MAXIMAL ORTHOGONAL LATIN RECTANGLES

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1. INTRODUCTION

Let r, n be positive integers, $r \leq n$. An $r \times n$ latin rectangle, usually on the set of symbols $\{1, 2, \ldots, n\}$, is a rectangular array $A = (a_{ij}), i = 1, \ldots, r; j = 1, \ldots, n$, with the property that every symbol appears in exactly one cell of each row, and in at most one cell of every column. Two $r \times n$ latin rectangles $A = (a_{ij}), B = (b_{ij})$ are *orthogonal* if $|\{(a_{ij}, b_{ij}) : i = 1, \ldots, r; j = 1, \ldots, n\}| = rn$, i.e. if no two of the ordered pairs (a_{ij}, b_{ij}) are equal.

It is a well-known classical result of M.Hall [H] that any $r \times n$ latin rectangle with $r < n$ can be extended, by adding a new row, to an $(r + 1) \times n$ latin rectangle, and so, eventually, to an $n \times n$ latin square. On the other hand, one can easily find examples of pairs of orthogonal $r \times n$ latin rectangles ($r < n$) which cannot be extended to a pair of orthogonal latin $(r+1) \times n$ rectangles. For example, the two 3×4 latin rectangles in Fig.1a are orthogonal but after the fourth row is adjoined to them - this can be done in one way only - the condition of orthogonality is violated. Similar can be said about the two 3×5 latin rectangles in Fig.1b.

1 2 3 4	1 3 2 4	1 2 3 4 5	1 2 3 4 5
4 1 2 3	3 2 4 1	2 1 5 3 4	4 5 1 2 3
3 4 1 2	4 1 3 2	4 5 1 2 3	5 3 2 1 4

(a) (b)

Fig.1

The research of the first author was supported by a grant from Kuwait University SM 143.

Call a set of t orthogonal $r \times n$ latin rectangles *maximal* if they cannot be extended to a set of t orthogonal $(r+1) \times n$ latin rectangles. In this article, our emphasis will be primarily on the case $t = 2$. We denote a pair of maximal orthogonal $r \times n$ latin rectangles by MOR(r, n). Our interest is in determining, for $n \geq 1$, the possible number of rows r in a MOR(r, n), i.e. the *spectrum* $\mathcal{M}(n)$ for MORs:

$\mathcal{M}(n) = \{r$: there exists MOR$(r, n)\}$.

Trivially, $\mathcal{M}(1) = \mathcal{M}(2) = \{1\}$, and it is an easy exercise to obtain $\mathcal{M}(3) = \{3\}, \mathcal{M}(4) = \{3, 4\}$.

In what follows the interval $[a, b]$ (the interval $(a, b]$, respectively) will denote the set of all integers $x : a \leq x \leq b$ ($a < x \leq b$, respectively). Let $P_n = (n/3, n]$. We conjecture that for sufficiently large n, $\mathcal{M}(n) = P_n$, and we present several theorems and constructions towards proving our conjecture.

2. A NONEXISTENCE RESULT

In this section we assume that all rectangular arrays are on the same set S of n symbols. Let $A = (a_{ij}), B = (b_{ij})$ be rectangular arrays of size $k \times n$ and $r \times n$, respectively. We denote by $C = A \circ B = (c_{ij})$ the rectangular array of size $(k + r) \times n$ obtained by adjoining the rows of B to those of A. That is, for any $j = 1, \ldots, n$, $c_{ij} = a_{ij}$ for $1 \leq i \leq k$, and $c_{ij} = b_{i-k,j}$ for $k + 1 \leq i \leq k + r$.

Let $A_1, \ldots, A_s$ be orthogonal $r \times n$ latin rectangles. Then $A_1, \ldots, A_s$ are said to be *jointly extendable* if there exist $1 \times n$ arrays $C_1, \ldots, C_s$ such that $A_1 \circ C_1, \ldots, A_s \circ C_s$ are orthogonal $(r + 1) \times n$ latin rectangles.

Before stating the main theorem of this section, we need one more piece of notation. Given an $r \times n$ array $A = (a_{ij})$ on the set S, we let $A(j) = S - \{a_{ij} : i = 1, \ldots, r\}$ for $j = 1, \ldots, n$.

Theorem 2.1. *Let $A_1, \ldots, A_s$ be orthogonal $r \times n$ latin rectangles. If $r \leq n/(2s)$ then $A_1, \ldots, A_s$ are jointly extendable.*

The following two auxiliary statements are needed for the proof of Theorem 2.1.

Lemma 2.2. *Let $A = (a_{ij})$ be an $r \times n$ array on S such that each row of A is a permutation of S. If $r \leq n/2$ then there exists a $1 \times n$ array $B = (b_1, \ldots, b_n)$ such that for $j = 1, \ldots, n$, $b_j \notin \{a_{ij} : i = 1, \ldots, r\}$.*

Proof. The statement of the lemma says that there exists a row $B = (b_1, \ldots, b_n)$ such that the element b_j of B differs from all elements in the j-th column of A, or, equivalently, that $\mathcal{A} = \{A(j) : j = 1, \ldots, n\}$ has a system of distinct representatives [BR]. To see this, we only need to verify P.Hall's condition [BR] that

$$|\bigcup_{j \in J} A(j)| \geq |J|$$

for any $J \subseteq \{1, \ldots, n\}$.

Since for $r \leq n/2$, $|A(j)| \geq n - r \geq n/2$ for $j = 1, \ldots, n$, P.Hall's condition is trivially satisfied for any $J \subset \{1, \ldots, n\}, |J| \leq n/2$. On the other hand, each element s of S occurs exactly once in each row of A, hence s occurs in A exactly r times. Thus s occurs in at least $n - r \geq n/2$ sets $A(j)$. Therefore
$|\bigcup_{j \in J} A(j)| \geq |J|$ for any $J \subseteq \{1, \ldots, n\}, |J| \geq n/2$, so P.Hall's condition is satisfied also in this case, and the proof of the lemma is complete. □

If we are given an $(r \times n)$ array A satisfying assumptions of Lemma 2.2, the row B guaranteed by the statement of Lemma 2.2 will be denoted by A^*.

