Switching Distance Graphs

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ABSTRACT. Let S be a set of graphs on which a measure of distance (a metric) has been defined. The distance graph $D(S)$ of S is that graph with vertex set S such that two vertices G and H are adjacent if and only if the distance between G and H (according to this metric) is 1. A basic question is the determination of which graphs are distance graphs. We investigate this question in the case of a metric which we call the switching distance. The question is answered in the affirmative for a number of classes of graphs, including trees and all cycles of length at least 4. We establish that the union and Cartesian product of two switching distance graphs are switching distance graphs. We show that each of K_3, $K_{2,4}$ and $K_{3,3}$ is not a switching distance graph.

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1 Introduction

One of the most fundamental problems in graph theory is the determination of whether two given graphs are isomorphic. If the graphs are not isomorphic, then the question arises as to how close to (or far from) being isomorphic the graphs are. A number of metrics have been defined on various classes of graphs. The edge slide distance was defined in [1] and [8], while the edge rotation distance was defined in [4]. These two metrics were further studied in [2], [6] and [7]. In [9], Zelinka introduced the induced subgraph distance metric, and in [10] introduced an analogous metric to study a distance between isomorphism classes of trees. In [11], various distances between isomorphism classes of graphs were compared. In this paper we define the switching distance metric and describe and study graphs associated with this metric.

For graph theory terminology we follow [3]. In particular, the *empty graph* of order n has n vertices and no edges. For two graph G_1 and G_2 with disjoint vertex sets the *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

Let G and H be two graphs having the same vertex set. We say that G *can be transformed into H by a switching operation* if G contains a subset U of vertices that either induces a complete graph and is such that $E(H) = E(G) - E(\langle U \rangle_G)$ or induces an empty graph and is such that $E(H) = E(G) \cup E(\langle U \rangle_{\overline{G}})$.

It is immediate that a graph G can be transformed into a graph H by a switching operation if and only if H can be transformed into G by a switching operation. More generally, we say simply that G can be *transformed* into H by switching operations, written $G \rightarrow H$, if there exists a sequence $F_0, F_1, \ldots, F_m$ ($m \geq 0$) of graphs such that $G = F_0$, $H = F_m$, and, if $m \geq 1$, F_i can be transformed into F_{i+1} by a switching operation for $i = 0, 1, \ldots, m-1$. It is readily seen that "G can be transformed into H" is an equivalence relation. Moreover, by switching on pairs of adjacent vertices, every graph G can be transformed into the empty graph having vertex set $V(G)$. Therefore the following holds:

Proposition 1 *Two graphs can be transformed into each other if and only if they have the same vertex set.*

Let G and H be two graphs having the same vertex set. We define the *switching distance* $d_s(G, H)$ between G and H as 0 if $G = H$ and, otherwise, as the smallest positive integer m for which there exists a sequence F_0, F_1, $\ldots$, F_m of graphs such that $G = F_0$, $H = F_m$, and F_i can be transformed into F_{i+1} by a switching operation for $i = 0, 1, \ldots, m-1$. By Proposition 1, this "distance" is a well-defined concept.

Let S be a set of (distinct) graphs having the same vertex set. Then the *switching distance graph* $D_s(S)$ of S is defined to be that graph with vertex set S such that two vertices G and H of $D_s(S)$ are adjacent if and only if $d_s(G, H) = 1$. We investigate here the question: Which graphs are switching distance graphs?

2 Which graphs are switching distance graphs?

In order to provide some answers to the question in the title of this section, we begin with a few straightforward observations.

Fact 1 *For $n \geq 4$, the n-cycle C_n is a switching distance graph.*

Proof: Let $V = \{v_0, v_1, \ldots, v_{n-2}\}$ be a set of $n-1$ vertices. Let H_0 be the empty graph with vertex set V, and let H_{n-1} be the complete graph with vertex set V. For $i = 1, 2, \ldots, n-2$, let H_i denote the graph obtained from H_{i-1} by adding the edge v_0v_i. Then $d_s(H_i, H_j) = 1$ if and only if $|i-j| = 1$. Thus $C_n \cong D_s(\{H_0, H_1, \ldots, H_{n-1}\})$. □

Proposition 2 *Every induced subgraph of a switching distance graph is again a switching distance graph.*

An immediate consequence of Fact 1 and Proposition 2 now follows.

