The spectrum of the values k for which there exists a complete k-arc in $PG(2, q)$ for $q \leq 23$

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ABSTRACT. Arcs and linear maximum distance separable (M.D. S.) codes are equivalent objects [25]. Hence, all results on arcs can be expressed in terms of linear M.D.S. codes and conversely. The list of all complete k-arcs in $PG(2, q)$ has been previously determined for $q \leq 16$. In this paper, (*i*) all values of k for which there exists a complete k-arc in $PG(2, q)$, with $17 \leq q \leq 23$, are determined; (*ii*) a complete k-arc for each such possible k is exhibited.

1 Introduction

In the projective plane $PG(2, q)$ over the Galois field $GF(q)$, a k-arc K is a set of k points, no three of which are collinear. The $(q+1)$ -arcs of $PG(2, q)$ are called *ovals*. A k-arc K is called *complete* if it is not contained in a $(k+1)$-arc of the same projective plane. For a detailed description of the most important properties of these geometric structures, we refer the reader to [6]. In [7], [8], and [24], the close relationship between the theory of complete k-arcs, coding theory and mathematical statistics is presented. Partly because of this relationship, in recent years the interest and research on the fundamental problem of determining the spectrum of the values of k for which there exist complete k-arcs in $PG(2, q)$ have increased considerably.

In this article, all values of k for which there exists a complete k-arc in the planes $PG(2, 17)$, $PG(2, 19)$, $PG(2, 23)$ are determined and an example is given in each case. For smaller values of q see Section 2.

The authors have also obtained some geometric and algebraic characterizations of such complete k-arcs but the details are too involved to be reported here. So they will be presented in another article.

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2 Notation and Background

With respect to the possible sizes of complete k-arcs in $PG(2,q)$ some general results are of particular interest. The cardinality of the largest arc in $PG(2,q)$ is denoted by $m_2(2,q)$, while the cardinality of the smallest complete arc is denoted by $n_2(2,q)$. In this context, also the cardinality of the second largest complete k-arc in $PG(2,q)$ is significant. Let us denote this by $m_2'(2,q)$. The definition of $m_2'(2,q)$ can be formulated as an embedding theorem: any complete k-arc having more than $m_2'(2,q)$ points can be embedded in an *oval*. We list some results that will be helpful in the following sections.

Theorem 2.1. (Bose [2]) $m_2(2,q) = \begin{cases} q+1 & \text{if } q \text{ is odd} \\ q+2 & \text{if } q \text{ is even} \end{cases}.$

Theorem 2.2. (Segre [18]) *In $PG(2,q)$, q odd, every oval is an irreducible conic.*

Theorem 2.3. (Segre [19]) *Any q-arc of $PG(2,q)$, q odd, is contained in a conic.*

Theorem 2.4. (Blokhuis [1]) $n_2(2,p) > \sqrt{3p} + 1/2$, *$p$ prime.*

Theorem 2.5. *In $PG(2,q)$, q odd, there exist*

(a) (Lombardo Radice [12], Pellegrino [15]) *complete $((q+3)/2)$-arcs;*

(b) (Korchmáros [10], Pellegrino [14]) *complete $((q+5)/2)$-arcs;*

(c) (Korchmáros [10], Pellegrino [16]) *complete $((q+7)/2)$-arcs for $q = 4t-1$, $t \neq 2^r$, or $q = 2p-1$, p odd prime.*

The catalogue of *all* complete k-arcs in $PG(2,q)$ (up to projectivity) is known for $q \leq 16$. For $q \leq 9$, the lists can be found in [6]; for $q = 11$ and $q = 13$ the lists were computed by Gordon [5]; independently, P. Lisonek [11] and T. Penttila et al. [17] have listed the full catalogue of complete k-arcs in $PG(2,16)$. Table 2.1 summarizes the results on this topic. In the table, an entry $k(n)$ indicates that there are n projectively distinct complete k-arcs for the given order q.