Let now $A = (a_{ij}), B = (b_{ij})$ be orthogonal $r \times n$ latin rectangles and let $C = (c_j)$ be a row such that $D = A \circ C$ is a latin rectangle. Define a latin $r \times n$ rectangle $D^{\perp}$ as follows. Let $c_j = x$. The element in the i-th row and j-th column of $D^{\perp}$ equals b_{il} provided $a_{il} = x$.

Lemma 2.3. *If $r \leq n/4$ then $A \circ C$ and $B \circ E^*$, where $E = B \circ D^{\perp}$, are orthogonal latin rectangles.*

Proof. As C is a permutation of S and A is a latin rectangle (i.e. any row of A is also a permutation of S), each row of $D^{\perp}$, and thus also each row of E is a permutation of S. Since E is a $(2r \times n)$ array, we can apply Lemma 2.2 to obtain a row E^*. Since B is a part of E, $B \circ E^*$ is a latin rectangle, and since $D^{\perp}$ is also a part of E, $A \circ C$ and $B \circ E^*$ are orthogonal. □

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1 Let $A_1, \dots, A_s$ be s mutually orthogonal $r \times n$ latin rectangles with $r \le n/(2s)$. We will construct row arrays $B_1, \dots, B_s$ such that $A_1 \circ B_1, \dots, A_s \circ B_s$ will be orthogonal as well. Set $B_1 = A_1^*$ (note that $r \le n/4$, so Lemma 2.2 applies). Clearly, $A_1 \circ B_1$ is a latin rectangle. For $1 < t \le s$, set $B_t = E_t^*$, $E_t = (A_1 \circ B_1)^\perp \circ \cdots \circ (A_{t-1} \circ B_{t-1})^\perp \circ A_t$. The matrix E_t satisfies the conditions of Lemma 2.2, therefore E_t^* exists. Since A_t is a part of E_t, $A_t \circ E_t^*$ is a latin rectangle while the fact that $(A_j \circ B_j)^\perp$ is a part of E_t implies that $A_t \circ E_t^*$ is orthogonal to $A_j \circ B_j$ for any $j = 1, \dots, t-1$. The proof of the theorem is complete. □

Remark. A technique similar to that in the proof of Theorem 2.1 was used in [HKR] to prove that if $A_1, \dots, A_s$ are latin rectangles of size $r_1 \times n, \dots, r_s \times n$, respectively, where $r_1 + \cdots + r_s \le n/2$ then there exists a row array B such that $A_1 \circ B, \dots, A_s \circ B$ are latin rectangles as well.

Corollary 2.4. *If $r \le n/4$ then any pair of orthogonal $r \times n$ latin rectangles are jointly extendable.*

3. Some direct constructions

Lemma 3.1. *Let $k, q \ge 3, k \le q < 2k$, and suppose there exists a pair of mutually orthogonal latin squares (MOLS) of order k, and of order q, respectively. Then there exist a MOR$(k, k+q)$.*

Proof. Let A, A' and B, B' be a pair of MOLS of order k and q, on the sets $\{1, 2, \dots, k\}$ and $\{k+1, k+2, \dots, k+q\}$, respectively. Let B_1, B_1' be a pair of orthogonal $k \times q$ latin rectangles obtained from B and B', respectively, by deleting their last $q - k (\ge 0)$ rows. Let

$X = \|A|B_1\|, Y = \|A'|B_1'\|$

be two $k \times (k+q)$ arrays obtained by juxtaposition. Clearly, X and Y are orthogonal. To see that they cannot be extended to orthogonal $(k+1) \times (k+q)$ latin rectangles, assume the contrary, and let $C = (c_1, \dots, c_k, c_{k+1}, \dots, c_{k+q})$, $D = (d_1, \dots, d_k, d_{k+1}, \dots, d_{k+q})$ be the row added to X, and to Y, respectively. Then none of $c_1, \dots, c_k, d_1, \dots, d_k$ can equal either of $1, 2, \dots, k$ (due to latinicity), thus $\{1, 2, \dots, k\} \subseteq \{c_{k+1}, \dots, c_{k+q}\}$, and $\{1, 2, \dots, k\} \subseteq \{d_{k+1}, \dots, d_{k+q}\}$. Since $q < 2k$, by the pigeonhole principle there exists a pair (c_p, d_p) such that $\{c_p, d_p\} \subseteq \{1, 2, \dots, k\}$, a contradiction with our orthogonality assumption. Thus X, Y form a MOR$(k, k+q)$. □

The following theorem is a direct corollary of Lemma 3.1.

Theorem 3.2. *Let $k \geq 3, k \neq 6, n \geq 2k, n - k \neq 6$. Then there exists a MOR(k, n) whenever $\frac{n}{3} < k \leq \frac{n}{2}$.*

Remark. Using Lemma 3.1 and Theorem 3.2, we obtain that there exists a MOR(k, n) whenever $\frac{1}{3}n < k \leq \frac{1}{2}n, k \geq 3$ except possibly when $(k, n) \in \{(6,12),(6,13),(6,14),(6,15),(6,16),(6,17),(5,11),(4,10)\}$. However, concerning the last two cases, the existence of a pair of incomplete MOLS of order 6 with a hole of size 2 (see [ACD]) shows that a slight modification of Lemma 3.1 ensures the existence of a MOR(5,11) and of a MOR(4,10) as well.

