An Upper Bound for the *(n,5)* - cages

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Abstract

A *(n,5)* - cage is a minimal graph of regular degree n and girth 5. Let *f(n,5)* denote the number of vertices in a *(n,5)* - cage. The best known example of an *(n,5)* - cage is the Petersen graph, the *(3,5)* - cage. The *(4,5)* - cage is the Robertson graph, the *(7,5)* - cage is the Hoffman-Singleton graph, the *(6,5)* - cage was found by O'Keefe and Wong [1] and there are three known *(5,5)* - cages. No other *(n,5)* - cages are known for $n \geq 8$. In this paper, we will use a graph structure called remote edges and a set of mutually orthogonal Latin squares to give an upper bound of *f(n,5)* for $n = 2^k + 1$.

1. Introduction

All graphs considered in this paper are simple (i.e., contain no loops or multiple edges), undirected and finite. For the most part, the terminology follows that of Bondy and Murty [2]. Let $G = (V,E)$ be a graph. $V(G)$ and $E(G)$ denote the vertex set and edge set respectively. A graph G is k - regular if $d(v) = k$ for all $v \in V(G)$. The girth of G is the length of a shortest cycle in G. A pair of edges are called remote edges if the distance between them is at least *3*. A subset S of $V(G)$ is an independent set of G if no two vertices of S are adjacent in G.

The problem of finding cages was not extensively studied until 1963 when P. Erdös and H. Sachs [3] used a nonconstructive method to show the existence of cages. The following Theorem by Tutte can be found in [4]. Wong gives an excellent survey on this subject (see[5]).

Theorem 1. Given $k \geq 3$ and $g \geq 3$, there exists a k - regular graph with girth g on $(k-1)\{\frac{(k-1)^{g-1}-1}{k-2} + \frac{(k-1)^{g-2}-1}{k-2} + 1\}$ vertices.

A Latin square of order s is identified as an $s \times s$ square. The s^2 cells are occupied by s distinct symbols such that each symbol occurs exactly once in each row and once in each column. Two Latin squares of the same order are said to be orthogonal if, on superposition, each symbol of the first square occurs exactly once with each symbol of the second square. A set of Latin squares (all of the same order), any two of which are orthogonal, is said to be a set of mutually orthogonal Latin squares. A set of *s-1* mutually orthogonal Latin

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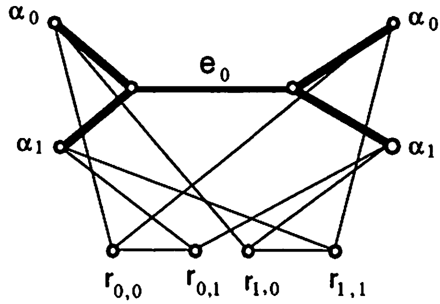
squares of order s is said to be a complete set of mutually orthogonal Latin squares. Bose and Nair [6] proved that for any given prime power s, there exists a complete set of orthogonal Latin squares of order s.

2. An Upper Bound of $f(n,5)$ where $n = 2^k + 1$

In this section we construct a family of regular graphs of degree $n = 2^k + 1$ and girth 5. First, we observe from the Petersen graph, the $(3,5)$ - cage, and the Robertson graph, the $(4,5)$ - cage, that there exist $(n - 1)/2$ remote edges with $n - 1$ leaves that are appropriately joined to either a set of independent vertices or a set of remote edges (see Figure 1). We call this set of remote edges and independent vertices (or edges) an *RI* structure. It was these two examples that motivated the present work.



The Petersen graph

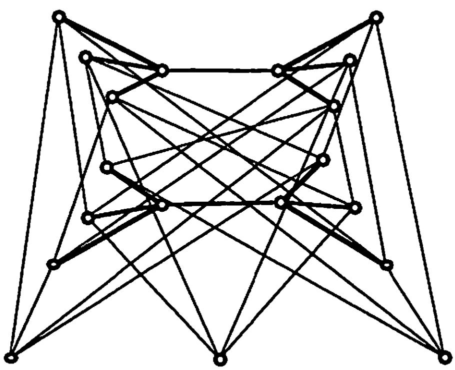

The Robertson graph

Figure 1.

