On k-packings of Graphs

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Abstract

A set $\mathcal{P} \subseteq V(G)$ is a k-packing of a graph G if for every pair of vertices u, v in $\mathcal{P}$, $d(u,v) \geq k+1$. We define a graph G to be *k-equipackable* if every maximal k-packing of G has the same size. In this paper, we construct, for $k \geq 1$, an infinite family $\mathcal{F}_k$ of k-equipackable graphs, recognizable in polynomial time. We prove further that for graphs of girth at least $4k+4$, every k-equipackable graph is a member of $\mathcal{F}_k$.

1 Introduction

A set $\mathcal{P} \subseteq V(G)$ is a *k-packing* of a graph G if, for every pair of vertices u, v in $\mathcal{P}$, $d(u,v) \geq k+1$. This concept seems to have been introduced by Meir and Moon [9], who defined the *k-packing number* of G to be the number $\pi_k(G)$ of vertices in any largest k-packing of G. We call a graph G *k-equipackable* if every maximal k-packing of G has the same size. We note that whereas the problem of determining the k-packing number of a graph is known to be hard in general, and remains so even when the graphs is restricted to certain classes (see, for example, [1], [2], [3], [8]), in the case of a k-equipackable graph it can be determined by a greedy algorithm, for $k \geq 1$.

The case that has received the most attention in the literature is when $k = 1$. A 1-packing of a graph G is called an *independent set* and the 1-packing number is called the *independence number* of G. Graphs which are 1-equipackable are called *well-covered*, a concept introduced by Plummer [10]. Some progress has been made on characterizing well-covered graphs subject to certain additional

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conditions (see in particular [11] for an excellent survey of different approaches and progress), but a characterization of all well-covered graphs appears out of reach at present.

In this paper, we describe in Section 2 how, for given $k \geq 1$, an infinite family $\mathcal{F}_k$ of k-equipackable graphs can be constructed and present a polynomial algorithm for deciding whether a given graph is in $\mathcal{F}_k$. We then prove, in Section 3, that every k-equipackable graph of girth $4k+4$ or more is a member of $\mathcal{F}_k$. This gives a complete characterization of k-equipackable graphs of girth $4k+4$ or more. We remark that an approach based on limiting the girth of the graph is one which has proved fruitful in characterizing well-covered graphs (see [4], [5]).

A related concept to k-packing, is k-domination. A *k-dominating set* of a graph G is a set $\mathcal{D} \subseteq V(G)$ such that for every vertex $v \in V(G)$, there is at least one vertex $x \in \mathcal{D}$ such that $d(v, x) \leq k$. The minimum size of a k-dominating set is called the *k-domination number* of G and denoted by $\gamma_k(G)$. In section 4, we show that every graph in the family $\mathcal{F}_{2k}$ has the additional property that its $2k$-packing number is equal to its k-domination number.

We use the following definitions and notation. A vertex of degree 1 is called a *leaf* and a vertex of degree 2 or more that is adjacent to a leaf is called a *stem*. For any pair of vertices u, v of a connected graph G, the *distance* between u and v is the length of the shortest $[u, v]$-path in G and denoted by $d(u, v)$. The maximum value of $d(u, v)$, taken over all pairs $u, v \in V(G)$, is called the *diameter* of G and denoted by diam G. A path of length diam (G) is called a *diametrical path*.

The set of vertices at distance j from a vertex v is denoted by $N_j(v)$ and the set at distance at most j from v is denoted by $N_j[v]$. For $u, v \in V(G)$, $u \neq v$, $N_j[u, v]$ denotes the set $N_j[u] \cup N_j[v]$. For $S \subseteq V(G)$, $\langle S \rangle$ denotes the subgraph of G induced by the set S.

All graphs considered in this paper are finite. Since a graph G is k-equipackable if and only if each of its connected components is k-equipackable, all graphs are assumed to be connected. Further, since any graph of diameter k or less is k-equipackable, with $\pi_k(G) = 1$, we shall restrict our discussion to graphs G for which diam $G > k$. Throughout this paper, we shall denote $\lfloor k/2 \rfloor$ by r.

2 Construction of $\mathcal{F}_k$

Let k be a positive integer, and let $r = \lfloor k/2 \rfloor$. Let G be a connected graph with diam $G > k$. An induced subgraph $\langle V' \rangle$ of G is called a *k-basic* subgraph if V' contains

(a) a central vertex b such that $V' = N_r[b]$, in the case when k is even; two adjacent central vertices b, b' such that $V' = N_r[b, b']$, in the case when k is odd;

(b) a vertex x such that $d(x, b) = r$ and, for all $u \in V(G) - V'$, every $[x, u]$-path contains the vertex b in the case when k is even and the edge bb' in the case when k is odd.