q	$n_2(2,q)$	Size k of the known complete arcs with $n_2(2,q) < k < m_2'(2,q)$	$m_2'(2,q)$	$m_2(2,q)$
2	4(1)	-	4	4(1)
3	4(1)	-	4	4(1)
4	6(1)	-	6	6(1)
5	6(1)	-	6	6(1)
7	6(2)	-	6	8(1)
8	6(3)	-	6	10(1)
9	6(1)	7(1)	8(1)	10(1)
11	7(1)	8(9), 9(3)	10(1)	12(1)
13	8(2)	9(30), 10(21)	12(1)	14(1)
16	9(6)	10(1944), 11(113), 12(32)	13(1)	18(2)

Table 2.1. Orders of complete k-arcs in $PG(2,q)$, $q \leq 16$

Two k-arcs in $PG(2,q)$ are *equivalent* if there is a projectivity which maps one onto the other. We are interested in the equivalence classes with respect to this relation, i.e. the orbits of k-arcs under the action of the group of projectivities of $PG(2,q)$.

As a consequence of Theorems 2.1 - 2.4, a complete non-oval k-arc exists in $PG(2,p)$, p prime, only if

$$\sqrt{3p} + 1/2 < k < p.$$

Lemma 1. ([4]). *The number of orbits of 5-arcs under the group of the projectivities of the projective plane over a field of prime order $p > 5$ is:*

$$O_{(5,p)} = \frac{p^2 + 10p + 35 + 20(-3/p) + 30(-1/p) + 24(5/p)}{120}$$

where (m/p) is the Legendre symbol.

Since a coordinate system of a projective plane $PG(2,q)$, $q > 2$, is defined by a 4-arc and since the group of projectivities is transitive on the coordinate systems of $PG(2,q)$, we can map every complete k-arc of $PG(2,q)$ onto an equivalent k-arc which contains the canonical coordinate system

$$\{U_1 \equiv (1,0,0), U_2 \equiv (0,1,0), U_3 \equiv (0,0,1), U \equiv (1,1,1)\}$$

by an element of $PGL(3,q)$.

Lemma 2. *In $PG(2,p)$, for any prime $p > 5$, choose $O_{(5,p)}$ of its mutually non-equivalent $5-arcs$, $K_1, ..., K_{O_{(5,p)}}$ containing the four points of the canonical coordinate system. Then every complete k-arc is equivalent to some complete k-arc which contains a $5-arc$ $K_j \in \{K_1, ..., K_{O_{(5,p)}}\}$.*

Proof: Using Lemma 1 and the transitivity properties of the group of projectivities of $PG(2,p)$, the result follows immediately.

Applying Lemma 2, an exhaustive computer search to produce a complete k-arc for each admissible k in a fixed $PG(2,p)$, can be performed by trying to complete in turn every one of the 5-arcs $K_1, ..., K_{o_{(5,p)}}$ by adding appropriate points of the plane.

Our computer program tried to find the complete k-arcs by an exhaustive backtrack search. The procedure followed to obtain our complete k-arcs in $PG(2,p)$ was as follows. Lists were stored of the lines through each point of the plane, and of the points lying on each line. There were also two 2-dimensional Boolean arrays used to describe the status of every point and line of the plane with respect not only to the arc currently under consideration, but also to arcs formed by the first 6, 7, 8, ... points of that arc. In each case, it was recorded whether or not a given point had the property of lying on no secant of the arc, and whether or not a given line was exterior. When a point was being added to the arc, the program considered in turn the lines through this point. If such a line was exterior to the old arc, it had to be noted as not exterior to the new arc: in the contrary case, it was necessary to go through the list of points on that line, and note that each of them now lay on a secant. When the last point that had been added to the arc was to be deleted, the properties of the remaining arc were still available in the store. The programs are available from the authors.

In the following sections we describe the results of our computer search on the spectrum of the values of k for which complete k-arcs in $PG(2,p)$, $p = 17, 19, 23$, exist.