Fact 2 *For $n \geq 1$, the path P_n on n vertices is a switching distance graph.*

Fact 3 *For $n \geq 1$, the star $K_{1,n}$ is a switching distance graph.*

Proof: Let H denote the empty graph with vertex set $\{v_1, v_2, \ldots, v_{n+1}\}$. For $i = 1, 2, \ldots, n$, let H_i denote the graph obtained from H by adding the edge v_iv_{i+1}. Then $d_s(H, H_i) = 1$ for all i and $d_s(H_i, H_j) = 2$. Thus $K_{1,n} \cong D_s(\{H, H_1, \ldots, H_n\})$. □

Fact 4 *For $n \geq 1$, the n-cube Q_n is a switching distance graph.*

Proof: Let $V = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$ be a set of $2n$ vertices, and let $E = \{u_iv_i \mid 1 \leq i \leq n\}$. For $j = 0, 1, \ldots, n$, let $\mathcal{G}_j$ be the set of all graphs $G = (V_j, E_j)$ having the vertex set $V_j = V$ and edge set $E_j \subseteq E$ with $|E_j| = j$. Then for each graph $G = (V_j, E_j)$ in $\mathcal{G}_j$, there corresponds a binary n-tuple $(a_1, a_2, \ldots, a_n)$ where a_i is 1 if $u_iv_i \in E_j$ and 0 otherwise. Hence, letting $S = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_n$, there is a one-to-one correspondence between the vertices in $D_s(S)$ and the vertices in Q_n. Furthermore, if $G = (V_j, E_j) \in \mathcal{G}_j$ and $H = (V_i, E_i) \in \mathcal{G}_i$, then $d_s(G, H) = 1$ if and only if $|i-j| = 1$ and $|E_i \cap E_j| = min\{i, j\}$. Thus, $D_s(S) \cong Q_n$. □

Fact 5 *$K_{2,3}$ is a switching distance graph.*

Proof: Let $V = \{v_1, v_2, v_3\}$ and let H_1 (G_1) denote the empty graph (complete graph, respectively) with vertex set V. For $i = 2, 3$, let $G_i = H_1 \cup \{v_1v_i\}$, and let $H_2 = G_1 - v_2v_3$. Then $K_{2,3} \cong D_s(\{H_1, H_2, G_1, G_2, G_3\})$. □

We make the following observation concerning complements of graphs.

Proposition 3 *If G and H are two graphs having the same vertex set, then $d_s(G, H) = d_s(\overline{G}, \overline{H})$.*

Proof: If $d_s(G, H) = 0$, then $G = H$, implying that $\overline{G} = \overline{H}$. Assume, then, that $d_s(G, H) = m \geq 1$. Hence there exists a sequence $G = F_0, F_1, \ldots, F_m = H$ of graphs where F_i can be transformed into F_{i+1} by a switching operation for $i = 0, 1, \ldots, m - 1$. Observe that $\overline{F_i}$ can be transformed into $\overline{F_{i+1}}$ by the same switching set that transformed F_i into F_{i+1}. Thus the sequence

$$\overline{G} = \overline{F_0}, \overline{F_1}, \ldots, \overline{F_m} = \overline{H} \tag{1}$$

implies that $d_s(\overline{G}, \overline{H}) \leq d_s(G, H) = m$. Now by applying the above technique to the sequence (1), we have $d_s(\overline{\overline{G}}, \overline{\overline{H}}) \leq d_s(\overline{G}, \overline{H})$ or $m = d_s(G, H) \leq d_s(\overline{G}, \overline{H}) \leq m$, producing the desired result. □

The following lemmas will be useful in establishing that a number of large classes of graphs are switching distance graphs. Let G' denote the graph obtained from a graph G by adding a new vertex v and joining v to every vertex of G, so $G' \cong G + K_1$.

Lemma 1 *For any two graphs G and H having the same vertex set, $d_s(G, H) = d_s(G', H')$.*

Proof: A sequence of switching sets of vertices used to transform G into H may be used to transform G' into H', so $d_s(G', H') \leq d_s(G, H)$. Furthermore, we may assume without loss of generality that in a sequence of switching sets of vertices used to transform G' into H' none of the switching sets S contain v for otherwise we may obtain a feasible sequence of switching sets if we delete v from all those sets S. Hence $d_s(G, H) \leq d_s(G', H')$. □

Corollary 1 *For any two graphs G and H having the same vertex set, $d_s(G, H) = 1$ if and only if $d_s(G', H') = 1$.*

By repeatedly applying Corollary 1 we have:

Lemma 2 *Let G be a switching distance graph and m a positive integer. Then there is a set T of m-connected graphs such that $G \cong D_s(T)$.*

Lemma 3 *Let G_0 and H_0 be connected graphs with disjoint vertex sets, and let G_1 (H_1) be a connected graph having the same vertex set as G_0 (H_0, respectively). Then $d_s(G_0 \cup H_0, G_1 \cup H_1) = 1$ if and only if either $G_0 = G_1$ and $d_s(H_0, H_1) = 1$ or $H_0 = H_1$ and $d_s(G_0, G_1) = 1$.*

Proof: Certainly, either $G_0 = G_1$ and $d_s(H_0, H_1) = 1$ or $H_0 = H_1$ and $d_s(G_0, G_1) = 1$ implies that $d_s(G_0 \cup H_0, G_1 \cup H_1) = 1$. Suppose, then, that $d_s(G_0 \cup H_0, G_1 \cup H_1) = 1$. Let S be the switching set of vertices that transforms $G_0 \cup H_0$ into $G_1 \cup H_1$. If S contains a vertex of G_0 and H_0, then it is evident that the subgraph of $G_0 \cup H_0$ induced by S is empty. It follows that the switching set S transforms $G_0 \cup H_0$ into a connected graph since each of G_0 and H_0 is connected. This contradicts the fact that $G_1 \cup H_1$ is disconnected. Hence either $S \subseteq V(G_0)$, in which case $H_0 = H_1$ and $d_s(G_0, G_1) = 1$, or $S \subseteq V(H_0)$, in which case $G_0 = G_1$ and $d_s(H_0, H_1) = 1$. □

Using Lemmas 2 and 3 we establish that the union and Cartesian product of two switching distance graphs are switching distance graphs.

Theorem 1 *Let G and H be switching distance graphs. Then*

(a) $G \cup H$ is a switching distance graph, and

(b) $G \times H$ is a switching distance graph.

Proof: (a) By Lemma 2, we can find sets S and T of connected graphs such that $D_s(S) \cong G$, $D_s(T) \cong H$, each graph in S has order m, each graph in T has order n and $m \neq n$. Then $G \cup H \cong D_s(S \cup T)$.

(b) By Lemma 2, we may assume that there exist disjoint sets S and T of connected graphs for which $D_s(S) \cong G$ and $D_s(T) \cong H$. Assume that $S = \{G_u \mid u \in V(G)\}$ with $d_s(G_u, G_w) = 1$ if and only if $uw \in E(G)$. Similarly, $T = \{H_v \mid v \in V(H)\}$. We show that

$$G \times H \cong D_s(\{G_u \cup H_v \mid u \in V(G),\, v \in V(H)\}).$$

By Lemma 3, $d_s(G_u \cup H_v, G_{u'} \cup H_{v'}) = 1$ if and only if either $G_u = G_{u'}$ and $d_s(H_v, H_{v'}) = 1$ or $H_v = H_{v'}$ and $d_s(G_u, G_{u'}) = 1$. Thus $d_s(G_u \cup H_v, G_{u'} \cup H_{v'}) = 1$ if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$, that is, there is an edge in $G \times H$ between (u, v) and (u', v'). This yields the desired result. □

By Proposition 2, we have the following corollary.

Corollary 2 *Let G and H be switching distance graphs with disjoint vertex sets. For each pair v, w with $v \in V(G)$ and $w \in V(H)$, the graph obtained from G and H by*

(*a*) *identifying* v *and* w*, or by*
(*b*) *joining* v *and* w *with an edge,*
is a switching distance graph.

Proof: That the graph described in (*a*) is a switching distance graph is immediate since it is an induced subgraph of the switching distance graph $G \times H$. To establish (*b*), consider the complete graph $F \cong K_2$ on two vertices v' and w'. Applying (*a*), form the new graph G' by taking G and F and identifying v and v'. We may now produce the graph described in (*b*) by taking G' and H and identifying w' and w. □

By repeated application of the above result, we have the following corollaries.

Corollary 3 *A graph is a switching distance graph if and only if each of its blocks is a switching distance graph.*

Corollary 4 *Every tree is a switching distance graph.*

3 Which graphs are not switching distance graphs?

We begin this section with the following lemma.