The vertex b is called an *r-branchpoint* of $\langle V' \rangle$ and the vertex x a *remote* vertex of $\langle V' \rangle$.

We say that G belongs to the family $\mathcal{F}_k$ if $V(G)$ can be partitioned into a finite number $m \geq 2$ of pairwise disjoint subsets $V_1, V_2, ..., V_m$ such that, for $i = 1, 2, ..., m$, $\langle V_i \rangle$ is a k-basic subgraph of G.

An example of a graph G in $\mathcal{F}_k$ for each of the values $k = 2$ and $k = 3$ is shown in Figures 1(a) and 1(b), respectively. In these figures, an r-branchpoint of each k-basic subgraph is denoted by a filled square (in both cases, $r = 1$) and the remote vertices are denoted by filled circles.

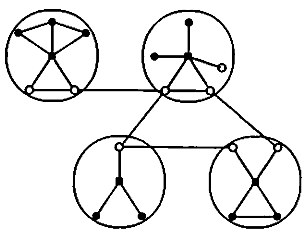


Fig.1(a)

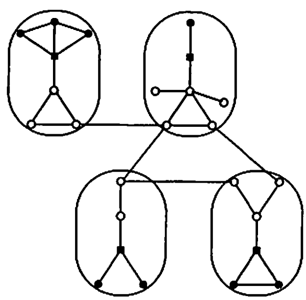


Fig.1(b)

We can make some immediate deductions concerning the structure of a k-basic subgraph B of a graph G in $\mathcal{F}_k$.

Lemma 2.1 *Let $G \in \mathcal{F}_k$ and B denote a k-basic subgraph of G. Suppose that b is an r-branchpoint and x a remote vertex of B. Additionally, in the case when $k = 2r+1$, let b' denote a vertex adjacent to b such that for all $u \in V(G) - V(B)$, every $[x, u]$-path contains the edge bb' and $B = \langle N_r[b, b'] \rangle$. Then*

(i) *diam $B = k$, and every vertex of $V(G) - V(B)$ is at distance at least $k+1$ from x;*

(ii) *b is the only r-branchpoint of B;*

(iii) *in the case when k is odd, b' is also uniquely determined.*

Proof. By property (a), diam $B \leq k$. Let $u \in V(G) - V(B)$. Then the shortest $[u, x]$-path contains b when $k = 2r$ and bb' when $k = 2r + 1$. However, $d(u, b) \geq r + 1$ when $k = 2r$, and $d(u, b) \geq r + 2$ when $k = 2r + 1$. Hence, in either case, $d(u, x) = d(u, b) + d(b, x) \geq k + 1$. Thus diam $B = k$, establishing (i).

Suppose that b, c are both r-branchpoints of B. Let x be a remote vertex corresponding to b, and let $u \in V(G) - V(B)$ be such that $d(u, x) = k + 1$. Suppose first $k = 2r$. Let P denote a shortest $[u, x]$-path. Then P contains b, which is thus the vertex of P at distance $r + 1$ from u. Let w be the vertex of P adjacent to u. Then $w \in B$ and hence $d(w, c) \leq r$. Since $u \notin B$, we have $d(w, c) = r$. Let Q_1, Q_2 denote respectively a shortest $[w, c]$-path and a shortest $[c, x]$-path. Then since $d(c, x) \leq r$, the $[u, x]$-path $Q = uw + Q_1 + Q_2$ has length at most $2r + 1 = k + 1$. Thus Q has length exactly $k + 1$ and is a shortest $[u, x]$-path. But then Q contains b, and the $[b, x]$ section of Q is a shortest $[b, x]$-path. Hence b is the vertex of Q at distance r from x and $r + 1$ from u, and so $b = c$. Thus the r-branchpoint of a $2r$-basic subgraph is unique.

In the case when $k = 2r + 1$, a similar argument establishes both (ii) and (iii). □

In general, a k-basic subgraph may contain more than one remote vertex, as in the examples shown in Figure 1. However, in the special case when $k = 1$ (so that $r = 0$), a remote vertex of a 1-basic subgraph coincides with its 0-branchpoint. Hence, in this case, each 1-basic subgraph contains a unique remote vertex, and this vertex is necessarily a leaf of G. Thus a 1-basic subgraph can be very simply described: it is isomorphic to K_2.