3 The spectrum of the sizes of the complete k-arcs in $PG(2,17)$

We summarize the results of this section in the following

Theorem 3.1. *The spectrum of the sizes of the complete k-arcs in* $PG(2,17)$ *is:* $\{10, 11, 12, 13, 14, 18\}$.

Proof: From Theorems 2.1 and 2.5, it follows that complete k-arcs with $k = 10, 11$, and 18 exist, and that $m_2(2,17) = 18$; also, by Theorems 2.3 and 2.4, non-oval complete k-arcs exist only if $7 < k < 17$.

Our search was finished without finding complete k-arcs with $k \leq 9$ and $k = 15, 16$, but we found complete k-arcs with $k \in \{10, 11, 12, 13\}$, and a unique (up to projectivity) complete 14-arc. So we are able to present a complete k-arc H_k for each $k \in \{10, 11, 12, 13, 14\}$. Since every such k-arc H_k contains the point set $S = \{U_1 = (1,0,0), U_2 = (0,1,0), U_3 = (0,0,1), U_4 = U = (1,1,1), U_5 = (1,2,3), U_6 = (1,3,2)\}$, below we only list the points of the $(k-6)$-*arcs* $C_k := H_k \backslash S$, for all $k \in \{10, 11, 12, 13, 14\}$.

$C_{10} = \{(1,4,5), (1,5,4), (1,6,13), (1,10,7)\}$.
$C_{11} = \{(1,4,5), (1,5,4), (1,6,13), (1,14,6), (1,16,9)\}$.
$C_{12} = \{(1,4,5), (1,5,4), (1,6,14), (1,7,11), (1,9,12), (1,12,9)\}$.
$C_{13} = \{(1,4,10), (1,5,6), (1,7,11), (1,8,4), (1,11,7), (1,12,5), (1,16,15)\}$.
$C_{14} = \{(1,4,10), (1,5,6), (1,7,11), (1,10,13), (1,11,7), (1,12,5), (1,14,9), (1,16,15)\}$.

We remark that the existence (and the uniqueness) of the complete 18-arc is a simple consequence of Theorems 2.1 and 2.2. So the proof of our Theorem now ends.

Remark 1: It has recently come to the attention of the authors that Penttila and Royle [17] have completed an exhaustive computer search for complete arcs in the projective plane under consideration, generally from alternate viewpoint, finding (up to equivalence) 560 complete 10-arcs, 2644 complete 11-arcs, 553 complete 12-arcs, 8 complete 13-arcs, a unique complete 14-arc, and so they reached the same conclusion about the possible sizes.

4 The spectrum of the sizes of the complete k-arcs in $PG(2,19)$

As in the previous section, we summarize the results in the following

Theorem 4.1. *The spectrum of the sizes of the complete k-arcs in $PG(2,19)$ is:* $\{10, 11, 12, 13, 14, 20\}$.

Proof: From Theorems 2.1 and 2.5, it follows that complete k-arcs with $k = 11, 12, 13, 20$ exist, and that $m_2(2,19) = 20$. Also, by Theorems 2.3 and 2.4, non-oval complete k-arcs exist only if $8 < k < 19$.

Our search was finished without finding complete k-arcs with $k \leq 9$ and $k = 15, 16, 17$ and 18. But we found complete k-arcs with $k \in$\{10, 11, 12, 13, 14\}. So we are able to present a complete k-arc H_k for each $k \in$\{10, 11, 12, 13, 14\}. Below we only list the points of the $(k-6)$ -arcs $C_k := H_k \backslash S$, for all $k \in$\{10, 11, 12, 13, 14\}, where S is the same set as in Section 3.

$C_{10} = \{(1,4,8), (1,6,16), (1,7,10), (1,8,7)\}$.
$C_{11} = \{(1,4,5), (1,5,4), (1,6,8), (1,7,16), (1,15,11)\}$.
$C_{12} = \{(1,4,5), (1,5,4), (1,6,8), (1,7,16), (1,8,6), (1,14,11)\}$.
$C_{13} = \{(1,4,5), (1,5,4), (1,6,8), (1,7,16), (1,8,13), (1,11,14), (1,16,6)\}$.
$C_{14} = \{(1,4,5), (1,5,4), (1,6,8), (1,8,6), (1,11,14), (1,13,16), (1,14,11), (1,16,13)\}$.

