The p-Competition Graphs of Strongly Connected and Hamiltonian Digraphs

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ABSTRACT. Competition graphs were first introduced by Joel Cohen in the study of food webs and have since been extensively studied. Graphs which are the competition graph of a strongly connected or Hamiltonian digraph are of particular interest in applications to communication networks. It has been previously established that every graph without isolated vertices (except K_2) which is the competition graph of a loopless digraph is also the competition graph of a strongly connected digraph. We establish an analogous result for one generalization of competition graphs, the p-competition graph. Furthermore, we establish some large classes of graphs, including trees, as the p-competition graph of a loopless Hamiltonian digraph and show that interval graphs on $n \geq 4$ vertices are the 2-competition graphs of loopless Hamiltonian digraphs.

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1 Introduction

Competition graphs were first introduced in 1968 by Cohen [3] in connection to the study of food webs and have since found many applications. One such example is the assignment of frequencies to transmitters in radio communication networks. Since it is desirable that a message initiated somewhere in the network be able to reach all stations, typically the digraphs for these networks are strongly connected. *Which graphs are the competition graphs of strongly connected digraphs?* Answers to this question are provided by Fraughnaugh et al. [4]. The area of competition graphs has been extensively researched, for example by Brigham and Dutton [1, 2], Lundgren and Maybee [7], Raychaudhuri and Roberts [8], and Roberts and Steif [9] and has generated related topics such as niche graphs, tolerance competition graphs and p-competition graphs. The p-competition graph was first introduced by Kim, McKee, McMorris, and Roberts [6]. This paper generalizes the work of Fraughnaugh et al. [4] in considering the question *which graphs are the p-competition graphs of loopless strongly connected and Hamiltonian digraphs?*

For definitions not given here, the reader is referred to Golumbic [5]. We use $V(G)$ and $E(G)$ to denote the vertex set and edge set of a graph G respectively. We use $V(D)$ and $A(D)$ to denote the vertex set and arc set of a digraph D respectively. We let $\mathrm{In}_D(x)$ denote the inset of a vertex x in a digraph D and $i(G)$ denote the set of isolated vertices in a graph G.

2 Preliminaries

The *p-competition graph* of a digraph D, denoted $C_p(D)$, is a graph on the same vertex set with vertices x and y adjacent in $C_p(D)$ if and only if there are $k \geq p$ vertices, $v_1, \ldots, v_k$, such that (x, v_i), (y, v_i) are arcs in D for all i. If $p = 1$, then $C_p(D)$ is called the *competition graph* of D. A *p-edge clique cover* (p-ECC) of a graph G is a family of subsets of $V(G)$, $\{S_1, \ldots, S_k\}$ (repetitions allowed) such that $\{x, y\} \in E(G)$ if and only if x and y appear together in at least p of the sets. Observe that if $G = C_p(D)$, then $\{\mathrm{In}_D(x) \mid x \in V(D)\}$ is a p-ECC for G. When $p = 1$, a p-ECC is called an *edge clique cover.* One should be careful to make the following distinction between a 1-ECC and a p-ECC for $p \geq 2$. While in a 1-ECC the sets are necessarily cliques in the graph, for $p \geq 2$, the sets in a p-ECC are not necessarily cliques. Every intersection of p sets in the family is either a clique or the empty set (see Figure 1).

We let $\Theta_E^p(G)$ denote the minimum cardinality of a p-ECC for the graph G. Kim, et al. [6] proved that if G has n vertices and $\Theta_E^1(G) \leq n - p + 1$, then G is the p-competition graph of an arbitrary digraph (possibly with loops). The same authors also proved that a graph G with n vertices is a p-competition graph of an arbitrary digraph if and only if $\Theta_E^p(G) \leq n$. This generalizes a result by Brigham and Dutton [1], who originally established the result for $p = 1$.