Theorem 2.2 *Let $G \in \mathcal{F}_k$ and let $B_1, B_2, ..., B_m$ denote the complete collection of k-basic subgraphs of G. Then G is k-equipackable and $\pi_k(G) = m$.*

Proof. Let $\mathcal{P}$ be a maximal k-packing of G. Since diam $B_i = k$, $1 \leq i \leq m$, each k-basic subgraph B_i contains at most one vertex of $\mathcal{P}$. Suppose there exists a k-basic subgraph B_1, say, that contains no vertex of $\mathcal{P}$. Let b denote the r-branchpoint and x denote a remote vertex of B_1. Then since $x \notin \mathcal{P}$, there exists a vertex $s \in \mathcal{P}$ such that $d(s, x) \leq k$. However, $s \in V(G) - V(B_1)$ and hence $d(s, x) \geq k + 1$, by Lemma 2.1. This contradiction establishes that B_1 contains at least one vertex of $\mathcal{P}$. Hence each of the sets B_i, $i = 1, 2, ..., m$, contains exactly one vertex of $\mathcal{P}$. However, every vertex of $\mathcal{P}$ is in at least one of the sets B_i, for $1 \leq i \leq m$. Hence $\pi_k(G) = m$. □

It follows from Theorem 2.2 that for a graph $G \in \mathcal{F}_k$, $\pi_k(G)$ is given by the number of k-basic subgraphs in a collection partitioning G.

Lemma 2.3 *Let $G \in \mathcal{F}_k$ and x, y be end-vertices of a diametrical path Q in G. Then we can find a remote vertex x' of a k-basic subgraph of G such that $d(x', y)$=diam G.*

Proof. Since diam $G > k$, the vertices x, y are in distinct k-basic subgraphs, B, B' say. Suppose that x is not a remote vertex of B. Let au be the edge of Q such that $a \in V(B)$ and $u \in V(G) - V(B)$. Then $d(x, a) \leq k$ and hence

$d(x,u) \leq k+1$. However, B contains at least one remote vertex, x' say. Then $d(x',u) \geq k+1$, by Lemma 2.1 (i). Hence $d(x',y)$=diam G. □

Now suppose that $G \in \mathcal{F}_k$ and Q is a diametrical path in G starting at a remote vertex x of a k-basic subgraph B. Since Q contains a vertex of $V(G) - V(B)$, it contains the unique r-branchpoint b, say, of B and, in the case when k is odd, the central edge bb' of B as well. Thus b can be identified as the vertex of Q at distance r from x (and when k is odd, b' is the vertex of Q at distance $r+1$ from x). We also note that since Q is a diametrical path, it follows that the induced subgraph $\langle V(G) - V(B) \rangle$ is connected. Hence it either belongs to $\mathcal{F}_k$, or it has diameter k and consists of a single basic subgraph.

These renarks are the basis of the following polynomial algorithm for deciding, for a given positive integer k and a given graph G, whether G belongs to $\mathcal{F}_k$ and, in this case, determining $\pi_k(G)$. Suppose first that $k = 2r$.

Algorithm

Step 1. Find a shortest path between each pair of vertices of G and determine diam G. If G is not connected or diam $G \leq k$, then conclude $G \notin \mathcal{F}_k$ and stop. Otherwise, set $G = G_1$ and all vertices unlabelled.

Step 2. Suppose that we have identified a sequence of subgraphs $G_1, G_2, ..., G_{i-1}$ and vertices $b_1, b_2, \ldots, b_{i-1}$, $i \geq 2$, such that for $j \neq l$, $N_r[b_j] \cap N_r[b_l] = \emptyset$, every vertex of $N_r[b_j]$ has a permanent label j, $1 \leq j \leq i-1$, and all other vertices of G are unlabelled. Let U_i be the set of unlabelled vertices of G. If $U_i = \emptyset$, then conclude that $G \in \mathcal{F}_k$ and $\pi_k(G) = i-1$; then stop. Otherwise, let $G_i = \langle U_i \rangle$. Find diam G_i and a shortest path between each pair of vertices in G_i. If G_i is not connected or diam $G_i < k$, then conclude that $G \notin \mathcal{F}_k$ and stop.

Step 3. Let D_i be the set of all vertices that are endpoints of diametrical paths in G_i. Choose a distinguished vertex $y \in D_i$ and let $D_i(y) = \{u \in D_i : d(y,u) = \text{diam } G_i\}$.

Step 4. Choose $x \in D_i(y)$ and let $Q[x,y]$ denote a shortest $[x,y]$-path in G_i. Give the vertex of $Q[x,y]$ at distance r from x the temporary label b_i; give all other vertices of G in $N_r[b_i]$ the temporary label i.

Step 5. If any one of these vertices is already labelled, then delete all temporary labels and set $D_i(y) := D_i(y) - \{x\}$. If now $D_i(y) = \emptyset$, then conclude that $G \notin \mathcal{F}_k$ and stop; otherwise, return to the start of step 4.