First, we use the *RI* structure to construct an n - regular graph on $2n^2 - 3n + 1$ vertices, where $n = 2^k + 1$. Since $n - 1 = 2^k$ is a prime power, there exists a complete set of mutually orthogonal Latin squares of order $n - 1$. Let $L_1, L_2, \ldots, L_{n-2}$ be a complete set of mutually orthogonal Latin squares defined by

$$L_t = \begin{pmatrix} l^t_{0,0} & l^t_{0,1} & \cdots & l^t_{0,n-2} \\ l^t_{1,0} & l^t_{1,2} & \cdots & l^t_{1,n-2} \\ \cdot & \cdot & \cdots & \cdot \\ l^t_{n-2,0} & l^t_{n-2,1} & \cdots & l^t_{n-2,n-1} \end{pmatrix} \quad \text{and} \quad L_0 = \begin{pmatrix} \alpha_0 & \alpha_0 & \cdots & \alpha_0 \\ \alpha_1 & \alpha_1 & \cdots & \alpha_1 \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{n-2} & \alpha_{n-2} & \cdots & \alpha_{n-2} \end{pmatrix}$$

where $l^t_{i,j} = \alpha_j \alpha_t + \alpha_i$, $\alpha_0 = 0, \alpha_1, \ldots, \alpha_{n-2}$ are the elements of a finite field GF_2^k, $1 \le t \le n-2$ and $0 \le i, j \le n-2$.

Construction

Step #1: Define $V(G)$.

Let $m = (n-3)/2$ and let $e_0, e_1, \ldots, e_m$ denote a set of remote edges with each end vertex being joined to $n-1$ vertices. These $n-1$ vertices are called leaves. Label the leaves by $\alpha_0, \alpha_1, \ldots, \alpha_{n-2}$. For $i = 0, 1, \ldots, n-2$ and $j = 0, 1, \ldots, n-2$, let $r_{i,j}$ denote an independent vertex and let $I_j = \bigcup_{i=0}^{n-2} r_{i,j}$ and $I = \bigcup_{j=0}^{n-2} I_j$. At this point, the graph has $(n-1)^2 + 2n(n-2)/2 = 2n^2 - 3n + 1$ vertices (see Figure 2). Each end vertex of the remote edges has degree n and each leaf has degree 1.

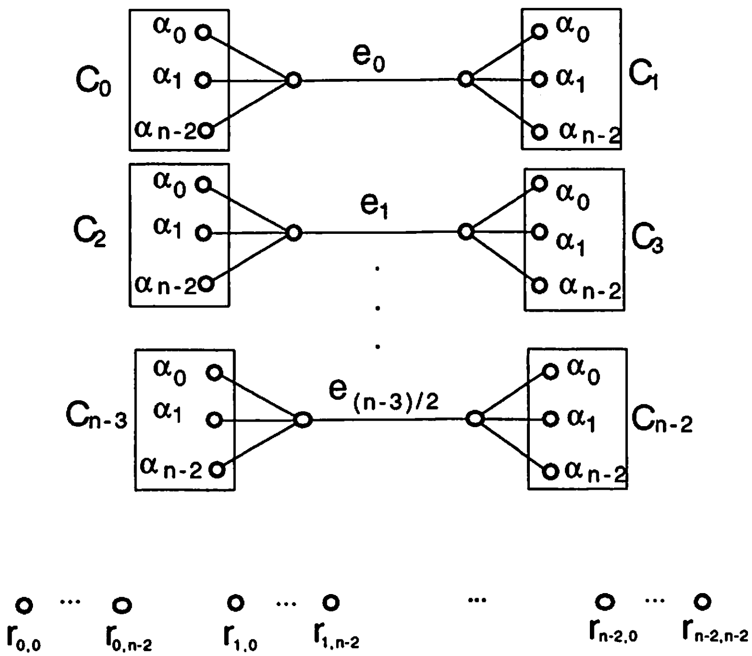


Figure 2. *RI* Structure

Step #2: Define $E(G)$.

Let C_h be a set of leaves joined to an end vertex of a remote edge and let $r_{0,i}$ be joined to α_i in C_h for $0 \le h \le n-2$ where $0 \le i \le n-2$. Now each leaf has degree 2 and $r_{0,i}$ has degree $n-1$ for $0 \le i \le n-2$. We next define edges between $C = \bigcup_{i=0}^{n-2} C_i$ and $I' = I - I_0$ using the complete set of mutually orthogonal Latin squares $L_1, L_2, \ldots, L_{n-2}$. Let $r_{i,j}$ be joined to α_t of C_h for $0 \le h \le n-2$ if and only if $\alpha_t = l^i_{j,h}$ where $0 \le j, t \le n-2$ and $1 \le i \le n-2$. Clearly, any vertex in I_j is adjacent to one vertex of C_i and any vertex of C_i is adjacent to one vertex of I_j. It follows that all the vertices in C are of degree n and all vertices in I are of degree $n-1$.

Step #3: Adding a matching in I.