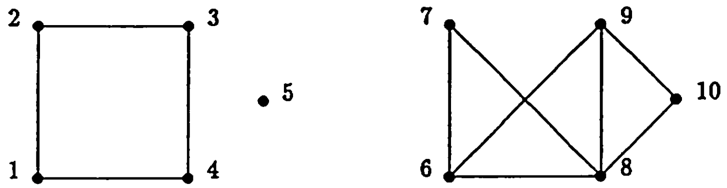

Figure 1.
The family of sets $\{1,2,3,4\}$, $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, $\{1,4\}$ is a 2-ECC for the graph on the left. The family of sets $\{6,7,8\}$, $\{6,8,9\}$, $\{8,9,10\}$ is a 1-ECC for the graph on the right.

3 The p-Competition Graphs of Strongly Connected Digraphs

If $\Theta^p_E(G) \leq n$, *is* G *necessarily the p-competition graph of a loopless strongly connected digraph*? The following result of Fraughnaugh, et al.[4] tells us the answer is no.

Proposition 3.1. *For* $p = 1$, $G \neq K_2$ *is the p-competition graph of a loopless strongly connected digraph if and only if* $\Theta^p_E(G) + i(G) \leq n$.

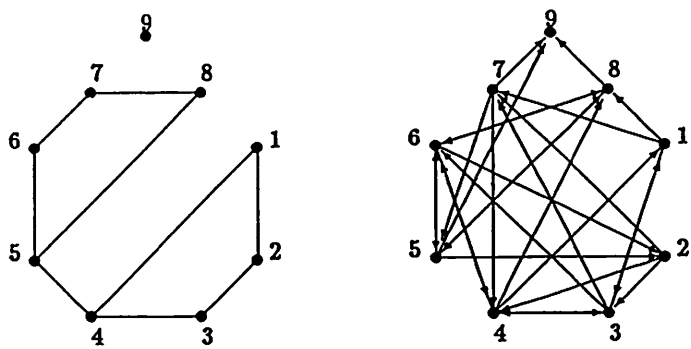

Figure 2.
G is the 1-competition graph of a loopless digraph, but not one that is strongly connected. G is the 2-competition graph of a loopless strongly connected digraph. The digraph is not strongly connected but G is its 2-competition graph. Adding the arc (9,8) makes the digraph strongly connected, but does not change its 2-competition graph.

Recall that if a digraph is strongly connected, then every vertex has an incoming and an outgoing arc. Consider the graph in Figure 2. A 1-ECC for this graph has at least 9 sets. Thus every vertex in the digraph must have at least two incoming arcs, one for each endpoint of an edge in the graph. The isolated vertex is not in a minimum 1-ECC but must have an outgoing arc in order that D be strongly connected. This outgoing arc creates a competition between the isolated vertex and some other vertex in the graph, a contradiction. This graph is the 2-competition graph of a loopless strongly connected digraph (see the digraph in Figure 2). Notice that by adding 9 to the inset of 8 makes the digraph strongly connected, but 9 competes at most once with any other vertex.

Theorem 3.2. *Let $p \geq 2$ and G be a graph which is the p-competition graph of some loopless digraph. Then G is the p-competition graph of some strongly connected loopless digraph.*

Proof: Observe that if G has $|V(G)|$ isolated vertices then let D be a directed cycle on $|V(G)|$ vertices and $C_p(D) = G$ for D strongly connected. Therefore we may assume G has at least one edge.

Let D be a loopless digraph with fewest number of strongly connected components such that $C_p(D) = G$. Let $D_1, D_2, \ldots, D_k$ be the topological ordered strongly connected components of D. Recall that an ordering of the strongly connected components of $D, D_1, D_2, \ldots, D_k$ is *topological* if and only if whenever $x \in D_i$ and $y \in D_j$ exist such that there is an arc from x to y, then $i \leq j$ (see Golumbic [5] for a proof that such a topological ordering exists).

If $D_1 = D_k$ we are done so assume $k \geq 2$. Since G has at least one edge, we can assume that D_1 contains a vertex x with at least one outgoing arc (if D_1 does not contain such a vertex then $D_2, \ldots, D_k$, D_1 is a topological ordering of the strongly connected components). We then have three cases.