Step 6. Find the component of $G - b_i$ containing x. If this component contains only vertices that have the label i, then $\langle N_r[b_i] \rangle$ is a k-basic subgraph of G, with b_i as r-branchpoint and x as a remote vertex. Make the temporary labels permanent; return to start of step 2 and increment $i := i+1$.

Step 7. If diam $G_i = k$, then find the component of $G - b_i$ containing y (this checks whether we have a k-basic subgraph if the roles of x and y are reversed).

If this component contains only vertices that have the label i, then make the temporary labels permanent; return to the start of step 2 and increment $i := i+1$.

Step 8. Conclude that $G \notin \mathcal{F}_k$ and stop.

This algorithm can easily be adapted for the case where $k = 2r+1$, $r \geq 0$. All that is required is that when we label b_i in step 4, we also give the vertex of Q at distance $r+1$ from x_i the label b'_i and substitute $N_r[b_i, b'_i]$ for $N_r[b_i]$ throughout. Similarly, we replace the graph $G - b_i$ by the graph $G - b_i b'_i$. Finally, when we reverse the roles of x_i and y in step 7, we also interchange the labels b_i and b'_i.

It is easily seen that the algorithm is polynomial. Suppose that G has order n. Then the shortest path between any pair of vertices can be found by Floyd's algorithm [6] in $O(n^3)$ operations. Steps 4, 6 and 7 can each be accomplished by a breadth-first search taking $O(n^2)$ operations. Thus the whole algorithm requires at most $O(n^4)$ operations and we have the following result.

Lemma 2.4 *It can be determined in polynomial time whether a given graph G with diam $G > k$ belongs to $\mathcal{F}_k$.*

3 Characterization

In this section, we prove that for graphs of girth at least $4k+4$, all k-equipackable graphs belong to $\mathcal{F}_k$ and are thus recognizable in polynomial time. The proof is divided into several lemmas, all but one of which hold at girth lower than $4k+4$. We give in each case the minimum girth required by the proof, thus giving some partial information on k-equipackable graphs with lower girth. The first result, that if a k-equipackable graph contains more than one k-basic subgraph, then all k-basic subgraphs are pairwise disjoint, holds without girth restriction.

Lemma 3.1 *Let G be a k-equipackable graph and let B_1, B_2 be distinct k-basic subgraphs of G. Then $V(B_1) \cap V(B_2) = \emptyset$.*

Proof. Let b_i be the r-branchpoint and x_i be a remote vertex of B_i, $i = 1, 2$. We show first that x_1, x_2 are distinct. Let $w \in V(B_1) - V(B_2)$. Then, by Lemma 2.1(i), $d(w, x_2) > k$. However, $d(w, x_1) \leq k$. Hence $x_1 \neq x_2$.

Suppose there is a vertex $u \in V(B_1) \cap V(B_2)$. Let $\mathcal{P}$ be a maximal k-packing of G containing u. Then, by Lemma 2.1(i). $\mathcal{P}' = \mathcal{P} \cup \{x_1, x_2\} - \{u\}$ is also a k-packing of G. But $|\mathcal{P}'| > |\mathcal{P}|$, contradicting the assumption that G is k-equipackable. Thus $V(B_1) \cap V(B_2) = \emptyset$. □

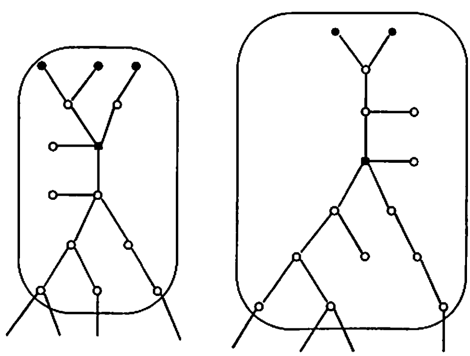
In the remaining results in this section, we establish that every vertex of a k-equipackable graph G of girth $4k+4$ or more is contained in a k-basic subgraph. Our proofs all require a girth restriction on G of at least $3k+3$.

It follows from Lemma 2.1 that for a graph G of girth at least $2r+2$, a k-basic subgraph B of G is a tree. In particular, a remote vertex x of B is a leaf of a subtree of depth r rooted at the r-branchpoint b of B, with the property that any path from a vertex $u \in V(G) - V(B)$ to x must pass through b.

We pointed out just before Theorem 2.2 that every 1-basic subgraph of a graph G in $\mathcal{F}_1$ is isomorphic to K_2, where one vertex of each K_2 is a leaf in G. At girth 4 or more, the structure of a graph G in $\mathcal{F}_2$ is also simple to describe. In this case, each 2-basic subgraph B is a star, with its central vertex as the 1-branchpoint and a leaf (that is also a leaf of G) as remote vertex. A join between two distinct 2-basic subgraphs is an edge incident with a leaf of each star.

