# Ryser Designs

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ABSTRACT. This paper provides an expository account, from a design-theoretic point of view, of the important result of Ryser that covering of the complete graph $K_v$ a total of $\lambda$ times by $v$ complete subgraphs can only be done in a very limited number of ways.

## 1 Introduction

This is a largely expository paper that deals with a particular class of *pairwise balanced designs*. We recall the definition of a PBD: we have $v$ varieties arranged in $b$ blocks. The varieties occur with frequencies $r_1, r_2, \ldots, r_v$, and these $r_i$ are selected from a set $R$; the blocks have lengths $k_1, k_2, \ldots, k_b$ (each $k_i < v$) and these $k_i$ are selected from a set $K$. Finally, each pair of varieties occurs a constant number, $\lambda$, of times in the blocks.

For example, consider the PBD comprising blocks 1234, 156, 25, 26, 35, 36, 45, 46. Here $v = 6$; $b = 8$; $r_1 = 2$, $r_2 = r_3 = r_4 = 3$, $r_5 = r_6 = 4$, $R = \{2,3,4\}$; $k_1 = 4$, $k_2 = 3$, $k_3 = k_4 = k_5 = k_6 = k_7 = k_8 = 2$, $K = \{2,3,4\}$; $\lambda = 1$.

A PBD can also be presented as an incidence matrix, $A$, of dimensions $v \times b$. The rows correspond to varieties and the columns to blocks. Entry $a_{ij} = 1$ if variety $i$ occurs in block $j$; otherwise $a_{ij} = 0$. Clearly $\sum k_i = \sum r_i = \#$ of entries 1 in matrix $A$.

We should conclude this section by noting that, in a PBD, we require all the blocks to be incomplete, that is, we require that no block contain all $v$ varieties. The case where complete blocks are allowed is trivial, since any PBD with pair count $\lambda$ that contains $t$ complete blocks corresponds (by deleting these blocks) to a PBD with only incomplete blocks that has pair count $\lambda - t$.

In the special case that $K$ contains only a single element $k$, then all blocks have the same length. Variety $i$ must occur with $\lambda(v-1)$ other varieties; but it also occurs with $r_i(k-1)$ other varieties. Hence $r_i(k-1) = \lambda(v-1)$, and all the $r_i$ have the same value $r = \lambda(v-1)/(k-1)$. Thus $R$ likewise contains only a single element, $r$. In this case, we speak of a *Balanced Incomplete Block Design* (BIBD) with parameters $(v, b, r, k, \lambda)$. As we have seen, $r(k-1) = \lambda(v-1)$. Also, counting the totality of 1's in the incidence matrix gives the familiar result $rv = bk$.

## 2 The Fisher Theorem for PBDs

It is easy to calculate the product of the incidence matrix $A$ by its transpose. We find

$$AA^T = \begin{pmatrix} r_1 & \lambda & \lambda & \lambda & \dots & \lambda \\ \lambda & r_2 & \lambda & \lambda & \dots & \lambda \\ \lambda & \lambda & r_3 & \lambda & \dots & \lambda \\ \lambda & \lambda & \lambda & r_4 & \dots & \lambda \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \lambda & \lambda & \lambda & \lambda & \dots & r_v \end{pmatrix}$$

Expanding along the first column of a matrix of size $(v+1) \times (b+1)$, we have

$$\det \left( \begin{array}{c|c} 1 & \mathbf{j}^T \\ \hline \mathbf{z} & AA^T \end{array} \right) = \det AA^T,$$

where $\mathbf{z}$ is a column vector of zeros, $\mathbf{j}$ is a column vector of 1's. Then (subtracting $\lambda$ times the first row from the other rows), we have

$$\det \left( \begin{array}{c|c} 1 & \mathbf{j}^T \\ \hline -\lambda\mathbf{j} & D_1 \end{array} \right) = \det AA^T,$$

where $D_1$ is a diagonal matrix with successive entries $r_1 - \lambda, r_2 - \lambda, r_3 - \lambda, \dots, r_v - \lambda$.

