcal{P}\},$$
$$X(\mathcal{P}) = \cup\{X(P) : P \in \mathcal{P}\}.$$

Theorem 1. *Any graph G with $\delta(G) \geq \delta \geq 2$ and girth$(G) \geq g \geq 3$ contains a cycle with at least* $\max\left\{\left[\delta(\delta-1)^{\lfloor (g+1)/4\rfloor} - \delta\right]/2, \delta(\delta-1)^{\lfloor g/4\rfloor} - \delta\right\}$ *chords.*

Proof: The result is trivial for $\delta = 2$ so we assume $\delta \geq 3$. For any longest path $P = v_1v_2\cdots v_m$ in G, $N_G(v_1) \subseteq V(P)$, so that, $X(P) = N_G(v_1)$. Among all longest paths in G, choose $P = v_1v_2\cdots v_m$ with $d_P(v_1, w) = \ell$ as large as possible where $w \in N_G(v_1)$, so that, $X(P) \subseteq \{v_1, \ldots, v_{\ell+1}\}$ and $v_1v_{\ell+1} \in E(G)$.

For $k \in \mathbb{N}$, let $T^k(P) = T(T^{k-1}(P))$ and $X_k(P) = X(T^{k-1}(P))$ where $T^0(P) = \{P\}$ so that $X_1(P) = X(P)$.

Let $Q = w_1w_2\cdots w_m \in T^k(P)$ where $\{w_1, \ldots, w_{\ell+1}\} = \{v_1, \ldots, v_{\ell+1}\}$, $(w_{\ell+1}, \ldots, w_m) = (v_{\ell+1}, \ldots, v_m)$ and there exists a (w_1, v_1)-walk in G of length $2k$. By our choice of P, $X(Q) = N_G(w_1) \subseteq \{w_1, \ldots, w_{\ell+1}\} = \{v_1, \ldots, v_{\ell+1}\}$. Hence for $R = Q - w_{j-1}w_j + w_1w_j = x_1x_2\cdots x_m$ ($j \neq 2$), $\{x_1, \ldots, x_{\ell+1}\} = \{v_1, \ldots, v_{\ell+1}\}$, $(x_{\ell+1}, \ldots, x_m) = (v_{\ell+1}, \ldots, v_m)$ and there exists an (x_1, v_1)-walk in G of length $2k+2$ (append $x_1 = w_{j-1}w_jw_1$ to the previous walk). Note this implies that there exists an (x, v_1)-walk in G of length $2k+1$ for $x \in X(Q) \subseteq X_{k+1}(P)$ (append xw_1 to the previous walk) and $X_{k+1}(P) \subseteq \{v_1, \ldots, v_{\ell+1}\}$.

Let $\mathcal{T} = \bigcup_{k=0}^{t-1} T^k(P)$ and $\mathcal{X} = \bigcup_{k=1}^{t} X_k(P)$, so that, $\mathcal{X} \subseteq \{v_1, \ldots, v_{\ell+1}\}$.

First, let $x \in X_{k_1}(P)$, $y \in X_{k_2}(P)$ with $x = y$. Now, there exists an (x, v_1)-walk in G of length $2k_1 - 1$ and a (y, v_1)-walk in G of length $2k_2 - 1$. Then, there exists a closed walk (so cycle) in G of positive length (at most)

$2k_1 + 2k_2 - 2 \leq 4t - 2 \leq g - 1$ provided $t \leq \lfloor (g+1)/4 \rfloor$; a contradiction. Hence, all vertices in $\mathcal{X}$ are distinct provided $t \leq \lfloor (g+1)/4 \rfloor$.

Next, let $Q \in T^{k_1}(P)$ have initial vertex w, $R \in T^{k_2}(P)$ have initial vertex x with $P \neq Q = R$, so that, $w = x$. Now, there exists a (w, v_1)-walk in G of length $2k_1$ and an (x, v_1)-walk in G of length $2k_2$. Then, there exists a closed walk (so cycle) in G of positive length (at most) $2k_1 + 2k_2 \leq 4t - 4 \leq g - 1$ provided $t \leq \lfloor (g+3)/4 \rfloor$; a contradiction. Hence, all paths in $\mathcal{T}$ are distinct provided $t \leq \lfloor (g+3)/4 \rfloor$.

Finally, let $x \in X_{k_1}(P)$, $y \in X_{k_2}(P)$ with $xy \in E(G)$. As above, there exists a cycle in G of length at most $4t - 1 \leq g - 1$ provided $t \leq \lfloor g/4 \rfloor$; a contradiction. Hence, $\mathcal{X}$ is an independent set of distinct vertices provided $t \leq \lfloor g/4 \rfloor$.

If $t \leq \lfloor (g+1)/4 \rfloor$, all paths in $\mathcal{T}$ are distinct, while each path in $\mathcal{T} - T^{t-1}(P)$ generates at least $\delta - 1$ distinct paths of $\mathcal{T}$ and each path in $\mathcal{T}$ generates at least δ distinct vertices of $\mathcal{X}$, so that

$$|\mathcal{X}| = \sum_{k=1}^{t} |X_k(P)| \geq \delta \sum_{k=0}^{t-1} |T^k(P)| \geq \delta \sum_{k=0}^{t-1} (\delta - 1)^k = \delta \frac{(\delta-1)^t - 1}{\delta - 2}.$$

For $t = \lfloor (g+1)/4 \rfloor$, the cycle $C = v_1 \cdots v_{\ell+1} v_1$ in G has at least

$$\frac{|\mathcal{X}|(\delta-2)}{2} \geq \frac{\delta(\delta-1)^{\lfloor (g+1)/4 \rfloor} - \delta}{2} \text{ chords,}$$

since the vertices in $\mathcal{X}$ are distinct vertices of C, while, for $t = \lfloor g/4 \rfloor$ the cycle C has at least

$$|\mathcal{X}|(\delta - 2) \geq \delta(\delta-1)^{\lfloor g/4 \rfloor} - \delta \text{ chords,}$$

since $\mathcal{X}$ is an independent set of distinct vertices of C. □

As a consequence we obtain the following slight extension of the result of Ali and Staton [1].

Corollary 2. *Any graph G with $\delta(G) = \delta \geq 2$ contains a cycle with at least $\delta(\delta-2)/2$ chords. Moreover, if G contains no triangles, it contains a cycle with at least $\delta(\delta-2)$ chords.*

Proof: Use Theorem 1 with $g = 3$ and 4, respectively. □

Remark 3: As noted in [1], K_δ shows that the first part of the corollary is essentially best possible, while, $K_{\delta,\delta}$ shows that the second part of the corollary is best possible. For $\delta \geq 3$ and $g \geq 3$, Bollobás [2; pps 108–109] proved the existence of δ-regular graphs with girth at least g having $2[(\delta-1)^{g-1} - 1]/(\delta - 2)$ vertices. Any cycle in such a graph has at most $2(\delta-1)^{g-1}$ chords, which is approximately the fourth power of the lower bound given in Theorem 1.

References

[1] A.A. Ali and W. Staton, On the Extremal Question for Cycles with Chords, to appear.

[2] B. Bollobás, *Extremal Graph Theory*, Academic Press, New York, 1978.