A 3-basic subgraph B of a graph $G \in \mathcal{F}_3$ of girth 4 or more consists of a pair of stars with adjacent centres, b, b', say. These stars have the property that every leaf (and there must be at least one) of B adjacent to b is also a leaf of G, while a join between B and another 3-basic subgraph is an edge incident with a leaf of B adjacent to b'. In this case, b is the 1-branchpoint of B and every leaf of B adjacent to b (but no leaf adjacent to b') is a remote vertex of B.

The structure of a k-basic subgraph of a graph of girth $2r+2$ or more in $\mathcal{F}_k$ for larger values of k is a generalization of the case when $k=2$ (if k is even) or $k=3$ (if k is odd). An example of such a k-basic subgraph for each of the values $k=5$ and $k=6$ is shown in Figure 2.



5-basic subgraph   6-basic subgraph

Fig.2

At girth $2k+4$ or more, the structure of the subgraph $\langle N_{k+1}[u] \rangle$, for any vertex $u \in V(G)$, is a tree rooted at u. In this case, for vertices $w, z \in N_{k+1}[u]$, such that w precedes z on the unique $[u,z]$-path in $\langle N_{k+1}[u] \rangle$, we shall say that w is an *ancestor* of z and that z is a *descendant* of w.

Lemma 3.2 *Let G be a k-equipackable graph of girth at least $3k+3$ containing at least one k-basic subgraph. Then, for every $v \in V(G)$, there exists a k-basic subgraph B containing v.*

Proof. Suppose there exists a vertex of G that is not contained in any k-basic subgraph. Then we can find a vertex u and k-basic subgraph B_1 with r-branchpoint b_1, such that $d(u, b_1) = k - r + 1$ and $u \notin V(B)$ for all k-basic subgraphs B of G. It follows from the girth restriction that there is a unique shortest $[b_1, u]$-path in G. Denote this by $b_1 v_1 v_2 ... v_r u$ in the case when $k = 2r$, by $b_1 b_1' v_1 v_2 ... v_r u$ in the case when $k = 2r+1$, for $r \geq 1$, and by $b_1 b_1' u$ when $r = 0$. If $r = 0$, let $v_0 = b_1'$. Let $N_j'(u)$ denote the subset of $N_j(u)$ that contains just those vertices that are at distance $j+1$ from v_r, $j = 1, 2, ..., k+1$.

Consider the induced tree $\langle N_{k+1}[u] \rangle$ rooted at u. We construct a subset $S \subseteq N_{k+1}'(u)$ as follows. If $N_{k+1}'(u) = \emptyset$, put $S = \emptyset$. Otherwise, for each vertex $w \in N_{k-r+1}'(u)$ that has a descendant $z \in N_{k+1}'(u)$, select just one such descendant and put this vertex in S. Then for any $s, t \in S$, we have $d(s,t) \geq k+1$, by the girth restriction. Since also $d(s, v_r) > k$, for all $s \in S$, the set $S \cup \{v_r\}$ is a k-packing and hence can be extended to a maximal k-packing $\mathcal{P}$ of G.

We show that v_r is the only vertex of $N_k[u]$ in $\mathcal{P}$. Note that the only vertices of $N_k[u]$ that are at distance greater than k from v_r are those in $N_k'(u)$. Suppose $y \in N_k'(u)$ and let w be the ancestor of y in $N_{k-r+1}'(u)$. If w has a descendant in $N_{k+1}'(u)$, then w has one such descendant z, say, in S. However, $d(z,y) \leq d(z,w) + d(w,y) = 2r - 1 < k$, and hence $y \notin \mathcal{P}$. So we may assume that w has no descendant in $N_{k+1}'(u)$. This implies in particular that y is a leaf of G. Let w_0 be the predecessor of w on the unique $[u, w]$-path in $\langle N_{k+1}[u] \rangle$. Then for all $v \in V(G) - N_r[w_0]$, every $[v, y]$-path contains w_0. It follows in the case when $k = 2r$, that w_0 is the r-branchpoint and y a remote vertex of a k-basic subgraph $B = \langle N_r[w_0] \rangle$. However, $d(u, w_0) = r$ and hence $u \in V(B)$, contrary to hypothesis.