Lemma 4 *Let* G *and* H *be two graphs having the same vertex set* V*, and let* $U \subseteq V$*. Let* G_U *(*H_U*) be the subgraph of* G *(*H*, respectively) induced by the vertex set* U*. Then* $d_s(G_U, H_U) \leq d_s(G, H)$*.*

Proof: Let $m = d_s(G,H)$ and let $F_0, F_1, \ldots, F_m$ be the sequence of graphs such that $G = F_0$, $H = F_m$, and F_i can be transformed into F_{i+1} by a switching operation for $i = 0, 1, \ldots, m-1$. For $i = 0, 1, \ldots, m-1$, let F_i' be the subgraph of F_i induced by U. Then $F_0', F_1', \ldots, F_m'$ is a sequence of graphs such that $G_U = F_0'$, $H_U = F_m'$, and if $F_i' \neq F_{i+1}'$, then F_i' can be transformed into F_{i+1}' by a switching operation for $i = 0, 1, \ldots, m-1$. It follows that $d_s(G_U, H_U) \leq m$. □

Proposition 4 *The complete graph* K_3 *is not a switching distance graph.*

Proof: Assume, to the contrary, that there exist graphs H_1, H_2 and H_3 such that $K_3 \cong D_s(\{H_1, H_2, H_3\})$. Then there is an edge in one of the graphs H_i that is not an edge in the other two graphs. Without loss of generality, we may assume that H_1 contains an edge uv that is not an edge of H_2 or H_3. If the switching set S_2 used to transform H_1 into H_2 is the same as the switching set S_3 used to transform H_1 into H_3, then it is evident that $H_2 = H_3$, which is not the case. Without loss of generality, we may assume that $w \in S_2 - S_3 \neq \emptyset$. Let F_1, F_2 and F_3 be the subgraphs induced by $\{u, v, w\}$ in H_1, H_2 and H_3, respectively. Then $F_1 \cong K_3$, $F_2 \cong \overline{K_3}$ and $F_3 \cong K_3 - e$. Hence, by Lemma 4, $2 = d_s(F_2, F_3) \leq d_s(H_2, H_3) = 1$, which is impossible. □

Theorem 2 *Let G and H be two graphs having the same vertex set and such that $d_s(G, H) = 2$. Then there are at most three graphs that are at switching distance 1 from each of G and H.*

Proof: If $d_s(G, H) = 2$, there exists at least one graph F at switching distance 1 from each of G and H. Let A (B) be the switching set of vertices used to transform G into F (F into H, respectively). Necessarily, $|A|$, $|B| \geq 2$. We now prove three claims.

Claim 1 *If $A \cap B = \emptyset$, then there are precisely two graphs at switching distance 1 from each of G and H.*

Proof: Without loss of generality, we may assume that each of A and B induce a complete graph in G (and therefore an empty graph in H). Hence, $E(G) - E(\langle A \rangle_G) - E(\langle B \rangle_G) = E(H)$. Since $A \cap B = \emptyset$, it is evident that every edge of G (H) that does not have both ends in A or both ends in B is an edge of H (G, respectively).

Now let F' be the graph formed from G by switching on the set B. Then F' may be transformed into H using the switching set A. Hence F and F' are at switching distance 1 from each of G and H. We show that F and F' are the only two such graphs. If this is not the case, then there exists a graph F'', not equal to F or F', at switching distance 1 from each of G and H. Let S_1 (S_2) be the switching set of vertices used to transform G into F'' (F'' into H, respectively). Then $S_1 \neq A$ and $S_1 \neq B$. We show that either $A \subset S_1$ and $|A \cap S_2| \leq 1$ or $B \subset S_1$ and $|B \cap S_2| \leq 1$.

If $A \not\subset S_1$ and $B \not\subset S_1$, then there exist vertices $a \in A$ and $b \in B$ that do not belong to S_1. Since every edge of G (H) that does not have both ends in A or both ends in B is an edge of H (G, respectively), at least one of a and b, say a, cannot belong to S_2. Hence $a \notin S_1$ and $a \notin S_2$. But then the set A does not induce an empty graph in H, which produces a contradiction. Hence $A \subset S_1$ or $B \subset S_1$. If $A \subset S_1$ and $B \subset S_1$, then we must have $A \subset S_2$ and $B \subset S_2$. But then A (B) does not induce an empty graph in H, a contadiction. So either $A \subset S_1$ or $B \subset S_1$. If $A \subset S_1$ and $|A \cap S_2| \geq 2$, then A would not induce an empty graph in H. Hence if $A \subset S_1$, then $|A \cap S_2| \leq 1$. Similarly, if $B \subset S_1$, then $|B \cap S_2| \leq 1$. We deduce, therefore, that either $A \subset S_1$ and $|A \cap S_2| \leq 1$ or $B \subset S_1$ and $|B \cap S_2| \leq 1$.