Expand along the first column of this last matrix, and we have the result

$$\det AA^T = \sum_{i=1}^{v}(r_i - \lambda)\left\{1 + \lambda\left(\frac{1}{r_1 - \lambda} + \frac{1}{r_2 - \lambda} + \dots + \frac{1}{r_v - \lambda}\right)\right\}.$$

But $r_i > \lambda$ (if $r_i = \lambda$, then variety $i$ would have to occur in $\lambda$ blocks, each of which contained all other varieties; but these would be complete blocks, which are not allowed).

Hence $\det AA^T > 0$, and so the $v \times v$ matrix $AA^T$ has rank $v$.

Now suppose, if possible, that $b < v$; then $AA^T$ is the product of 2 matrices each of rank $\leq b$ (the smaller dimension). But we know that the rank of a product can not exceed the lesser of the ranks of the matrices being multiplied. Hence rank $AA^T \leq b < v$. This is a contradiction and so we have the

**Theorem.** *In a PBD, $v \leq b$, that is, the number of blocks is at least equal to the number of varieties.*

In graph-theoretic terms, this means that, if the complete graph $K_v$ is covered by a collection of complete graphs to a total of $\lambda$ times, then the number of subgraphs $\geq v$.

H.J. Ryser was particularly interested in the extreme case when $v$ and $b$ were equal. So we shall call such designs *Ryser Designs.* (Actually, Ryser looked at the duals of these designs, but is seems considerably simpler to look at the designs *per se.* The amount of labour involved becomes considerably less if we concentrate on the more natural design point of view and try to minimize the amount of matrix manipulation.)

## 3 The Ryser Theorem

Henceforth, we shall be considering the case $v = b$. From the discussion in Section 1, it appears natural to introduce the qualities $y_i = (k_i - 1)/(v-1)$, $i$ ranging from 1 to $v$. $\mathbf{Y}$ will denote the column vector formed from the quantities $y_i$.

**Lemma 3.1.** *If $A$ is the incidence matrix of the PBD, then $A\mathbf{Y} = \lambda \mathbf{j}$, where $j$ is the vector with all entries unity.*

**Proof:** $A\mathbf{Y} = \frac{1}{v-1}A(k_1 - 1, k_2 - 1, \ldots, k_v - 1)^T$.

When we apply row $i$ of $A$ to the entries in the vector $(k_1 - 1, k_2 - 1, \ldots, k_v - 1)^T$, it will pick up an amount $k_j - 1$ if and only if variety $i$ occurs in block $j$; so it will accumulate an amount equal to the number of other elements that occur with variety $i$.

Since this is just $\lambda(v-1)$, we have

$$\begin{aligned} A\mathbf{Y} &= \frac{1}{v-1}(\lambda(v-1), \lambda(v-1), \ldots, \lambda(v-1))^T \\ &= (\lambda, \lambda, \ldots, \lambda)^T = \lambda \mathbf{j}. \end{aligned}$$

We now introduce a new matrix

$$B = \left( \begin{array}{c|c} \sqrt{-\lambda} & \mathbf{Y}^T \\ \hline \sqrt{-\lambda}\mathbf{j} & A \end{array} \right),$$

where $\sqrt{-\lambda}$ is a formal symbol having $\sqrt{-\lambda}\sqrt{-\lambda} = -\lambda$.

Then, using Lemma 3.1, we have

$$BB^T = \left(\begin{array}{c|c} -\lambda + \mathbf{Y}^T\mathbf{Y} & -\lambda \mathrm{j}^T + \mathbf{Y}^T A^T \\ \hline -\lambda \mathrm{j} + A\mathbf{Y} & -\lambda \mathrm{jj}^T + AA^T \end{array}\right)$$
$$= \left(\begin{array}{c|c} -\lambda + \sum y_i^2 & 0 \\ \hline 0 & D_1 \end{array}\right),$$

where $D_1 = \operatorname{diag}(r_1 - \lambda, r_2 - \lambda, \ldots, r_v - \lambda)$, as before.