In the case when $k = 2r+1$, suppose first that w_0 also has no descendant in $N_{k+1}'(u)$. Let w_0' be the predecessor of w_0 on the $[u, w]$-path in $\langle N_{k+1}[u] \rangle$. Then for all $v \in V(G) - N_r[w_0, w_0']$, every $[v, y]$-path contains the edge $w_0' w_0$. Hence $\langle N_r[w_0, w_0'] \rangle$ is a k-basic subgraph containing u, contrary to hypothesis. On the other hand, if w_0 has a descendant in $N_{k+1}'(u)$, then it has one such descendant z, say, such that $z \in S$. However, $d(z,y) \leq d(z, w_0) + d(w_0, y) = 2r+1 = k$, and hence $y \notin \mathcal{P}$.

We conclude that v_r is the only vertex of $N_k[u]$ in $\mathcal{P}$. Let x be a remote vertex of B_1. Then $d(u,x) = k+1$. Further, for any $s \in S$, $d(s,x) > k$. Hence $\mathcal{P}' = \mathcal{P} \cup \{x, u\} - \{v_r\}$ is also a k-packing of G, but with $|\mathcal{P}'| > |\mathcal{P}|$. This contradicts the assumption that G is k-equipackable and thus establishes the result. □

Lemmas 3.1 and 3.2 together give the following.

Corollary 3.3 *Let G be a k-equipackable graph of girth at least $3k+3$ containing at least one k-basic subgraph. Then $G \in \mathcal{F}_k$.* □

Lemma 3.4 *Let G be a k-equipackable graph of girth $4k+4$ or more. Then G contains a leaf.*

Proof. Suppose that G is leafless. Then G is not a tree and hence G contains a cycle of length at least $4k+4$. Let $uv_1v_2...v_kw$ be any $(k+2)$-path in G. Let $N'_j(u)$, $N'_j(w)$ denote the subsets of $N_j(u)$ and $N_j(w)$ respectively, containing just those vertices at distance at least $j+1$ from v_{r+1}.

Since G is leafless and has girth at least $4k+4$, every vertex $z \in N'_{k-r+1}(u) \cup N'_{k-r+1}(w)$ has a descendant $z' \in N'_{k+1}(u) \cup N'_{k+1}(w)$. We construct a subset $S \subseteq N'_{k+1}(u) \cup N'_{k+1}(w)$ by selecting exactly one such descendant z' of each vertex $z \in N'_{k-r+1}(u) \cup N'_{k-r+1}(w)$ and putting this vertex in S. By the girth restriction, $d(s,t) \geq k+1$, for any $s,t \in S$. Since also $d(s,v_{r+1}) > k$, for all $s \in S$, the set $S \cup \{v_{r+1}\}$ is a k-packing and hence can be extended to a maximal k-packing $\mathcal{P}$ of G. However, $\mathcal{P}' = \mathcal{P} \cup \{u,w\} - \{v_{r+1}\}$ is also a k-packing of G, but with $|\mathcal{P}'| > |\mathcal{P}|$, contrary to the assumption that G is k-equipackable. Hence G contains a leaf. □

Lemma 3.5 *Let G be a k-equipackable graph of girth $4k+2$ or more that contains a leaf. Then G contains a k-basic subgraph.*

Proof. The result is clear when $k = 1,2$ and hence we shall assume that $k \geq 3$.

Suppose first that G is a tree. Let x,y be leaves such that $d(x,y) = \operatorname{diam} G$. Let b be the vertex of the unique $[x,y]$-path such that $d(x,b) = r$. Then b is the r-branchpoint of a k-basic subgraph containing x as a remote vertex.

We may thus assume that G contains a cycle of length at least $4k+2$. Let $V_0(G)$ denote the subset of vertices of G that lie on at least one cycle. Suppose that G contains no k-basic subgraph. For each leaf x, let $D(x)$ denote the minimum distance of x from any vertex of $V_0(G)$. Let c be the maximum value of $D(x)$, taken over all leaves x, and let $L = \{x : D(x) = c\}$. For each leaf $x \in L$, let $n(x)$ denote the number of vertices in $V_0(G)$ at distance c from x. Choose a leaf $x_0 \in L$ such that $n(x_0)$ is minimum.

Let x_c be a vertex on a cycle C such that $d(x_0,x_c) = c$ and let $x_0x_1...x_c$ be a shortest $[x_0,x_c]$-path. If $c \geq k+1$, denote the $[x_0,x_{k+1}]$ section of this path by R; otherwise, extend the path by adjoining a section $x_cx_{c+1}...x_{k+1}$ of the cycle C to give the path R.

Consider first the induced tree $\langle N_{k+1}[x_{k+1}] \rangle$ rooted at x_{k+1}. Let $N'_j(x_{k+1})$ denote the subset of $N_j(x_{k+1})$ containing just those vertices at distance $j+1$ from x_k, $j = 1,2,...,k+1$. Construct a subset $S_1 \subseteq N'_{k+1}(x_{k+1})$ as follows. For each vertex $w \in N'_{k-r+1}(x_{k+1})$ that has a descendant $z \in N'_{k+1}(x_{k+1})$, we select just one such descendant and put this vertex in S_1. Then for any $s,t \in S_1$, we have $d(s,t) > k$ and $d(s,x_k) > k$.