Theorem 3.3. *Let $r \leq \frac{n}{3} (r \neq 2, 6)$, and suppose there exists MOR(k, r). Then there exists a MOR(n − r + k, n).*

Proof. It is well known [H1] that a pair of MOLS of order r can be embedded in a pair of MOLS of order n if and only if $r \leq \frac{n}{3}$. Consider a pair of MOLS of order n, say, X, X', with a pair of MOLS of order r, say A, A', embedded in the lower left-hand corner of X, X', respectively. Replace now A, A' with two $k \times r$ latin rectangles B, B' forming a MOR(k, r) (by placing B, B' to occupy the first k rows of A, A', respectively), and delete the last $r - k$ rows of X, X'. Since B, B' form a MOR, the resulting two $(n - r + k) \times n$ orthogonal latin rectangles are clearly maximal. □

Corollary 3.4. *There exists a MOR(r, n) for $\frac{7}{9}n \leq r \leq n$, except possibly for $(r, n) \in \{(30,36),(33,39),(36,42),(39,45),(42,48),(45,51)\}$.*

Proof. Follows from Theorem 3.2, the Remark following it, and from Theorem 3.3. □

Suppose A and B form a MOR(r, n). The definition of a MOR(r, n) says that no matter how row C (D, respectively) is adjoined to A (B, respectively), to form two $(r + 1) \times n$ latin rectangles, there will be at least one cell $(r + 1, j)$ such that the ordered pair (c_j, d_j) already occurs among the ordered pairs obtained by superimposition of A and B, and thus violates orthogonality. If, however, this is true for *every* cell $(r + 1, j)$, the corresponding MOR is said to be *strong*. Note that, in particular, for each $r \neq 2, 6$, the MOR$(n, 2n)$ constructed in Lemma 3.1 is strong.

Theorem 3.5. *If there exist two orthogonal latin squares of order $2n$ and a $MOR(r, 2n)$ then there exists a $MOR(3n + r, 6n)$.*

Proof. Let X, Y, Z be three pairwise disjoint $2n$-element sets. Consider the following two $(3n + r) \times 6n$ latin rectangles M, M':

$$M = \begin{pmatrix} A_1 & C_1 & B_1 \\ B & A' & C \\ C_2 & B_2 & A_2 \end{pmatrix}, \quad M' = \begin{pmatrix} A_1' & B_1' & C_1' \\ B' & C' & A' \\ C_2' & B_2' & A_2' \end{pmatrix}$$

where $A_1 A_1'$ [C_1, B_1', and B_1, C_1', respectively] is a strong MOR($n, 2n$) (from Theorem 3.1) based on X [based on Z, Y, resp., and on Y, Z, resp.]. Further, B, B' [A, C', and C, A', resp.] are two MOLS of order $2n$ based on Y [based on X, Z, and on Z, X, resp.]. Finally, A_2, A_2' [B_2, B_2', and C_2, C_2', resp.] is a MOR(r, n) based on X [based on Y, and on Z, resp.].

Clearly, M and M' are orthogonal $(3n + r) \times 6n$ latin rectangles. Let us show that M, M' cannot be extended to a pair of $(3n + r + 1) \times 6n$ orthogonal latin rectangles.Consider the entries in the cell $(3n + r + 1, i)$ $(i = 1, \ldots, 2n)$ of M and M'. They cannot both belong to Y since B, B' is a pair of MOLS of order $2n$, and they cannot both belong to X since A_1, A_1' is a strong MOR($n, 2n$).Assuming that the $(3n + r + 1, i)$ entry of M belongs to X and that of M' belongs to Z (or vice versa) leads to a contradiction since A, C' is a pair of MOLS of order $2n$ (and so is C, A'). This leaves only the possibility that both $(3n + r + 1, i)$ entries of M, M' belong to Z. This must be true for all $i = 1, \ldots, 2n$ which is impossible since C_2, C_2' is a MOR(r, n) based on Z. □

Theorem 3.6. *If there exists a set of 3 MOLS of order n then there exists a $MOR(n + 2t, 2n)$ for $t = 1, \ldots, \lfloor \frac{n}{4} \rfloor$.*

Proof. Start with the strong MOR($n, 2n$) as obtained in Lemma 3.1 (from two MOLS(n)):
$X = \|A \mid B\|, Y = \| A' \mid B' \|$
where $A = (a_{ij}), A' = (a_{ij}')$ are based on $I = \{1, \ldots, n\}$, and $B = (b_{ij}), B' = (b_{ij}')$ are based on $J = \{n + 1, \ldots, n\}$. Moreover, we have $b_{ij} = a_{ij} + n$, and $b_{ij}' = a_{ij}' + n$. Let $T_1, \ldots, T_n$ [$T_1', \ldots, T_n'$, respectively] be the set of cells of simultaneous transversals of A, A' [of B, B', respectively], and suppose T_s contains the cells $(i_{1s}, 1), (i_{2s}, 2), \ldots, (i_{ns}, n)$

(then T_s' will contain the cells $(i_{1s}+n,1),\ldots,(i_{ns}+n,n)$). For a set $S\subseteq\{1,\ldots,n\}$, let $A_S=(a_{ij}{}^S), B_S=(b_{ij}{}^S)$ be the $n\times 2n$ latin rectangles obtained from A,B by interchanging the elements of T_s for all $s\in S$, i.e. $a_{ij}{}^S=b_{ij}$ if $(i,j)\in T_S, s\in S$, and $=a_{ij}$ otherwise, while $b_{ij}{}^S=a_{ij}$ if $(i,j)\in T_S, s\in S$, and $=b_{ij}$ otherwise. The latin rectangles A_S' and B_S' are defined similarly.