In order to add one to the degree of every vertex in I, we add new edges $(r_{i,j}, r_{i,j+1})$ where $0 \le i \le n-2$ and $j \in \{0,2,4,\ldots,n-3\}$. The resulting graph, G, is regular of degree n on $2n^2 - 3n + 1$ vertices.

Examples

(a) $n = 3$.

The Petersen graph (see Figure 1) can be constructed by $L_0 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $L_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(b) $n = 5$ (see Figure 3).

$$L_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{pmatrix}, L_1 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 2 & 3 & 1 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 1 & 3 \\ 3 & 1 & 0 & 2 \end{pmatrix}, L_3 = \begin{pmatrix} 0 & 3 & 1 & 2 \\ 1 & 2 & 0 & 3 \\ 2 & 1 & 3 & 0 \\ 3 & 0 & 2 & 1 \end{pmatrix}$$

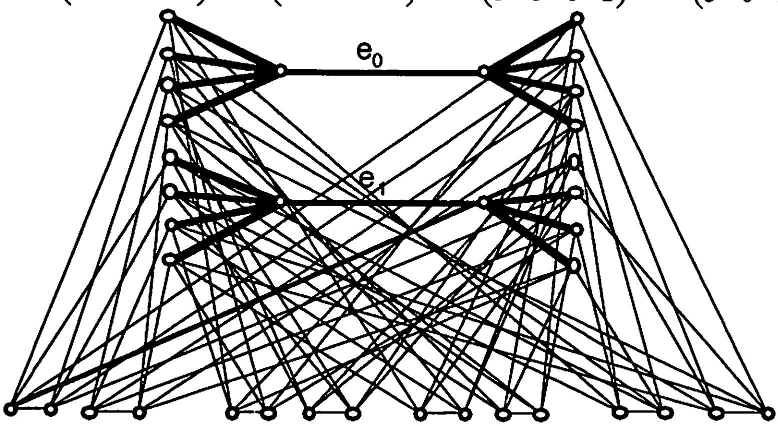


Figure 3.

Theorem 2. $f(n,5) \leq 2n^2 - 3n + 1$, if $n = 2^k + 1$.

Proof: It suffices to show that the graph, G, constructed above, has girth 5. Since any vertex of I is adjacent to one vertex in C_0 and one vertex in C_1, there are many 5 - cycles in G. To complete the proof, we now show that G is a triangle free and 4 - cycle free graph.

Case 1. Verify that G contains no triangle.

Suppose there is a triangle in G. Note that there is no triangle in $V(G) - I$ since there are no edges in $G[C]$, the induced graph on vertex set C. This implies that any triangle must contain two adjacent vertices in I, say $r_{i,j}$ and $r_{i,j+1}$ where $j \in \{0,2,4,...,n-3\}$. That is, there exists a vertex in C, say $u \in C_h$, for some $0 \leq h \leq n - 2$, joined to two vertices in C_i, which contradicts the fact that every vertex in C_i is adjacent to one vertex of I_j. Thus, G is a triangle free graph.

Case 2. Verify that G contains no 4 - cycle.

Suppose there exists a 4 - cycle, C^4, in G. Since $G[I]$ contains only a perfect matching, there are only three possibilities for the location of the four vertices of C^4.

(a) Two adjacent vertices of C^4 belong to I.

This implies the other two vertices of C^4 belong to C, which contradicts the fact that there are no edges in $G[C]$.

(b) Two vertices in some C_h are adjacent to the same vertex of I.

This is impossible since any vertex of I is adjacent to only one vertex of C_h.

(c) C^4 contains two nonadjacent vertices of I and two vertices in two different leaf sets, say C_{h1} and C_{h2}, where $h1 \neq h2$.