Consider next the induced tree $\langle N_{k+1}[x_0]\rangle$ rooted at x_0. Let $N'_j(x_0)$ denote the subset of $N_j(x_0)$ containing just those vertices at distance j or more from x_k, $j = 1, 2, ..., k+1$. Construct a subset $S_2 \subseteq N'_{k+1}(x_0)$ as follows. If $N'_{k+1}(x_0) = \emptyset$, let $S_2 = \emptyset$. Otherwise, for each vertex $w \in N'_{k-r+1}(x_0)$ that has a descendant $z \in N'_{k+1}(x_0)$, select just one such descendant and put this vertex in S_2. Then $d(s,t) > k$ and $d(s, x_k) > k$, for any $s, t \in S_2$. Further, for any $s \in S_1$, $t \in S_2$, we have $d(s,t) \geq k+1$, by the girth restriction. In any case, the set $\{x_k\} \cup S_1 \cup S_2$ is a k-packing of G and hence can be extended to a maximal k-packing $\mathcal{P}$ of G.

We now show that x_k is the only vertex of $N_k[x_0] \cup N_k[x_{k+1}]$ in $\mathcal{P}$. The proof that if G contains no k-basic subgraph, then x_k is the only vertex of $N_k[x_{k+1}]$ in $\mathcal{P}$ is identical to the proof given in Lemma 4.2 that v_r is the only vertex of $N_k[u]$ in $\mathcal{P}$. We shall therefore suppose that $y \in N_k[x_0]$ and $d(y, x_k) \geq k+1$. Let x_m denote the vertex closest to y on the $[x_0, x_k]$ section of R.

Suppose first that y has a descendant in $N_{k+1}(x_0)$. If one such descendant z, say, is in S_2, then since $d(y,z) < d(x_m, z) \leq k$, we have $y \notin \mathcal{P}$. If y has no descendant in S_2, then we must have $y \in N_j(x_0)$, where $k - r + 2 \leq j \leq k$. In this case, y has an ancestor $w \in N_{k-r+1}(x_0)$ and w has a descendant s, say, in S_2. However, $d(y,s) = d(y,w) + d(w,s) < k$ and hence again $y \notin \mathcal{P}$.

We may therefore suppose that y has no descendant in $N_{k+1}(x_0)$. Thus every path in $\langle N_{k+1}[x_0]\rangle$ starting at x_m and containing y terminates in a leaf at distance at most $k-m$ from x_m. Let y_0 be one such leaf (where possibly $y_0 = y$). Let $Q = x_0 x_1 ... x_m y_q ... y_1 y_0$ denote the unique $[x_0, y_0]$-path in $\langle N_{k+1}[x_0]\rangle$. Let v be a vertex of $V_0(G)$ at minimum distance from y_0. Then since $d(y_0, x_k) > d(x_0, x_k)$, it follows that $d(y_0, x_m) > d(x_0, x_m)$. Hence the vertex of Q nearest to v is one of the vertices y_j, $1 \leq j \leq q$, since otherwise $D(y_0) = d(y_0, v) > c$, contradicting the definition of c. Let y_i be the vertex of Q at minimum distance from v (where possibly $y_i = v$). If there is more than one choice for v, then choose v so that i is minimum. Since $y_i \neq x_m$, it follows that v and x_c are distinct.

We note that $d(x_0, v) \geq c$. Suppose first that $d(x_0, v) = d(x_0, y_i) + d(y_i, v) > c$. It follows from the choice of x_0, that $d(y_0, v) = d(y_0, y_i) + d(y_i, v) \leq c$. Hence

$$d(y_0, y_i) < d(x_0, y_i). \tag{1}$$

Now suppose that $d(x_0, v) = c$. If $d(y_0, v) = c$, the shortest path from y_0 to a vertex $u \in V_0(G)$ at distance c must include y_i but no vertex of Q between y_i and x_0, since otherwise $d(x_0, u) < c$. But, then $n(x_0) > n(y_0)$, contradicting the choice of x_0. Hence this case cannot occur. We must therefore have $d(y_0, v) < c$, and the inequality (1) holds again.