We now describe an extension of X,Y to a pair of $(n+2t)\times 2n$ orthogonal latin rectangles. In order to keep the notation from getting out of hand, we will give a description for the case $S=\{1\}$, with the case of general S being handled similarly. Let X',Y' be two $n\times n$ (orthogonal) latin rectangles defined by

$X'=\|A_{\{1\}}|B_{\{1\}}\|, Y'=\|A_{\{1\}}'|B_{\{1\}}'\|,$

and consider two $(n+2)\times n$ latin rectangles $X'\circ C$ and $Y'\circ C'$ where

$C=(C_1,C_2), C'=(C_1',C_2')$ where

$C_1=(c_{11},\ldots,c_{1n},c_{1,n+1},\ldots,c_{1,2n}),$

$C_2=(c_{21},\ldots,c_{2n},c_{2,n+1},\ldots,c_{2,2n}),$

$C_1'=(c_{11}',\ldots,c_{1n}',c_{1,n+1}',\ldots,c_{1,2n}')$

$C_2'=(c_{21}',\ldots,c_{2n}',c_{2,n+1}',\ldots,c_{2,2n}').$

Here $(c_{11},\ldots,c_{1n})$ is a projection of T_1 in A, $(c_{1,n+1},\ldots,c_{1,2n})$ is a projection of T_1' in B, $(c_{21}',\ldots,c_{2n}')$ is a projection of T_1 in A', and $(c_{2,n+1}',\ldots,c_{2,2n}')$ is a projection of T_1' in B'.

The rows C_2 and C_1' are now determined as follows. The first n elements of C_2 will all be elements of J; they must be chosen in such a way that the pairs $(c_{21},c_{21}'),\ldots,(c_{2n},c_{2n}')$ do not occur as ordered pairs in $(A_{\{1\}},A_{\{1\}}')$, the set of ordered pairs resulting when $A_{\{1\}}$ and $A_{\{1\}}'$ are superimposed. The first components $c_{21},\ldots,c_{2n}$ can in turn be selected so as to satisfy this requirement if there exists an SDR for the family $U_1,\ldots,U_n$ where $U_j=I\setminus V_j$, and $V_j=\{x:(x,c_{2j}'\in(A_{\{1\}},A_{\{1\}}')\}$. Since we have clearly $|U_1|=\cdots=|U_n|=q$ (say), and $(U_1,\ldots,U_n)$ induces a regular bipartite graph on $I\cup J$, an SDR of the required kind is guaranteed to exist (as long as $q>0$).

The last n elements $c_{2,n+1},\ldots,c_{2,2n}$ of C_2 are determined similarly except that they will all be elements of I. The first (and last) n elements for the row C_1 are determined in a similar manner. The obtained pair of $(n+2)\times 2n$ orthogonal latin rectangles is clearly maximal.

We can proceed in this fashion (i.e. keep extending a pair of $(n+2t)\times 2n$ latin rectangles by adding another two rows) as long as the sets analogous to $U_1, \ldots, U_n$ described above are nonempty, i.e. while $q > 0$. This will clearly be the case when $t \leq \lfloor \frac{n}{4} \rfloor$. □

We illustrate Theorem 3.6 starting with a (strong) MOR(5,10) to obtain MOR(7,10) and a MOR(9,10).

1 2 3 4 5 6 7 8 9 10	1 4 2 5 3 6 9 7 10 8
5 1 2 3 4 10 6 7 8 9	4 2 5 3 1 9 7 10 8 6
4 5 1 2 3 9 10 6 7 8	2 5 3 1 4 7 10 8 6 9
3 4 5 1 2 8 9 10 6 7	5 3 1 4 2 10 8 6 9 7
2 3 4 5 1 7 8 9 10 6	3 1 4 2 5 8 6 9 7 10

MOR(5,10)

6 2 3 4 5 1 7 8 9 10	6 4 2 5 3 1 9 7 10 8
5 1 7 3 4 10 6 2 8 9	4 2 10 3 1 9 7 5 8 6
4 5 1 2 8 9 10 6 7 3	2 5 3 1 9 7 10 8 6 4
3 9 5 1 2 8 4 10 6 7	5 8 1 4 2 10 3 6 9 7
2 3 4 10 1 7 8 9 5 6	3 1 4 7 5 8 6 9 2 10
1 4 2 5 3 6 9 7 10 8	8 10 7 9 6 4 1 3 5 2
7 10 8 6 9 5 3 1 4 2	1 3 5 2 4 6 8 10 7 9

MOR(7,10)

6 7 3 4 5 1 2 8 9 10	6 9 2 5 3 1 4 7 10 8
5 1 7 8 4 10 6 2 3 9	4 2 10 8 1 9 7 5 3 6
9 5 1 2 8 4 10 6 7 3	7 5 3 1 9 2 10 8 6 4
3 9 10 1 2 8 4 5 6 7	5 8 6 4 2 10 3 1 9 7
2 3 4 10 6 7 8 9 5 1	3 1 4 7 10 8 6 9 2 5
1 4 2 5 3 6 9 7 10 8	8 10 7 9 6 4 1 3 5 2
7 10 8 6 9 5 3 1 4 2	1 3 5 2 4 6 8 10 7 9
4 2 5 3 1 9 7 10 8 6	9 6 8 10 7 5 2 4 1 3
10 8 6 9 7 3 1 4 2 5	2 4 1 3 5 7 9 6 8 10

MOR(9,10)

Theorem 4.1. *For every m and every $t, 1 \le t \le m+1$ there exists a $MOR(2m+1, 2m+1+t)$.*

Proof. Let $M = Z_{2m+t+1}$. Consider a function $f : [0, 2m] \to M$ defined by $f(i) = i, 0 \le i \le m$ and $f(i) = i + t, m+1 \le i \le 2m$. Clearly, f is a 1-1 mapping of $[0, 2m]$ onto $[0, m] \cup [m+1+t, 2m+t]$. Now let $A = (a_{ij}), B = (b_{ij}), 0 \le i \le 2m, 0 \le j \le 2m+t$ be defined by $a_{ij} = i + j, b_{ij} = i + j + f(i)$ where the sums are taken in the group M. Obviously, A is a latin rectangle. To prove that B is a latin rectangle as well it suffices to show that the mapping $g : [0, 2m] \to M, g(i) = i + f(i)$ is one-to-one, i.e. that $i \ne k$ implies $g(i) \ne g(k)$. This is clear when $i, k \in [0, m]$ or $i, k \in [m+1, 2m]$. Let now, say, $0 \le i \le m$ and $k = m + s, 1 \le s \le m$. Assuming $g(i) = g(k)$ gives $2i = 2(m+s) + t$ in M, and it follows that $2i = 2m + t + 1 + 2s - 1 = 2s - 1$, still in M. But if $0 \le i \le m$ and $1 \le s \le m$, we cannot have $2i = 2s - 1$ in $M = Z_{2m+t+1}$. Hence g is 1-1, and B is a latin rectangle.