Without loss of generality, we may assume that $A \subset S_1$ and $|A \cap S_2| \leq 1$. Now let $v \in S_1 - A$. Then v must be adjacent to every vertex of A in

G. Hence v is not adjacent to any vertex of A in F''. Since $|A \cap S_2| \leq 1$, there is therefore at least one vertex a of A that is not adjacent to v in H. However every edge of G (H) that does not have both ends in A or both ends in B is an edge of H (G, respectively). In particular, the edge va of G is also an edge of H, producing a contradiction. Hence F and F' are at switching distance 1 from each of G and H. □

Claim 2 *If* $|A \cap B| = 1$, *then there are at most three graphs at switching distance* 1 *from each of* G *and* H.

Proof: Let $A \cap B = \{v\}$. Let F' be the graph formed from G by switching on the set B. Then F' may be transformed into H using the switching set A, so F and F' are at switching distance 1 from each of G and H. Observe that every edge of G (H) that does not have both ends in A or both ends in B is an edge of H (G, respectively). Let $A_v = A - \{v\}$ and let $B_v = B - \{v\}$. If $|A_v| \geq 2$ or $|B_v| \geq 2$, then it is not too difficult to see that F and F' are the only two graphs at switching distance 1 from each of G and H. On the other hand, if $|A_v| = 1$ and $|B_v| = 1$, then let $A_v = \{a\}$ and let $B_v = \{b\}$. If $ab \in E(G)$ and $va, vb \notin E(G)$, or if $ab \notin E(G)$ and $va, vb \in E(G)$, then let F'' be the graph obtained from G by switching on the set $\{a, b\}$. Note that F'' may be transformed into H using the switching set $A \cup B = \{a, b, v\}$. Hence F'' is at switching distance 1 from each of G and H. In this case it is not too difficult to see that F, F' and F'' are the only such graphs. If the subgraph of G induced by a, b and v is not as described above, then it is evident that F and F' are the only two graphs at switching distance 1 from each of G and H. □

Claim 3 *If* $|A \cap B| \geq 2$, *then there are at most three graphs at switching distance* 1 *from each of* G *and* H.

Proof: Without loss of generality, we may assume that A induces a complete graph in G. Since $|A \cap B| \geq 2$, it is evident that B induces an empty graph in F. It follows that, if $A - B \neq \emptyset$ and $B - A \neq \emptyset$, then F is the only graph at switching distance 1 from each of G and H. Suppose, then, that either $A \subset B$ or $B \subset A$. Suppose, firstly, that $A \subset B$. Then if $|A| \geq 3$ or $|B - A| \geq 2$, it is evident that F is the only graph at switching distance 1 from each of G and H. On the other hand, if $|A| = 2$ or $|B - A| = 1$, then let $A = \{a_0, a_1\}$ and let $B - A = \{b\}$. For $i = 0, 1$, let $S_i = \{a_i, b\}$ and let F_i be the graph formed from G by switching on the set S_i. Then F_i may be transformed into H using the switching set S_{1-i}. Hence F, F_0 and F_1 are all at switching distance 1 from each of G and H. In this case, F, F_0 and F_1 are the only such graphs. Suppose, next, that $B \subset A$. Then using a similar proof as in the case where $A \subset B$, we may show that there are at most three graphs at switching distance 1 from each of G and H. □

By Claims 1, 2 and 3, there are at most three graphs that are at switching distance 1 from each of G and H. This completes the proof of Theorem 2. □

An immediate corollary now follows.

Corollary 5 *$K_{2,4}$ is not a switching distance graph.*

The proof of Theorem 2 also shows us which graphs give rise to a switching distance graph that is isomorphic to $K_{2,3}$. Let G and H be two graphs having the same vertex set V and such that $d_s(G,H) = 2$. If F_1, F_2 and F_3 are three graphs at switching distance 1 from each of G and H, then it follows from the proof of Theorem 2 that there exists a subset S of V of cardinality 3 such that $E(G) - E(\langle S\rangle_G) = E(H) - E(\langle S\rangle_H)$. Let $S = \{u, v, w\}$. Since $d_s(G,H) = 2$, there are only two possibilities for the subgraph induced by S in G and H. Firsly, $E(\langle S\rangle_G) = \emptyset$ and $E(\langle S\rangle_H) = \{uv, uw\}$, in which case $E(\langle S\rangle_{F_1}) = \{uv, uw, vw\}$, $E(\langle S\rangle_{F_2}) = \{uv\}$ and $E(\langle S\rangle_{F_3}) = \{uw\}$, and secondly, $E(\langle S\rangle_G) = \{uv, uw, vw\}$ and $E(\langle S\rangle_H) = \{uv\}$, in which case $E(\langle S\rangle_{F_1}) = \{uv, uw\}$, $E(\langle S\rangle_{F_2}) = \{uv, vw\}$ and $E(\langle S\rangle_{F_3}) = \emptyset$. Hence there can be no graph F with vertex set V that is distinct from G and H at switching distance 2 from each of G and H and at switching distance 1 from each of F_1, F_2 and F_3. Therefore we have the following corollary of Theorem 2.