Set $u = -\lambda + \sum y_i^2$, and we may write

$$BB^T = \operatorname{diag}(u, r_1 - \lambda, r_2 - \lambda, \ldots, r_v - \lambda).$$

Suppose that we now define a diagonal matrix

$$D = \operatorname{diag}\left(\frac{1}{\sqrt{u}}, \frac{1}{\sqrt{r_1 - \lambda}}, \frac{1}{\sqrt{r_2 - \lambda}}, \ldots, \frac{1}{\sqrt{r_v - \lambda}}\right),$$

and form a new matrix $K = DB$. Then

$$KK^T = DBB^T D^T = D(BB^T)D.$$

This is the product of 3 diagonal matrices and so we have

**Lemma 3.2.** *$KK^T = I$, where $I$ is the identity matrix.*

Now let us write down the rows of $K$. They are:

Row 1: $\sqrt{\frac{-\lambda}{u}}, \frac{y_1}{\sqrt{u}}, \frac{y_2}{\sqrt{u}}, \ldots, \frac{y_v}{\sqrt{u}}$

Row 2: $\sqrt{\frac{-\lambda}{u}}, \frac{a_{11}}{\sqrt{r_1-\lambda}}, \frac{a_{12}}{\sqrt{r_2-\lambda}}, \ldots, \frac{a_{1v}}{\sqrt{r_v-\lambda}}$

Row 3: $\sqrt{\frac{-\lambda}{u}}, \frac{a_{21}}{\sqrt{r_1-\lambda}}, \frac{a_{22}}{\sqrt{r_2-\lambda}}, \ldots, \frac{a_{2v}}{\sqrt{r_v-\lambda}}$

etc.

The relation $KK^T = I$ tells us that each row is of unit length and all rows are orthogonal, that is, $\mathbf{R}_i \cdot \mathbf{R}_j = \delta_{ij}$, where the dot indicates the usual inner product. If we do this computation as a check, we naturally find nothing new. However, we also have $K^T K = I$, since the inverse of matrix $K$ is unique. Thus

$$\mathbf{C}_i \cdot \mathbf{C}_j = \delta_{ij},$$

where $\mathbf{C}_i$ denotes the $i$th column of $K$. This does give new results.

Use column 1 of $K$ with column $j$ of $K (j > 1)$, and we find

$$\sqrt{\frac{-\lambda}{u}} \frac{y_j}{\sqrt{u}} + \frac{\sqrt{-\lambda}}{r_1 - \alpha} a_{1j} + \frac{\sqrt{-\lambda}}{r_2 - \lambda} a_{2j} + \cdots = 0.$$

We obtain

$$\sum_\alpha \frac{a_{\alpha j}}{r_\alpha - \lambda} = \frac{-y_j}{u}.$$

Now use column $j$ of $K$ with itself; we obtain

$$\frac{y_j^2}{u} + \frac{a_{1j}^2}{r_1 - \lambda} + \frac{a_{2j}^2}{r_2 - \lambda} + \cdots = 1.$$

Hence, we have

$$1 - \frac{y_j^2}{u} = \sum_\alpha \frac{a_{\alpha j}^2}{r_\alpha - \lambda} = \sum_\alpha \frac{a_{\alpha j}}{r_\alpha - \alpha},$$

using the fact that each $a_{\alpha j}$ is either 0 or 1 and so $a_{\alpha j}^2 = a_{\alpha j}$. We thus have

$$1 - \frac{y_j^2}{u} = -\frac{y_j}{u}.$$

In short, no matter what value $j$ takes, $y_j$ satisfies the relation

$$y_j^2 - y_j - u = 0.$$

Thus, $y_j$ satisfies the quadratic equation $y^2 - y - u = 0$ and so there are only 2 possible values for $y$, namely, $y_1$ and $y_2$, where $y_1 + y_2 = 1$, $y_1 y_2 = -u$. We immediately deduce

**Ryser's Theorem** [4]. *In a PBD with $v = b$, there are only 2 possible block lengths $k_1$ and $k_2$, and $k_1 + k_2 = v + 1$.*

**Proof:** Since $y_1 + y_2 = 1$, we have

$$\frac{k_1 - 1}{v - 1} + \frac{k_2 - 1}{v - 1} = 1,$$

whence $k_1 + k_2 = v + 1$.