Next we show that our two $(2m+1) \times (2m+1+t)$ latin rectangles A and B are orthogonal. Assume the contrary, and let there exist pairs $(i, j) \ne (r, s)$ such that $a_{ij} = a_{rs}$ and $b_{ij} = b_{rs}$, i.e. $i + j = r + s$ and $i + j + f(i) = r + s + f(r)$. Thus, $i \ne r$ but $f(i) = f(r)$, contradicting the fact that f is 1-1. Thus A and B are orthogonal.

It remains to be shown that A, B cannot be extended to a pair of orthogonal rectangles by adding a row. We prove a much stronger version, namely that it is impossible to add the elements $a_{2m+1,0}$ and $b_{2m+1,0}$.

Let $x = 2m + l, 1 \le l \le t$, and consider the set $B_x = \{y \in M : x = a_{ij}$ and $y = b_{ij}$ for some $i, j, 0 \le i \le 2m, 0 \le j \le 2m+t\}$. Now, $y \in B_x$ is and only if $y = i + j + f(i)$ for $i + j = 2m + l, 1 \le l \le t$. If $0 \le i \le m$ then $y = 2i + j = 2i + (2m + l - i) = i + 2m + l$. In other words, if $i \in [0, m]$ then $y \in [2m+l, 2m+t] \cup [0, m+l-t-1] \subseteq M$ (note that the last interval is empty when $l = 1$ and $t = m+1$). If $m+1 \le i \le 2m$ then $y = 2i + t + j = 2i + t + (2m + l - i) = i + 2m + t + l = i + l - 1$ in M, and therefore $y \in [m+l, 2m+l-1]$ in this case. Summing up, if $x = 2m + l, 1 \le l \le t$ then $B_x = [m+l, 2m+t] \cup [0, m+l-t-1]$.

Now let us determine the entries b_{i0} (i.e. we put $j = 0$). We know that $b_{i0} = i + f(i)$; if $0 \le i \le m$ then $b_{i0} = 2i$ and if $i = m + s, 1 \le s \le m$

then $b_{i0} = 2(m+s)+t = 2s-1$ (in M). We see that $b_{i0} \in [0,2m]$ for each $i, 0 \le i \le 2m$. Note that the same is true for the entries a_{i0}.

Finally, assume that there were entries $a_{2m+1,0}$ and $b_{2m+1,0}$ which would appear in the corresponding cells of the new row to be added to A and B, respectively. It follows from our earlier considerations that both, $a_{2m+1,0}$ and $b_{2m+1,0}$ must belong to the set $[2m+1, 2m+t]$. But then, putting $x = a_{2m+1,0}$ we see that $b_{2m+1,0} \notin B_x$, i.e. $b_{2m+1,0} \in [m+l-t, m+l-1]$. However, $l \le t \le m+1$ implies that $m+l-1 < 2m+1$ (as always $m+l-t \ge 0$), and therefore $[2m+1, 2m+t] \cap [m+l-t, m+l-1] = \emptyset$. Thus there is no way to choose $b_{2m+1,0}$ for any given $a_{2m+1,0}$. Thus A, B form a MOR, and the proof is complete. □

The next example is an application of Theorem 4.1 with $m = 2, t = 3$ and $M = Z_8 = \{0, 1, \ldots, 8\}$ to produce MOR(5,8).

Example 4.2. MOR(5,8)

0 1 2 3 4 5 6 7	0 1 2 3 4 5 6 7
1 2 3 4 5 6 7 0	2 3 4 5 6 7 0 1
2 3 4 5 6 7 0 1	4 5 6 7 0 1 2 3
3 4 5 6 7 0 1 2	1 2 3 4 5 6 7 0
4 5 6 7 0 1 2 3	3 4 5 6 7 0 1 2

Theorem 4.3. *If there exists a MOR(k,n) then there exists a MOR$(2n+1+k, 3n+1)$.*

Proof. Let A, A' form a MOR(k,n) on elements $x_1, \ldots, x_n \notin Z_{2n+1}$. Define now a pair of latin rectangles $B = (b_{ij}), B' = (b_{ij}')$ as follows (arithmetic operations on subscripts are in Z while the remaining operations are in Z_{2n+1}):

(i) $0 \le i, j \le 2n$. Put $M_i = \{i-t : t \in [1,n]\}, M_i' = \{i+t : t \in [1,n]\}$ note that $M_i \cap M_i' = \emptyset$, $M_i \cup M_i' = \{i\} \in Z_{2n+1}$. Then define $b_{ij} = 2i-j$ if $j \notin M_i$, and $b_{ij} = x_t$ if $j = i-t \in M_i$, $b_{ij}' = 2j-i$ if $j \notin M_i'$, and $b_{ij}' = x_t$ if $j = i+t \in M_i'$.

(ii) $b_{i,2n+j} = i+j, b_{i,2n+j}' = i+2j$ for $0 \le i \le 2n, 1 \le j \le n$.

(iii) $b_{2n+i,j} = 2i+j, b_{2n+i,j}' = i+j$ for $1 \le i \le k, 0 \le j \le 2n$.

(iv) $b_{2n+i,2n+j} = a_{ij}, b_{2n+i,2n+j}' = a_{ij}'$ for $1 \le i \le k, 1 \le j \le n$.

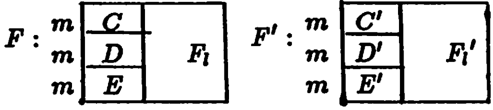
(Note that $a_{ij} = x_j, 1 \leq j \leq n$.)