Case 1: $D_1 = \{x\}$. Then $\text{In}(x) = \emptyset$ and $\text{Out}(x) \neq \emptyset$. Let y be a vertex such that there is an arc from x to y. Let D_y denote the strongly connected component containing y. Create D' by adding the arc (y, x) to D. The set of vertices competing with x at least p times is unchanged. Since y is the only vertex with an arc to x, the set of vertices competing with y is unchanged. Thus $C_p(D) = G$ implies $C_p(D') = G$ and D' has fewer strongly connected components than D, a contradiction.

Case 2: $|D_1| \geq 2$ and there exists $y \in D_k$, $x \in D_1$ such that there is no arc from x to y. Suppose $|D_k| < p$. If there exists $q \notin D_k$ such that $(q, y) \in A$, create D' by adding (y, q) to D. Since D is loopless, y competes at most $(p-1)$ times with any vertex (namely at most $(p-2)$ times for a vertex in D_k, and once for q). Therefore the set of vertices competing with y at least p times is unchanged. Then $C_p(D) = G$ implies $C_p(D') = G$. Letting D_q denote the strongly connected component containing q, we observe that $D_k \cup D_q$ is strongly connected in D', i.e., D' has fewer strongly connected components than D, a contradiction.

Thus all arcs incoming at y originate in D_k. Create D' by adding (x, y) and (y, x) to D. Then the set of vertices competing with x at least p times in D has not changed since at most $(p-1)$ vertices have arcs to y (namely at most $(p-2)$ vertices in D_k and x). The set of vertices competing with y at least p times in D has not changed since y has at most $(p-1)$ outgoing arcs (namely to at most $(p-2)$ vertices in D_k and x). Thus $C_p(D) = G$ implies $C_p(D') = G$ and $D_1 \cup D_k$ is strongly connected in D', i.e., D' has fewer strongly connected components than D, a contradiction.

Thus $|D_k| \geq p$. Since $|D_1| \geq 2$, $\text{In}(x) \neq \emptyset$ and since $2 \leq p \leq |D_k|$, $\text{In}(y) \neq \emptyset$. Create D' from D by switching the insets of x and y and leaving all other arcs the same. No competitions have changed therefore $C_p(D) = G$ implies $C_p(D') = G$. Since $|D_1| \geq 2$ and $|D_k| \geq 2$, $D_1 \cup D_k$ is strongly connected in D', i.e., D' has fewer strongly connected components than D, a contradiction.

Case 3: $|D_1| \geq 2$ and for all $x \in D_1$ and all $y \in D_k$ there is an arc from x to y. Suppose $|D_k| < p$. Create D' by adding arc (y, x) to D. Then the set of vertices competing with y at least p times has not changed since y has arcs to at most $(p-1)$ vertices (namely at most $(p-2)$ in D_k and x). Since the set of vertices competing with x at least p times has not changed, $C_p(D) = G$ implies $C_p(D') = G$. Since $D_1 \cup D_k$ is strongly connected in D', D' has fewer strong components than D, a contradiction.

Thus $|D_k| \geq p$. Since every vertex in D_1 has an arc to every vertex in D_k, D_1 is a clique in G. Thus we can remove arcs between vertices strictly in D_1 and the p-competition graph is unchanged. Pick an arbitrary vertex $x \in D_1$. Create D' by deleting all arcs incoming at x and adding arc (y, x) to D. The set of vertices competing with y at least p times is unchanged since y is the only vertex with an arc to x. Thus $C_p(D) = G$ implies $C_p(D') = G$. Since $|D_1| \geq 2$ and $|D_k| \geq 2$, $D_1 \cup D_k$ is strongly connected in D', i.e., D' has fewer strongly connected components than D, a contradiction.

Since in each case the contradiction implies D has fewer than k strongly connected components where $k \geq 2$, we must have $k = 1$, i.e., D is strongly connected. Therefore every graph which is the p-competition graph of a loopless digraph is the p-competition graph of a strongly connected loopless digraph. □

4 Hamiltonian Digraphs: Constructions

The following constructions will be useful in characterizing several large classes of graphs as the p-competition graphs of loopless Hamiltonian digraphs.