In terms of graphs, we have the surprising result that, if $K_v$ is covered $\lambda$ times by $v$ complete subgraphs, then these subgraphs can be of only two sizes.

We conclude this section by pointing out that the quantity $u$ is negative. Indeed, we have

**Lemma 3.3.** $-\frac{1}{4} \le u < 0$.

**Proof:** Since $y_j^2 - y_j - u = 0$, we have

$$y_j = \frac{1}{2}\left(1 \pm \sqrt{1 + 4u}\right).$$

But $y_j$ is a real number and so $1 + 4u \ge 0$, that is, $u \ge -\frac{1}{4}$.

Also $u = -\lambda + \sum \left(\frac{k_i-1}{v-1}\right)^2$. Hence

$$\begin{aligned} u(v-1)^2 &= -\lambda(v-1)^2 + \sum(k_i-1)^2 \\ &= -\lambda v(v-1) + \lambda(v-1) + \sum k_i(k_i-1) - \sum(k_i-1) \end{aligned}$$

Now $\lambda\binom{v}{2} = \sum \binom{k_i}{2}$ = total number of pairs.

Hence $u(v-1)^2 = \lambda(v-1) - \sum k_i + v$.

But $\sum k_i = \sum r_i \geq \sum(1+\lambda) = v(1+\lambda)$, since each $r_i > \lambda$. Then

$$u(v-1)^2 \leq \lambda(v-1) - v(1+\lambda) + v = -\lambda$$

Thus $u \leq -\frac{\lambda}{(v-1)^2} < 0$.

## 4 The Frequency Relation

We have seen that there can only be two block lengths in the Ryser PBDs. Suppose that there are $f_1$ blocks of length $k_1$ and $f_2$ blocks of length $k_2$. First, we dispose of a special case.

**Lemma 4.1.** *If $k_1 = k_2$, then we have a Balanced Incomplete Block Design with parameters* $(4\lambda-1, 4\lambda-1, 2\lambda, 2\lambda, \lambda)$.

**Proof:** We have $k_1 = k_2 = \frac{k+1}{2}$. From Section 1, we see that the design is a BIBD with parameters $\left(v, v, \frac{v+1}{2}, \frac{v+1}{2}, \lambda\right)$. Then

$$\frac{v+1}{2}\frac{v-1}{2} = \lambda(v-1),$$

whence $v+1 = 4\lambda$, $v = 4\lambda - 1$.

Except in the special case of Lemma 4.1, we take $k_2 > k_1$. Then we can write down two frequency equations:

$$f_1 + f_2 = v,$$

$$f_1\binom{k_1}{2} + f_2\binom{k_2}{2} = \lambda\binom{v}{2}.$$

These two linear equations are easily solved to give $f_2$, the number of long blocks, as

$$f_2 = \lambda - \frac{(k_1-1)(k_1-2\lambda)}{v+1-2k_1}.$$

Note that $v + 1 - 2k_1 = k_2 - k_1 > 0$. Of course, $f_1 = v - f_2$.

## 5 The Case $\lambda = 1$

The case $\lambda = 1$ is rather special, but it illustrates the general procedure well. We write down the frequency equation

$$f_2 = 1 - \frac{(k_1 - 1)(k_1 - 2)}{v + 1 - 2k_1}$$

**Case 1.** $k_1 = 1$, $k_2 = v$; $f_2 = 1$, $f_1 = v - 1$. We refer to this as the *degenerate case*. There is one complete block $(1, 2, 3, \ldots, v)$ and $v - 1$ singletons.

Indeed, this degenerate case occurs for and $\lambda$. It uses $\lambda$ complete blocks and $v - \lambda$ singletons. In graph-theoretic terms, it corresponds to covering the complete graph $K_v$ a total of $\lambda$ times by using $\lambda$ copies of $K_v$ and $v - \lambda$ copies of $K_0$.