We remark that the existence (and the uniqueness) of the complete 20-arc is a simple consequence of Theorems 2.1 and 2.2. So the proof of our Theorem now ends.

Remark 2: As in the previous section, it has recently come to the attention of the authors that Penttila and Royle [17] have completed an exhaustive computer search for complete arcs in the projective plane $PG(2,19)$, generally from alternate viewpoint, finding (up to equivalence) 29 complete 10-arcs, 9541 complete 11-arcs, 30135 complete 12-arcs, 2232 complete 13-arcs, 70 complete 14-arcs, and so they reached the same conclusion about the possible sizes.

5 The spectrum of the sizes of the complete k-arcs in $PG(2,23)$

Theorem 5.1. *The spectrum of the sizes of the complete k-arcs in $PG(2,23)$ is:* $\{10,12,13,14,15,16,17,24\}$.

Proof: By application of our Lemmas 1 and 2 on orbits of arcs in projective planes, we were able to show that $n_2(2,23)=10$ and that (up to projectivity) there is a unique 10-arc in $PG(2,23)$. From Theorems 2.1 and 2.5, it follows that complete k-arcs with $k=13,14,15,24$ exist. Besides, we were able to produce complete k-arcs for $k\in\{12,16,17\}$. Since Kaneta showed that $m_2'(2,23)=17$ (see [9]), for the problem of determining the spectrum of the values of k for which there exist complete k-arcs in $PG(2,23)$ it remains to prove the existence of complete 11-arcs. Our search was finished without finding complete 11-arcs.

As in the previous sections, we present only one complete k-arc H_k for each possible size.

$$H_{10}=\{U_1,U_2,U_3,U,(1,2,3),(1,3,11),(1,4,2),(1,9,15),(1,11,19),(1,15,16)\}.$$

Below we only list the points of the $(k-6)$-arcs $C_k := H_k\backslash S$, for all $k\in\{12,13,14,15,16,17\}$, where S is the same set as in Section 3.

$C_{12}=\{(1,4,5),(1,5,4),(1,6,8),(1,7,11),(1,8,21),(1,11,16)\}$.
$C_{13}=\{(1,4,5),(1,5,4),(1,6,8),(1,7,11),(1,8,21),(1,11,9),(1,20,10)\}$.
$C_{14}=\{(1,4,5),(1,5,4),(1,6,8),(1,7,11),(1,8,21),(1,11,9),(1,12,22),(1,20,17)\}$.
$C_{15}=\{(1,4,5),(1,5,4),(1,6,8),(1,7,11),(1,9,16),(1,10,6),(1,11,9),(1,17,12),(1,20,17)\}$.

$C_{16} = \{(1,4,5),(1,5,4),(1,6,8),(1,7,11),(1,12,7),(1,14,16),(1,16,19),$
$(1,18,13),(1,19,22),(1,22,15)\}.$
$C_{17} = \{(1,4,5),(1,5,18),(1,6,13),(1,8,14),(1,9,7),(1,10,8),(1,12,20),$
$(1,14,11),(1,17,12),(1,19,21),(1,22,19)\}.$

Finally, the existence and the uniqueness of the complete 24-arc is a simple consequence of Theorems 2.1 and 2.2.

6 Concluding Remarks

The problem of compiling the full catalogue of complete k-arcs in $PG(2,q)$ for $q \geq 25$ seems to be extremely hard. For example, with respect to $m_2'(2,q)$, q small, only the following values are known (see [3], [8], [21]): $m_2'(2,25) = 21$, $m_2'(2,27) = 22$, $m_2'(2,64) = 57$.

Regarding the exact size of the smallest complete k-arcs in $PG(2,q)$ for $q \geq 25$ no exact results have been obtained. However, some results are known, see [13].

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