Let $C^4 = \{r_{i,j}, \alpha_s, r_{i',j'}, \alpha_t\}$ where $r_{i,j}$ and $r_{i',j'}$ are the nonadjacent vertices of I and $\alpha_s \in C_{h1}$ and $\alpha_t \in C_{h2}$ are the nonadjacent vertices of C. If one of $\{i,i\}$ equals zero, say $i = 0$, then $s = t$ by the definition of the adjacency of the vertices in I_0. This implies that there are two identical elements in the j' -th row of $L_{i'}$. This forces $i' = 0$ since $L_{i'}$ is a Latin square except in the case $i' = 0$. That is, both $r_{0,j}$ and $r_{0,j'}$, are joined to α_s. Clearly, $j = j'$ by the definition of adjacency of $r_{0,j}$ and $r_{0,j'}$, contradicting the fact that $v_{i,j}$ and $v_{i',j'}$ are two different vertices. Thus, neither i nor i' could equal zero. By the definition of adjacency of $r_{i,j}$ and $r_{i',j'}$, $l^{i}_{j,h1} = \alpha_s$, $l^{i'}_{j',h1} = \alpha_s$, $l^{i}_{j,h2} = \alpha_t$ and $l^{i'}_{j',h2} = \alpha_t$. It follows that $\alpha_i \alpha_{h1} + \alpha_j = \alpha_{i'} \alpha_{h1} + \alpha_{j'}$ and $\alpha_i \alpha_{h2} + \alpha_j = \alpha_{i'} \alpha_{h2} + \alpha_{j'}$ and, in turn, that $(\alpha_i - \alpha_{i'})(\alpha_{h1} - \alpha_{h2}) = 0$. Since $h1 \neq h2$ and $\alpha_{h1} \neq \alpha_{h2}$ and GF_2^k is a field, we must conclude $\alpha_i = \alpha_{i'}$. This implies $\alpha_j = \alpha_{j'}$, and it follows that $i = i'$ and $j = j'$, a contradiction with $r_{i,j} \neq r_{i',j'}$. Therefore, G is a 4 - cycle free graph. □

3. A further Discussion of the Upper Bound of $f(n,5)$

In the *RI* structure, there is no edge in $G[C]$. This implies that there is room for more edges to be added in order to give a better upper bound for $f(n,5)$. We now alter the *RI* structure used in Theorem 2 to improve the upper bound by n.

Theorem 3. $f(n,5) \leq 2n^2 - 4n + 2$, if $n = 2^k + 1$.

To alter the *RI* structure, we first remove the vertices $r_{0,i}$ $(0 \leq i \leq n - 2)$ and then add a perfect matching among vertices in C. Before proving this theorem, we need the following two lemmas.

Lemma 1. In any complete set of orthogonal Latin squares $L_1, L_2, \ldots, L_{n-2}$, if $l^m_{i,j} = l^m_{s,t}$ then $l^m_{i,t} = l^m_{s,j}$ where $0 \leq i, j, s, t \leq n - 2$, $1 \leq m \leq n - 2$ and $n - 1$ is prime power.

The proof is straight forward from the definition of L_m.

Lemma 2. If G is a graph on n vertices $(n = 2^k$ and $k \geq 2)$ and G is of regular degree $n/2 - 1$, then G contains a perfect matching.

Proof: If G is a disconnected graph, then $G = K_{n/2} + K_{n/2}$ and clearly G contains a perfect matching. Otherwise, we claim that G must be a 2 - connected graph. Suppose, to the contrary, that c is a cut vertex of G and let G_1 and G_2 be a pair of components of $G - \{c\}$. If there exists $x \in G_1$ and $y \in G_2$ such that $(x,c) \notin E(G)$ and $(y,c) \notin E(G)$, then $|V(G)| \geq |N(x)| + |N(y)| + 1 = n/2 + n/2 + 1 = n + 1$, a contradiction. This implies that c has to be joined to every vertex of G_1 or G_2. For example, assume c is joined to every vertex of G_1. Since G is an $(n/2 - 1)$ - regular graph, c is not adjacent to any vertex of G_2, a contradiction with the choice of c.

Let L be the longest path of G. By a theorem of Dirac (see[2]), G has a cycle of length at least $2(n/2 - 1) = n - 2$. It follows that $|L| \geq n-1$. If $|L| = n$, the existence of a perfect matching is trivial. If $|L| = n - 1$, there exists a vertex not on L which is joined to a vertex on L. Since $|L| = n - 1$ is odd, there exists two disjoint paths with even lengths in G and these two paths contain all n vertices of G. This implies a perfect matching in G. □

After $r_{0,i}$ for $0 \leq i \leq n - 2$ are removed, a perfect matching joining α_t of C_s to α_t of C_{s+3} and joining α_t of C_t to α_t of C_{t+1} where $0 \leq s, t \leq n-2$ and $s \equiv 0$ and $t \equiv 1 \pmod 4$, is added. Since only one perfect matching is added, there is no triangle or 4 - cycle in C. In fact, there is no triangle in G since α_t appears only once in each row of L_i. The only possible 4- cycle must involve a newly added edge and an edge in I, say $C^4 = \{\alpha^s_i, \alpha^t_i, r^{i2}_m, r^{i1}_m\}$. We now redefine the existed perfect matching in I to avoid such 4 - cycles. Note that for any vertex $r_{m,i}$ its adjacency to vertices of C is decided by the i-th row of L_m. To avoid such 4 - cycles, $r_{m,i}$ should not be joined by any vertex $r_{m,j}$ such that $l^m_{i,s} =$