Corollary 6 *$K_{3,3}$ is not a switching distance graph.*

4 An upper bound on the switching distance between two graphs

The minimum number of cliques partitioning the edge set $E(G)$ of a graph G is an upper bound on the distance of G from the empty graph, but the distance can be much smaller. For example, the stars $K_{1,n}$ have switching distance 2 (from the empty graph and also from each other) and partition number $n-1$. Erdős, Goodman and Pósa [5] proved that the edge set of any graph on n vertices can be partitioned into at most $n^2/4$ edge-disjoint complete subgraphs which are triangles and edges. Switching each of them, we get the empty graph. This gives an upper bound of $n^2/2$ on the switching distance between two graphs having the same vertex set consisting of n vertices. We show that this upper bound can be improved to $4n - 2log_4 n + 2$.

For a positive integer n, we denote by $R(n)$ the "inverse" of the Ramsey function; that is, $R(n)$ is the largest integer k such that every 2-coloring of the edges of the complete graph K_n on n vertices contains a monochromatic K_k.

Theorem 3 *If G is a graph with n vertices, then*

$$d_s(G, \overline{K_n}) \leq 2n - 2R(n) + 1.$$

Proof: Suppose $R(n) = k$. Then G contains an independent set of cardinality k or a clique of cardinality k. If G contains a clique on k vertices, then switch on this set to produce an independent set of cardinality k. Having obtained an independent set S on k vertices, $k < n$, we choose any vertex v not in S, and transform $S \cup \{v\}$ to an independent set in (at most) two switching steps as follows. First we switch on the set S_v of neighbors of v in S to produce a new graph in which v and its neighbors S_v in S induce a complete graph. Then we switch on the set $S_v \cup \{v\}$. Continuing in this way, G can be transformed into an empty graph on n vertices by at most $2n - 2R(n) + 1$ switching operations. □

Corollary 7 *Let $\mathcal{G}_n$ be the set of all graphs having the same vertex set V, where $|V| = n$. For any two G, $H \in \mathcal{G}_n$,*

$$d_s(G,H) \leq 4n - 2log_4 n + 2.$$

Proof: By Theorem 3, $d_s(G,H) \leq d_s(G,\overline{K_n}) + d_s(\overline{K_n},H) \leq 4n - 4R(n) + 2 \leq 4n - 2log_4 n + 2$, since $R(n) \geq \frac{1}{2} log_4 n$. □

5 Concluding remarks

In the course of this investigation we encountered a number of problems which we have yet to settle. A partial listing of these problems follows.

1. Characterize the class of switching distance graphs.
2. Are there minimal forbidden induced subgraphs other than K_3 , $K_{2,4}$ and $K_{3,3}$? (If there is an infinite family of minimal forbidden subgraphs, then the recognition problem of switching distance graphs becomes more interesting.)
3. What is the complexity of determining the switching distance between two graphs?
4. Let $\mathcal{G}_n$ be the set of all graphs having the same vertex set. Consider the switching distance graph S_n of $\mathcal{G}_n$. Prove that the radius and the diameter of S_n are of the form $cn+o(n)$ as n gets large and determine that value of c. Are the empty graph and the complete graph central in the sense that the largest distance from them is the radius?
5. What is the expected switching distance between two randomly chosen graphs of order n? What is the expected switching distance of a random graph from the complete graph?
6. Given a switching distance graph H, determine the smallest $n = n(H)$ such that H is the switching distance graph of a set of (distinct) graphs on n vertices.

7. Another interesting measure of distance can be defined among the isomorphism classes of graphs, that is, for two graphs G and H of the same order, find a sequence of switching operations that will transform G into a graph which is isomorphic (but not necessarily identical) to H.

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