Lemma 4.1. *Let G be a connected graph such that G is the p-competition graph of a loopless Hamiltonian digraph. Adding a pendant vertex x to G results in a graph G' which is also the p-competition graph of a loopless Hamiltonian digraph.*

Proof: Let D be a loopless Hamiltonian digraph such that $C_p(D) = G$. Let $v_1, v_2, \ldots, v_n$ denote a Hamiltonian cycle in D. Create D' from D as follows. Add vertex x. Let v_i denote the vertex adjacent to x in G'. Let $\mathrm{In}_{D'}(x) = \mathrm{In}_D(v_{i-1})$. Let $\mathrm{In}_{D'}(v_{i-1}) = \{x, v_i\}$. Observe that $v_1, v_2, \ldots, v_{i-2}, x, v_{i-1}, v_i, \ldots, v_n$ is a Hamiltonian cycle in D'. Since G is connected v_i has outgoing arcs to at least p other vertices of D. Let x have an outgoing arc to $p-1$ of these vertices. Then $C_p(D') = G'$, where D' is Hamiltonian. □

Corollary 4.2. *Let T be a tree. If T has a subtree which is the p-competition graph of a loopless Hamiltonian digraph then T is the p-competition graph of a loopless Hamiltonian digraph.*

Proof: This follows from Lemma 4.1, since we may add pendant vertices successively to the subtree, obtaining T. □

Lemma 4.3. *Let T be a tree which is the p-competition graph of a loopless Hamiltonian digraph D. Adding a pendant vertex x to a vertex of degree $d \geq 2$ results in a tree T' that is the $(p+1)$-competition graph of a loopless Hamiltonian digraph.*

Proof: Let D be a loopless Hamiltonian digraph such that $C(D) = T$. Let $v_1, v_2, \ldots, v_n$ denote a Hamiltonian cycle in D. Create D' from D as follows. Add x to D. Let $\mathrm{In}_{D'}(x) = V$. Let v_i denote the vertex adjacent to x in T'. Let v_j and v_k denote two vertices adjacent to v_i in T and T'. Since v_i and v_j are adjacent in T, v_i and v_j have arcs to at least p common vertices. Let x have an arc to these vertices. Since v_i and v_k are adjacent in T, v_i and v_k have arcs to at least p common vertices, at least one of which, v_m, has no arc from v_j, since T is a tree. Let x have an arc to this vertex. Then no previous competitions have changed since all vertices have arcs to x, while x and v_i compete at least $(p+1)$ times. Then $v_1, v_2, \ldots, v_{m-1}, x, v_m, \ldots, v_n, v_1$ is a Hamiltonian cycle in D'. Therefore $C_p(D') = T'$, where D' is Hamiltonian. □

A *branch* of a tree is a path of the tree with the vertex at one end adjacent to an internal vertex (or a vertex with at least three neighbors including the vertex of the branch). A maximal branch has the further property that the other end vertex is a pendant vertex.

Lemma 4.4. *Let T be a tree which is the p-competition graph of a loopless Hamiltonian digraph. Let T' be a tree produced from T by adding a branch of l new vertices ($l \geq 2$). Then T' is the k-competition graph of a loopless Hamiltonian digraph for $p \leq k \leq p+l-1$.*

Proof: The proof is by induction on l. If $l = 2$, observe T' is a p-competition graph of a loopless Hamiltonian digraph by Corollary 4.2. To show T' is a $(p+1)$-competition graph of a loopless Hamiltonian digraph, add the first vertex of the branch as indicated in Lemma 4.3 creating a tree which is the $(p+1)$-competition graph of a loopless Hamiltonian digraph. Then by Corollary 4.2, T' is the $(p+1)$-competition graph of a loopless Hamiltonian digraph.