$l^m_{j,s+3}$ if $s \equiv 0 \pmod{4}$ or $l^m_{i,t} = l^m_{j,t+1}$ if $t \equiv 1 \pmod{4}$. By Lemma 1, there are $(n-2)/2$ vertices in I_m to which $r_{m,i}$ can not be joined. In other words, there are $(n-1)/2-1$ vertices in I_m which can be joined by an arbitrary vertex $r_{m,i}$ without creating a 4 - cycle. This, together with Lemma 2, implies that we can always add a perfect matching in I_m without creating 4 - cycles. □

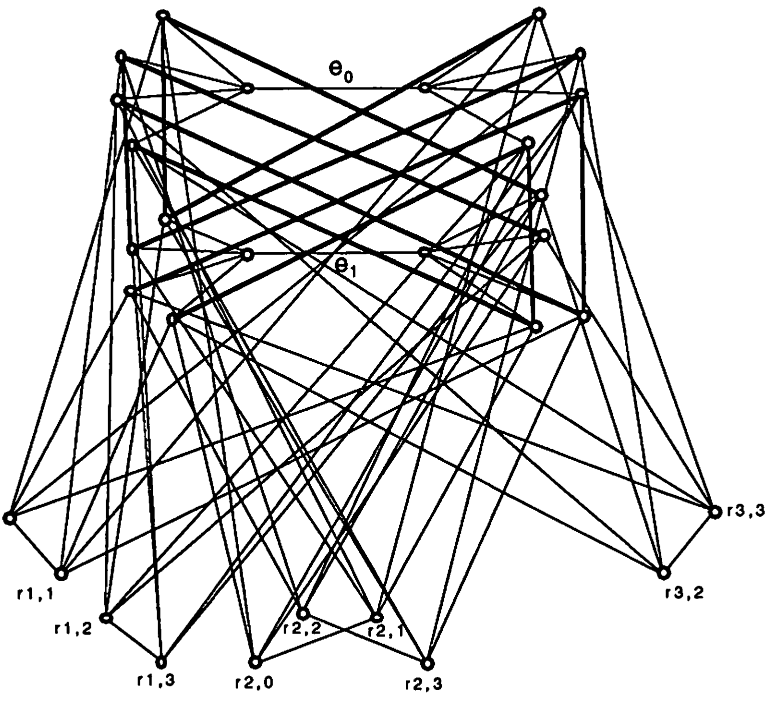


Figure 4.

From Theorem 3, $f(5,5) \leq 32$. The unique $(5,5)$ - cage is the Robertson-Wegner graph on 30 vertices. However, the following further modifications can be made to obtain the Robertson-Wegner graph based on the structure used in Theorem 3. First we remove two vertices from I_3, and then add two more edges between C_0 and C_2 and add two more edges between C_1 and C_3. Secondly we have to redefine the edges in I by the following: $(r_{1,0},r_{1,1})$, $(r_{1,2},r_{1,3})$, $(r_{2,0},r_{2,2})$, $(r_{2,1},r_{2,3})$ and $(r_{3,2},r_{3,3})$. The resulting graph shown in Figure 4 is the Robertson-Wegner graph on 30 vertices. From this example, one can see that it may be possible to improve the upper bound given in Theorem 3, since there is still room to add edges in C and reduce the number of vertices in I. In general, implementation of this improved upper bound requires the following steps.

Step #1: Construct a regular graph without triangles and 4 - cycles in C.

Step #2: Redefine the edges in I so that 4 - cycle can be avoided. Unfortunately, we have not been able to find a proper way of redefining the edges of I in general. We propose the following conjecture.

Conjecture. $f(2^k+1,5) \le (2^{k+1}-2^l)2^l + 2^k$ for $k \ge 2$ and $1 \le l \le (k-1)/2$.

Note that the upper bound given in Theorem 3 is as good as the best result proved by O'Keefe and Wong [5] in 1984, in which they *proved* $f(n,5) \le 2(n-2)^2$ for $n \ge 7$ and $n-2$ a prime power. However, the *RI* structure shows a very explicit graph structure that may lead to further improvement on the upper bound of $f(n,5)$. For example, if the above Conjecture is true, then the coefficient of the linear term in O'Keefe and Wong's upper bound could be changed from a constant 8 to $(3 + 2^l)$ and therefore the upper bound would be reduced to $2n^2 - (3+2^l)n + (2^l - 1)$ where $n = 2^k + 1$ and $2l \le k-1$.

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