**Case 2.** $k_1 = 2$.

Here $k_2 = v - 1$, $f_2 = 1$, $f_1 = v - 1$. This gives a PBD with one long block $(1, 2, 3, \ldots, v - 1)$ and $v - 1$ blocks of length 2, namely, $(i, v)$, where $i$ ranges from 1 to $v - 1$.

In graph-theoretic terms, we cover $K_v$ by using $K_{v-1}$ together with all the $K_2$s (that is, edges) that emanate from the point not in the $K_{v-1}$.

Case 3. $k_1 > 2$.

Here $f_2 = 1 - \frac{(k_1-1)(k_1-2)}{v+1-2k_1} < 1$.

But $f_2$ is an integer, and so $f_2 = 0$ and $f_1 = v$. Hence

$$\frac{(k_1 - 1)(k_1 - 2)}{v + 1 - 2k_1} = 1,$$

whence $v = k_1^2 - k_1 + 1$.

Thus we have $v$ short blocks forming a BIBD with parameters $(k_1^2 - k_1 + 1, k_1^2 - k_1 + 1, k_1, k_1, 1)$ in the case $k_1 > 2$. The BIBD with $k_1 = 2$ is provided by Lemma 4.1.

The result of this section, that $K_v$ can only by covered (non-trivially) by a near-pencil (Case 2) or a finite projective geometry (Case 3 and Lemma 4.1) was first obtained (for the dual situation) by Erdös and de Bruijn in 1948 [3].

## 6 Some Results for General $\lambda$

We have seen that, for $\lambda = 1$, there can be a single long block. Let us now investigate whether this can occur for $\lambda > 1$. Suppose

$$f_2 = 1 = \lambda - \frac{(k - 1)(k - 2\lambda)}{v + 1 - 2k}, f_1 = v - 1.$$

For convenience, we have set $k_1 = k$. Then

$$\frac{(k-1)(k-2\lambda)}{v+1-2k} = \lambda - 1,$$

whence $v = \frac{(k-1)(k-2)}{\lambda-1} + 1$.

We then have 1 block of length $v+1-k$ and $v-1$ blocks of length $k$. Let $r_a$ be the frequency of elements from the long block in the short blocks; let $r_b$ be the frequency of elements not in the long block. Then

$$f_b(k-1) = \lambda(v-1),$$

$$f_b = \frac{\lambda(v-1)}{k-1} = \frac{\lambda(k-2)}{\lambda-1}.$$

If $\lambda$ of these other elements occurred in each of the $v-1$ short blocks, the number of pairs from these elements would be

$$\binom{\lambda}{2}(v-1) = \frac{\lambda(k-1)(k-2)}{2} = \lambda\binom{k-1}{2}.$$

But there are exactly $k-1$ elements not in the long block, and so this is the proper number of pairs. Any deviation from an equal number of elements per block would increase the pair count [5]; hence we have

**Lemma 6.1.** *The elements not in the single long block intersect the short blocks in exactly $\lambda$ elements.*

We now look at the elements in the long block; for any element, we have

$$(v-k) + r_a(k-1) = \lambda(v-1)$$

Then

$$\begin{aligned} r_a(k-1) &= \lambda(v-1) - (v-1) + (k-1) \\ &= (\lambda-1)(v-1) + k - 1 \\ &= (k-1)(k-2) + k - 1. \end{aligned}$$

Thus $r_a = k - 2 + 1 = k - 1$.