Assume the statement is true for the addition of a branch on $l < n$ new vertices and consider the addition of a branch on $l = n$ new vertices. By the induction hypothesis, the addition of the first $(l-1)$ vertices produces a tree that is the k-competition graph of a loopless Hamiltonian digraph for $p \le k \le p+l-2$. Then by Corollary 4.2, T' is the k-competition graph of a loopless Hamiltonian digraph for $p \le k \le p+l-2$. It remains to be shown that T' is a $(p+l-1)$-competition graph of a loopless Hamiltonian digraph.

Let $v_1, v_2, \dots, v_l$ denote the consecutively labeled vertices of the branch such that v_1 is adjacent to an internal vertex, v_0, of T. Let D be a loopless Hamiltonian digraph such that $C_p(D) = T$. Create D' from D as follows. Direct an arc from all vertices of T to $v_1, \dots, v_{l-1}$. Observe this preserves all adjacencies from $C_p(T)$ in $C_{p+l-1}(T')$.

Since v_0 is an internal vertex, there exists vertices t and u adjacent to v_0 in T. Let S denote a set of $p+1$ vertices, p of which t and v_0 have arcs directed toward in D and 1 of which u and v_0 have an arc directed toward, but t does not. Direct an arc from all vertices in the branch to all vertices of S.

For $i = 1, \dots, l-1$, direct an arc from v_i to all vertices v_k $(k = 0, 1, \dots, l)$ except v_i and v_{i-1}. Direct an arc from v_l to all vertices v_k $(k = 0, 1, \dots, l-3)$. Observe that nonconsecutively labeled vertices of the branch, v_i and v_k, compete at most $(p+1)+(l+1-4)$ times in D' (namely for $(p+1)$ vertices of S and all vertices of the branch except v_i, v_{i-1}, v_k and v_{k-1}). Consecutively labeled vertices of the branch, v_i and v_{i+1}, compete at least $(p+1)+(l+1-3)$ times (namely for $(p+1)$ vertices of S and all vertices of the branch except v_{i-1}, v_i and v_{i+1}). Observe that v_1 and v_0 compete at least $(p+1)+(l-2)$ times, while for all other vertices v_i of the branch, v_i and v_0 compete at most $(p+1)-(l-3)$ times (since v_0 has no arc to v_l and v_l has no arc to v_{l-2}).

Now consider an arbitrary vertex $v \in T$ other than v_0 and a vertex v_i of the branch. Since T is a tree, v can have at most p arcs to vertices of S, while v_i has arcs to $l-2$ vertices in the set $\{v_0, \dots, v_{l-1}\}$. Therefore v and v_i compete at most $p+l-2$ times. Thus $C_{p+l-1}(D') = T'$. Furthermore if $x_1, x_2, \dots, x_n$ denotes a Hamiltonian cycle in D and x_i is any vertex of S, then $x_1, x_2, \dots, x_{i-1}, v_1, v_2, \dots, v_l, x_i, x_{i+1}, \dots, x_n, x_1$ is a Hamiltonian cycle in D', completing the proof. □

The next lemma allows us to join two p-competition graphs of loopless Hamiltonian digraphs.

Lemma 4.5. *Let $p \geq 2$. If G_1 and G_2 are the p-competition graphs of loopless Hamiltonian digraphs, then $G_1 \cup G_2$ is the p-competition graph of a loopless Hamiltonian digraph.*

Proof: Let D_1 and D_2 be loopless Hamiltonian digraphs such that $C_p(D_1) = G_1$ and $C_p(D_2) = G_2$. Let $v_1, v_2, \ldots, v_{n_1}$ and $x_1, x_2, \ldots, x_{n_2}$ denote a Hamiltonian cycle in each digraph respectively. Let v_i be an arbitrary vertex in D_1 and x_i an arbitrary vertex in D_2. Create digraph D from D_1 and D_2 as follows. Let $\text{In}_D(v_i)$ be the inset of x_i in D_2; similarly, let $\text{In}_D(x_i)$ be the inset of v_i in D_1. Then $C_p(D) = G_1 \cup G_2$ and $v_1, v_2, \ldots, v_{i-1}, x_1, x_{i+1}, \ldots, x_{n_2}, x_1, \ldots, x_{i-1}, v_i, v_{i+1}, \ldots, v_{n_1}, v_1$ is a Hamiltonian cycle in D. □

5 Utilizing the Constructions

Before we can utilize the constructions of the previous section, we must establish a few examples of graphs as the p-competition graphs of loopless Hamiltonian digraphs.