It is a routine matter to verify that B and B' are indeed orthogonal $(2n+1+k) \times (3n+1)$ latin rectangles. Letting now $B_t' = \{b_{ij}' : x_t = b_{ij}\}$ for a fixed $t, 1 \leq t \leq n$ we see that $B_t' = \{b_{ij}' : j = i - t\} = \{i - t : i \in Z_{2n+1}\} = Z_{2n+1}$. This shows that in a purported extension of B, B' by one additional row, no element of Z_{2n+1} can appear in the $2n+1+j$-th column of this additional row for any $j = 1, \ldots, n$. Thus the last n elements of this new row added to B' would have to be x_i's which is impossible since A, A' form a MOR. □

We remark that the construction of Theorem 4.3 is very similar to that in [DK] used to obtain a pair of MOLS of order $3n+1$ (Theorem 11.4.5 of [DK]).

Theorem 4.4. *Let m be odd and l even such that $2m \leq l < 4m$. Then there exists a MOR$(3m, 3m+l)$.*

Proof. Consider the group Z_n where $n = m + l$. Let $F = (f_{ij}), F' = (f_{ij}'), 0 \leq i \leq 3m-1, 0 \leq j \leq n-1$ be given by
$f_{ij} = i + 2j, f_{ij}' = i + j$,
the sums being taken in the group Z_n. Since $n \geq 3m$ and n is odd, F, F' are orthogonal $3m \times n$ latin rectangles. Define now six $m \times m$ subarrays of F and F' as follows (we always assume $0 \leq i, j \leq m-1$):
$C = (c_{ij}), c_{ij} = f_{ij}, D = (d_{ij}), d_{ij} = f_{i+m,j}, E = (e_{ij}), e_{ij} = f_{i+2m,j}, C' = (c_{ij}'), c_{ij}' = f_{ij}', D' = (d_{ij}'), d_{ij}' = f_{i+m,j}', E' = (e_{ij}'), e_{ij}' = f_{i+2m,j}'$.
Thus we have:



Let $Z_m{}^\alpha$ and $Z_m{}^\beta$ be two disjoint groups isomorphic to Z_m (and both disjoint from Z_n introduced above). Put $Z_m{}^\alpha = \{\alpha_0, \alpha_1, \ldots, \alpha_{m-1}\}$, $Z_m{}^\beta = \{\beta_0, \beta_1, \ldots, \beta_{m-1}\}$, and let $\alpha_i + \alpha_j = \alpha_{i+j}, \beta_i + \beta_j = \beta_{i+j}, 0 \leq i, j \leq m-1$ (the sum in the subscript being mod m).

On $Z_m{}^\alpha$ define two $m \times m$ arrays $A = (a_{ij})$ and $A' = (a_{ij}'), 0 \leq i, j \leq m-1$ by $a_{ij} = \alpha_{i+j}, a_{ij}' = \alpha_{i+2j}$. Similarly we define two $m \times m$ arrays $B = (b_{ij}), B' = (b_{ij}')$ on $Z_m{}^\beta$ by $b_{ij} = \beta_{i+2j}, b_{ij}' = \beta_{i+j}$.

Again, since m is odd, A, A' are orthogonal latin squares, and so are B, B'.

Now let L and L' be two $3m \times (3m + l)$ arrays on the set of symbols $Z_m{}^\alpha \cup Z_m{}^\beta \cup Z_n$ defined as follows:

$$L = \begin{array}{|c|c|c|c|} \hline A' & B & C & \\ \cline{1-3} D & A & B & F_l \\ \cline{1-3} B' & E & A & \\ \hline \end{array} \qquad L' = \begin{array}{|c|c|c|c|} \hline C' & A & B' & \\ \cline{1-3} A & B & D' & F_l' \\ \cline{1-3} B & E' & A' & \\ \hline \end{array}$$

It is obvious from the construction that both L and L' are latin rectangles. To prove that they are orthogonal we have to show that the pairs of arrays $(A, B), (A', C'), (B, D'), (A, D)$ and (B', C) are orthogonal. This is obvious for the pair (A, B). Of the remaining four cases, consider (A, D); the others are similar. Thus, assume that $\alpha_{i+j} = \alpha_{r+s}$ and $d_{ij} = d_{rs}$ where $0 \leq i, j, r, s \leq m - 1$. Thus $i + j = r + s \ (mod\ m)$ and $i + m + 2j = r + m + 2s \ (mod\ n)$. Substituting $i + j = r + s + \epsilon m, \epsilon \in \{0, +1, -1\}$ into the second equation we obtain $m + j = m + \epsilon m + s \ (mod\ m)$, i.e., $j = s + \epsilon m \ (mod\ n)$. But $0 \leq j, s \leq m - 1$ and $n > m$ which implies that $\epsilon = 0, j = s$, and $i = r$. This proves orthogonality.

It remains to be shown that L, L' are maximal orthogonal. Assume to the contrary that $(\lambda_{3m,j})$ and $(\lambda_{3m,j}{}'), 0 \leq j \leq 3m + l - 1$ are two new rows which could be added to L and L' to form a new pair of orthogonal latin rectangles. The way in which L, L' were defined implies that $(Z_m{}^\alpha \cup Z_m{}^\beta) \cap \{\lambda_{3m,j} : 0 \leq j \leq 3m - 1\} = \emptyset$ and $(Z_m{}^\alpha \cup Z_m{}^\beta) \cap \{\lambda_{3m+j}{}' : 0 \leq j \leq 3m - 1\} = \emptyset$. Thus both sets $L_{3m} = \{\lambda_{3m+j} : j \geq 3m\}$ and $L_{3m}{}' = \{\lambda_{3m+j}{}' : j \geq 3m\}$ contain the $2m$-element set $Z_m{}^\alpha \cup Z_m{}^\beta$ as a subset. However, since $|L_{3m}| = |L_{3m}{}'| = l < 4m$, there must be a pair $(\lambda_{3m+j}, \lambda_{3m+j}{}')$ for some $j, 3m \leq j < 3m + l$ such that both λ_{3m+j} and $\lambda_{3m+j}{}'$ belong to $Z_m{}^\alpha \cup Z_m{}^\beta$. But by our construction, L, L' already contain all the $4m^2$ ordered pairs of elements of $Z_m{}^\alpha \cup Z_m{}^\beta$ (just consider the pairs of squares $(A, A'), (A, B), (B, A)$ and (B', B)). This contradiction proves maximality. $\square$