Now let us look at the $v-1$ short blocks. Pick a specific block and let $d_i$ be the number of $i$-elemet intersections with the other short blocks. We have:

$$\sum d_i = d_0 + d_1 + d_2 + d_3 + \cdots = v - 2.$$

$$\begin{aligned} \sum i d_i &= \lambda(r_b - 1) + (k-\lambda)(r_a - 1) \\ &= \frac{\lambda^2(k-2)}{\lambda-1} - \lambda + (k-\lambda)(k-2) \end{aligned}$$

$$\sum \binom{i}{2} d_i = \binom{k}{2}(\lambda-1) - \binom{k-\lambda}{2}.$$

Now it is well known that, for all $s$,

$$\begin{aligned} D(s) &= \frac{s(s+1)}{2}\sum d_i - s\sum id_i + \sum \binom{i}{2} d_i \\ &= \sum d_i \left(\frac{s(s+1)}{2} - si + \frac{i(i-1)}{2}\right) \\ &= \sum d_i \frac{s^2+s-2si+i^2-i}{2} \\ &= \sum d_i \frac{(s-i)(s-i+1)}{2} \geq 0. \end{aligned}$$

Let us form $D(\lambda-1)$. Then

$$\begin{aligned} D(\lambda-1) &= \frac{\lambda(\lambda-1)}{2}(v-2) - (\lambda-1)(k^2-\lambda k-2k+\lambda) \\ &\quad - \lambda^2(k-2) + \binom{k}{2}(\alpha-1) - \binom{k-\lambda}{2} \\ &= 2\lambda - k. \end{aligned}$$

But $f_2 = 1$ and hence $k > 2\lambda$.

Thus $D(\lambda-1) < 0$ and this is a contradiction since $D(s) \geq 0$ for all $s$. Hence we have

**Theorem 6.1.** *For $\lambda > 1$, it is not possible to have only 1 long block.*

Theorem 6.1 was originally given, from a matric point of view, by Bridges [2].

## 7 The Case $\lambda = 2$

We will now discuss the case $\lambda = 2$ from the design-theoretic point of view (cf. [4] for $\lambda = 2$, [1] for $\lambda = 3$). We start from the frequency relation

$$f_2 = 2 - \frac{(k_1-1)(k_1-4)}{v+1-2k_1}$$

The case $k_1 = 1$ gives the degenerate solution of 2 complete blocks and $v-2$ singletons.

The case $k_1 = 2$ yields $f_2 = 2 + \frac{2}{v-3}$. So $v-3 = 1$ or $v-3 = 2$. If $v = 4$, then $f_2 = 4$, $f_1 = 0$, and all blocks are long; we have the BIBD $(4,4,3,3,2)$. If $v = 5$, then $f_2 = 3$, $f_1 = 2$, $k_2 = 4$. So we need 3 blocks of length 4, 2 of length 2. Suppose an element occurs $\alpha$ times in the long blocks, $\beta$ times in the short blocks; then $3\alpha + \beta = 8$ and the only solution is $\alpha = \beta = 2$. Clearly we can not have 5 elements each occurring twice in the 2 short blocks.

The case $k_1 = 3$ yields $f_2 = 2 + \frac{2}{v-5}$. So $v - 5 = 1$ or 2, that is, $v = 6$ or 7. If $v = 6$, there are 4 blocks of length 4, 2 blocks of length 4. With $\alpha$ and $\beta$ again denoting frequencies in the long and short blocks, we have $3\alpha + 2\beta = 10$, whence $\alpha = \beta = 2$. But we can not have 6 elements occurring twice in the short blocks. On the other hand, if $v = 7$, there are 3 blocks of length 5, 4 blocks of length 3. With the usual notation, $4\alpha + 2\beta = 12$, whence $(\alpha, \beta) = (3, 0)$ or $(2, 2)$. Suppose there are $p$ varieties of type $(3, 0)$ and $q$ of type $(2, 2)$; then $p + q = 7$, $3p + 2q = 15$. It follows that $p = 1$, $q = 6$.