Since y_i has a descendant v on a cycle, it follows from the girth restriction that y_i has a descendant $z \in N_{k+1}(x_0)$. Now $d(y,z) \leq d(y_0, z) = d(y_0, y_i) + d(y_i, z) < d(x_0, z)$, by (1). Hence $d(y,z) \leq k$. Thus, if y_i has a descendant in S_2, then $y \notin \mathcal{P}$. If y_i has no descendant in S_2, then we must have $y_i \in N_j(x_0)$, where $k - r + 2 \leq j \leq k$. In this case, y_i has an ancestor $w \in N_{k-r+1}(x_0)$ and

w has a descendant s, say, in S_2. However, $d(y,s) = d(y,w) + d(w,s) < k$ and hence again $y \notin \mathcal{P}$.

We conclude that x_k is the only vertex of $N_k[x_0] \cup N_k[x_{k+1}]$ in $\mathcal{P}$. Hence $\mathcal{P}' = \mathcal{P} \cup \{x_0, x_{k+1}\} - \{x_k\}$ is also a k-packing of G, but with $|\mathcal{P}'| > |\mathcal{P}|$. This contradicts the assumption that G is k-equipackable, establishing the result. □

In conclusion, from Theorem 2.2, Lemmas 3.4 and 3.5 and Corollary 3.3, we deduce the following characterization of k-equipackable graphs of girth $4k+4$ or more.

Theorem 3.6 *Let G be a graph with diam $G > k$ and girth at least $4k+4$. Then G is k-equipackable if and only if $G \in \mathcal{F}_k$.* □

From Lemma 2.4, we can deduce the following corollary to Theorem 3.6.

Corollary 3.7 *Given a graph G with diam $G > k$ and girth at least $4k+4$, we can decide in polynomial time whether G is k-equipackable.* □

The lower bound on the girth of G in Lemma 3.4 is sharp. This follows from noting that if u, v are consecutive vertices in any k-packing of the cycle C_g, then $k+1 \leq d(u,v) \leq 2k+1$. Hence C_g has a maximal k-packing of size 2 when $2k+2 \leq g \leq 4k+2$, of size 3 when $3k+3 \leq g \leq 6k+3$, of size 4 when $4k+4 \leq g \leq 8k+4$, and so on. Thus C_g is k-equipackable if $g \leq 3k+2$ and also for the isolated case when $g = 4k+3$, for which C_g has maximal k-packings of size 3 only. For all other values of $g \geq 3k+3$, C_g has maximal k-packings of at least two different sizes.

In [5], it is shown that in the case when $k = 1$, the cycle C_7 is in fact the only well-covered graph of girth 7 that does not belong to $\mathcal{F}_1$. A similar result has been obtained by the authors and G.Gunther [7] in the case when $k = 2$. That is, the cycle C_{11} is the only 2-equipackable graph of girth 11 that is not a member of $\mathcal{F}_2$. We make the following conjecture.

Conjecture 1 *The cycle C_{4k+3} is the only k-equipackable graph of girth $4k+3$ that is not a member of $\mathcal{F}_k$.*

4 A remark on k-domination

In [9], it is proved that for any tree T, $\pi_{2k}(T) = \gamma_k(T)$, for $k \geq 1$. It is easily seen that the inequality $\pi_{2k}(G) \leq \gamma_k(G)$ holds in any graph G. For, suppose that $\mathcal{P}$ is a maximal $2k$-packing and $\mathcal{D}$ any minimal k-dominating set in G. Then every vertex $s \in \mathcal{P}$ is contained in a set $N_k[c]$, for some $c \in \mathcal{D}$. However, no set $N_k[c]$ contains more than one vertex of $\mathcal{P}$, for any $c \in \mathcal{D}$. Thus $|\mathcal{P}| \leq |\mathcal{D}|$.

It follows that if we can exhibit a $2k$-packing $\mathcal{P}$ and a k-dominating set $\mathcal{D}$ such that $|\mathcal{P}| = |\mathcal{D}|$ in a graph G, then $\mathcal{P}$ is a $2k$-packing of maximum size and $\mathcal{D}$ is a k-dominating set of minimum size.

Theorem 4.1 *Let $G \in \mathcal{F}_{2k}$. Then $\gamma_k(G) = \pi_{2k}(G)$.*

Proof. Let $B_1, B_2, ..., B_m$ denote the $2k$-basic subgraphs of G and b_i denote the k-branchpoint of B_i, $i = 1, 2, ..., m$. Then the set $\{b_1, b_2, ..., b_m\}$ is both a $2k$-packing and a k-dominating set of G, giving $\gamma_k(G) = \pi_{2k}(G)$. □

The proof of Theorem 4.1 establishes that graphs in $\mathcal{F}_{2k}$ have the additional property that they contain a set of vertices that is both a minimal k-dominating set and a maximal $2k$-packing. Meir and Moon [9] point out that this is not true in general in the case of trees.

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