Theorem 4.5. *Let m and l be odd, $m < l < 2m$. Then there exists a $MOR(3m, 3m+l)$.*

Proof. Put $n = 2m + l$ and let $D = (d_{ij}), D' = (d_{ij}'), 0 \leq i \leq 3m-1, 0 \leq j \leq n-1$ be two orthogonal $3m \times n$ latin rectangles where $d_{ij} = i + j, d_{ij}' = i + 2j$ (sums in Z_n, n odd). Consider the subarrays B_k, C_k, B_k', C_k' of D and D', $k = 1, 2, 3$ as depicted:

$$D = \begin{array}{c} \\ m \\ m \\ m \end{array} \begin{array}{|c|c|c|} \multicolumn{1}{c}{m} & \multicolumn{1}{c}{m} & \multicolumn{1}{c}{\ell} \\ \hline B_1 & C_1 & \\ \cline{1-2} B_2 & C_2 & D_l \\ \cline{1-2} B_3 & C_3 & \\ \hline \end{array} \qquad D' = \begin{array}{c} \\ m \\ m \\ m \end{array} \begin{array}{|c|c|c|} \multicolumn{1}{c}{m} & \multicolumn{1}{c}{m} & \multicolumn{1}{c}{\ell} \\ \hline B_1' & C_1' & \\ \cline{1-2} B_2' & C_2' & D_l' \\ \cline{1-2} B_3' & C_3' & \\ \hline \end{array}$$

Let now A, A' be a pair of orthogonal latin squares on the group $Z_m{}^\alpha$ (as in the previous proof) given by $A = (a_{ij}), a_{ij} = \alpha_{i+j}, A' = (a_{ij}'), a_{ij}' = \alpha_{i+2j}$ where $0 \leq i, j \leq m-1$, and the sums in subscripts are taken mod m. Consider the following two $3m \times (3m + l)$ arrays L, L' on $Z_n \cup Z_m{}^\alpha$:

$$L = \begin{array}{|c|c|c|c|} \hline A & B_1 & C_1 & \\ \cline{1-3} B_2 & A & C_2 & D_l \\ \cline{1-3} B_3 & C_3 & A & \\ \hline \end{array} \qquad L' = \begin{array}{|c|c|c|c|} \hline B_1' & A' & C_1' & \\ \cline{1-3} A' & B_2' & C_2' & D_l' \\ \cline{1-3} B_3' & C_3' & A' & \\ \hline \end{array}$$

Both L and L' are obviously latin. To prove their orthogonality it suffices to show that $A \perp (B_1' \cup B_2')$, and $A' \perp (B_1 \cup B_2)$ (in the obvious sense of this unusual symbol). We show just the first part, as the second one is similar. Let $\alpha_{i+j} = \alpha_{r+s}$ and $i+2j = r+2s+\delta m \pmod n$ where $\delta = 0$ or 1, according as we have elements from the same B_k' or from different B_k'-'s. Substituting $i + j = r + s + \epsilon m, \epsilon \in \{0, 1, -1\}$, into the second equation we obtain $j = s + \kappa m$ where $\kappa \in \{-1, 0, 1, 2\}$. But $0 \leq j, s \leq m - 1$, and from $j = s + \kappa m \pmod n$, $n \geq 3m$ we have $j = s$ and hence $i = r$. (Note that in the case $A' \perp (B_1 \cup B_2)$ we really need $l > m$, i.e. $n > 3m$ to prove the orthogonality.)

The fact that L, L' are maximal can be proved similarly as in Theorem 4.3. Indeed, let $(\lambda_{3m,j})$ and (λ_{3m}') be two new rows which could be added to L and L'. Again, if $L_{3m} = \{\lambda_{3m,j} : j \geq 3m\}, L_{3m}' = \{\lambda_{3m,j}' : j \geq 3m\}$ then it follows from the construction that $Z_m{}^\alpha \subseteq L_{3m}$ and $Z_m{}^\alpha \subseteq L_{3m}'$. But since $|L_{3m}| = |L_{3m}'| = l < 2m$, there is a $j \geq 3m$ such that both $\lambda_{3m,j}$ and $\lambda_{3m,j}'$ belong to $Z_m{}^\alpha$. This contradicts the fact that all m^2 ordered pairs of elements of $Z_m{}^\alpha$ have already been used up (just consider the pairs of squares A, A'). □

5. Small orders

A small improvement on Corollary 2.4 is offered by the following.

Lemma 5.1. *There exists no MOR(2,n) for any n.*

Proof. In view of Corollary 2.4 and our remarks in Section 1, we need to prove the statement only for $n = 5, 6, 7$. Let $A = (a_{ij}), B = (b_{ij})$ be orthogonal $2 \times n$ latin rectangles. We will show that there always exist row arrays C and D such that $A \circ C$ and $B \circ D$ are orthogonal $3 \times n$ latin rectangles. We may assume w.l.o.g. that $a_{1j} = b_{1j}$ for $j = 1, \ldots, n$. Set $C = (b_{21}, \ldots, b_{2n})$. Clearly, $A \circ C$ is latin as $a_{2j} \neq b_{2j}$. Let $E = B \circ (A \circ C)^{\perp} = (e_{ij})$. To complete the proof we need to show that $\mathcal{E} = \{E(j) : j = 1, \ldots, n\}$ has a system of distinct representatives. As the first row of $(A \circ C)^{\perp}$ equals C which equals the second row of B, we get $|E(j)| = n-3$ for $j = 1, \ldots, n$. Thus P. Hall's condition is trivially satisfied for any $J \subset \{1, \ldots, n\}, |J| \leq n - 3$. In addition, any element of the symbol set S occurs in at most 3 columns of E, hence
$|\bigcup_{j \in J} A(j)| = n \geq |J|$ for any $|J| \geq 4$. This proves the statement for $n = 6, 7$. When $n = 5$ this leaves still the case $|J| = 3$. Assuming in this case that $\mathcal{E}$ does not have a system of distinct representatives leads again to a contradiction (this case does require further analysis but we omit the details). □

Lemma 5.2. $\mathcal{M}(5) = \{3, 5\}$.