We now discuss this case. The long blocks all have the form $1xxxx$, $1xxxx$, $1xxxx$. The short blocks have the form $xxx, xxx, xxx, xxx$. The elements 2,3,4,5,6,7, occur twice each in both long and short blocks. So, if we remove the element 1 from the long blocks and add it to the short blocks, we have constructed a BIBD (7,7,4,4,2), a well known design which is just the complement of the BIBD (7,7,3,3,1). So this case does produce a design which can be written as

| | | |
|---|---|---|
| 12345 | 246 | 347 |
| 12367 | 257 | 356 |
| 14567 | | |

This is an example of a special design that is always derivable from the BIBD $(4\lambda - 1, 4\lambda - 1, 2\lambda, 2\lambda, \lambda)$ of Lemma 4.1. Simply delete one element from $2\lambda$ blocks and add it to the other $2\lambda - 1$ blocks (the current case is for $\lambda = 2$). The result is a design with $2\lambda - 1$ blocks of length $2\lambda + 1$, $2\lambda$ blocks of length $2\lambda - 1$.

Next, we come to the case $k_1 = 4$. Then $f_2 = 2$, $f_1 = v - 2$, $k_2 = v - 3$. With the usual $\alpha$, $\beta$ notation, we have $\alpha(v - 4) + 3\beta = 2(v - 1)$. We find the solutions $(\alpha, \beta) = (2, 2)$ or $(1, \frac{v+2}{3})$ or $(0, \frac{2v-2}{3})$.

Suppose there are $p$, $q$, $r$, elements of each type; then $p + q + r = v$, $2p + q = 2(v - 3)$. It follows that $q + 2r = 6$ and so $p - r = v - 6$. We may thus set $(p, q, r) = (v + r - 6, 6 - 2r, r)$. Now consider the distribution of the $p$ elements in the $v - 2$ short blocks; since each occurs twice, we have $2p \leq v - 2$, that is, $2v + 2r - 12 \leq v - 2$, $v \leq 10 - 2r$. Also $k_1 = 4$, $k_2 > 4$; hence $v \geq 8$. Thus we must have $r = 0$ or 1. In either case $q > 0$, and so some elements occur once in the long blocks, $\frac{v+2}{3}$ times in the short blocks. Hence $v \equiv 1 \pmod 3$ and so $v = 10$, $r = 0$, $q = 6$, $p = 4$. Now delete the 4 elements (1,2,3,4) that occur twice in the long blocks. We are left with a BIBD on 6 elements with parameters (6,10,5,3,2). For this design, we have

$$\sum d_i = 9, \ \sum i d_i = 12, \ \sum \binom{i}{2} d_i = 3.$$

Then

$$D(1) = \sum d_i - \sum i d_i + \sum \binom{i}{2} d_i$$
$$= d_0 + d_3 = 0.$$

But the 2 triples obtained from the 2 long blocks are disjoint, and this contradiction rules out the case $k_1 = 4$.

Finally, if $k_1 > 4$, the relation

$$f_2 = 2 - \frac{(k_1 - 1)(k_1 - 4)}{v + 1 - 2k_1}$$

shows that $f_2 < 2$. But $f_2 = 1$ is ruled out by Theorem 6.1. And $f_2 = 0$ means that all blocks are short blocks of length $k_1$ forming a BIBD with $(k_1 - 1)(k_1 - 4) = 2(v + 1 - 2k_1)$. This gives $v = 1 + \binom{k_1 - 1}{2}$ and the design is a BIBD $\left(1 + \binom{k_1}{2}, 1 + \binom{k_1}{2}, k_1, k_1, 2\right)$, $k_1 > 4$.

Summing up, the only solutions for $\lambda = 2$ turn out to be the biplanes (BIBDs with $v = b = 1 + \binom{k_1}{2}$, $r = k = k_1$, $\lambda = 2$) and the single design on 7 elements with 3 blocks of length 5, 4 blocks of length 3. It is worth noting that the biplanes come out in three ways: with $v = 4$, all blocks are "long" and of length 3; with $v = 7$, we have $k_1 = k_2 = 4$ in Lemma 4.1; with $k_1 > 4$, we get all the remaining biplanes (all blocks "short" and of length $k_1$).

## 8 Conclusion

We have derived some general results connected with Ryser Designs and shown how these can be used to carry out the discussion for $\lambda = 1$ and $\lambda = 2$. Further discussion will appear in a second paper.

## References


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