Proof. An exhaustive search establishes that there exists no MOR(4,5). This, together with the example given in the Introduction, and Lemma 5.1 establishes the claim. □

Lemma 5.3. $\mathcal{M}(6) = \{3, 4, 5\}$.

Proof. The existence of MOR(r,6) for r=3 follows from Lemma 3.1, and for r=5 from the existence of a pair of incomplete MOLS of order 6 with a hole of size 2 (see, e.g., [ACD]). The nonexistence of a MOR(6,6) is equivalent to the nonexistence of a pair of MOLS of order 6. An example of a MOR(4,6) is given below. □

4 2 3 1 5 6	1 3 2 4 6 5
3 1 5 6 4 2	3 2 1 6 5 4
2 6 1 5 3 4	2 1 3 5 4 6
1 3 4 2 6 5	5 6 4 1 2 3

Lemma 5.4. $\mathcal{M}(7) = \{3, 4, 5, 6, 7\}$.

Proof. Lemma 3.1 ensures the existence of a MOR(3,7) while Theorem 3.3 yields MOR(r,7) for r=6,7. Examples of MOR(r,7) for r=4 and 5 are given below. □

4 3 2 1 7 6 5	3 2 4 7 6 1 5
3 4 1 7 2 5 6	4 1 3 2 5 7 6
2 1 4 6 5 3 7	1 4 2 5 3 6 7
1 2 3 5 6 7 4	2 3 1 6 7 5 4

MOR(4,7)

1 2 3 4 5 6 7	1 5 2 6 3 7 4
7 1 2 3 4 5 6	5 2 6 3 7 4 1
6 7 1 2 3 4 5	2 6 3 7 4 1 5
5 6 7 1 2 3 4	6 3 7 4 1 5 2
4 3 5 6 7 2 1	4 7 1 5 2 3 6

MOR(5,7)

Lemma 5.5. $\mathcal{M}(8) = \{3, 4, 5, 6, 7, 8\}$

Proof. Again, MOR(r,8) for r=3,4 is obtained by Lemma 3.1, for r=7,8 by Theorem 3.3, and for r=6 by Theorem 3.6. An example of MOR(5,8) was given in Example 4.2. □

Lemma 5.6. $\{4, 5, 7, 8, 9\} \subseteq \mathcal{M}(9)$.

Proof. Theorem 3.2 implies the existence of a MOR(4,9). A MOR(r,9) for r=8,9 exists by Theorem 3.3. An example of a MOR(7,9) is given below. □

1 2 3 4 5 6 7 8 9	1 6 2 7 3 8 4 9 5
9 1 2 3 4 5 6 7 8	6 2 7 3 8 4 9 5 1
8 9 1 2 3 4 5 6 7	2 7 3 8 4 9 5 1 6
7 8 9 1 2 3 4 5 6	7 3 8 4 9 5 1 6 2
6 7 8 9 1 2 3 4 5	3 8 4 9 5 1 6 2 7
5 6 7 8 9 1 2 3 4	8 4 9 5 1 6 2 7 3
4 3 5 7 6 8 9 1 2	5 9 1 2 6 7 3 8 4

MOR(7,9)

Lemma 5.7. $\{4,5,7,8,9,10\} \subseteq \mathcal{M}(10)$.

Proof. The existence of a MOR(r,10) for r=4,5 is ensured by Theorem 3.2 and the remark following it, for r=9,10 by Theorem 3.3, and for r=7,9 by Theorem 3.6. The example of a MOR(8,10) given below was supplied to us by Don Kreher. □

0 1 8 9 6 7 4 5 2 3	0 1 8 9 6 7 4 5 2 3
1 0 9 8 7 6 5 4 3 2	2 3 0 1 8 9 6 7 4 5
2 3 0 1 8 9 6 7 4 5	4 5 2 3 0 1 8 9 6 7
3 2 1 0 9 8 7 6 5 4	6 7 4 5 2 3 0 1 8 9
4 5 2 3 0 1 8 9 6 7	3 2 1 0 9 8 7 6 5 4
5 4 3 2 1 0 9 8 7 6	1 0 9 8 7 6 5 4 3 2
6 7 4 5 2 3 0 1 8 9	7 6 5 4 3 2 1 0 9 8
7 6 5 4 3 2 1 0 9 8	5 4 3 2 1 0 9 8 7 6

MOR(8,10)

6. Conclusion and open problems

The results of previous sections lead us to the following conjecture on the spectrum for MORs:

Conjecture. *For sufficiently large n, $\mathcal{M}(n) = (\frac{n}{3}, n]$.*

(Here "sufficiently large" is not likely to mean "very large", rather something like $n \geq 30$.)

However, we are currently unable to prove, for example, that MOR(r, n) does not exist for $\frac{n}{4} \leq r \leq \frac{n}{3}$, nor are we able to "fill in" the spectrum completely within the conjectured range. The smallest open problem is to decide the existence of a MOR(6,9).

Acknowledgement

The authors thank Don Kreher for many stimulating conversations and valuable comments. Part of this research was done while the second author was visiting Kuwait University. He thanks the Department of Mathematics of Kuwait University for its hospitality.

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