Lemma 5.1. *Let $p \geq 2$. If G is a cycle on $n \geq p+3$ vertices, then G is the p-competition graph of a loopless Hamiltonian digraph.*

Proof: Let $v_1, v_2, \ldots, v_n$ denote the consecutively labeled vertices of G. Create digraph D as follows. Let

$$\text{In}_D(v_i) = \{v_{i+1 \bmod n}, v_{i+2 \bmod n}, \ldots, v_{i+p+1 \bmod n}\}.$$

Then v_i and $v_{i+1 \bmod n}$ compete p times, namely for $v_{i+2 \bmod n}, \ldots, v_{i+p \bmod n}$ and $v_{i+p+1 \bmod n}$. Consider nonconsecutive vertices v_i and v_k. There are 4 vertices for which v_i and v_k do not compete, namely v_i, v_k, $v_{i+1 \bmod n}$ and $v_{k+1 \bmod n}$. Thus v_i and v_k compete for at most $p-1$ vertices. Therefore $C_p(D) = G$ and $v_1, v_n, v_{n-1}, \ldots, v_2, v_1$ is a Hamiltonian cycle in D. □

Lemma 5.2. *The complete graph K_n on $n \geq p+2$ vertices is the p-competition graph of a loopless Hamiltonian digraph.*

Proof: Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertices of K_n. Create D as follows. Let $\text{In}_D(v_i) = V - \{v_i\}$. Then $C_p(D) = G$ and that $v_1, v_2, \ldots, v_n, v_1$ is a Hamiltonian cycle in D. □

Lemma 5.3. *The complete graph minus one edge on $n \geq 2p+1$ vertices is the p-competition graph of a loopless Hamiltonian digraph.*

Proof: Let G be the complete graph minus one edge for $n \geq 2p+1$. Label the vertices of G, $v_1, v_2, \ldots, v_n$ such that (v_1, v_n) is the missing edge. Create D as follows. For $1 \leq i < \lceil \frac{n}{2} \rceil$, let $\text{In}_D(v_i) = V - \{v_i, v_1\}$. For $\lceil \frac{n}{2} \rceil < i \leq n$, let $\text{In}_D(v_i) = V(G) - \{v_i, v_n\}$. Let $\text{In}_D(v_{\lceil \frac{n}{2} \rceil}) = V(G) - \{v_{\lceil \frac{n}{2} \rceil}\}$ if n is odd, and $V(G) - \{v_{\lceil \frac{n}{2} \rceil}, v_1\}$ if n is even.

Then all pairs compete at least $\lceil \frac{n}{2} \rceil - 1 \geq p$ times except for v_1 and v_n which compete at most once and

$$v_1, v_{\lceil \frac{n}{2} \rceil + 1}, v_{\lceil \frac{n}{2} \rceil + 2}, \ldots, v_n, v_{\lceil \frac{n}{2} \rceil}, v_{\lceil \frac{n}{2} \rceil - 1}, v_{\lceil \frac{n}{2} \rceil - 2}, \ldots, v_1$$

is a Hamiltonian cycle in D. □

Lemma 5.4. *If G is a path on $n \geq p + 3$ vertices, then G is the p-competition graph of a loopless Hamiltonian digraph.*

Proof: We need only verify this result for $n = p + 3$ by Corollary 4.2. Let $v_1, v_2, \ldots, v_n$ be the consecutively labeled vertices of G. Create D as follows: for $i \neq p+1$, let $\mathrm{In}_D(v_i) = V(G) - \{v_i, v_{i+1 \bmod n}\}$; let $\mathrm{In}_D(v_{p+1}) = V(G) - \{v_i, v_{i+1 \bmod n}, v_{i+2 \bmod n}\}$. Observe that $v_1, v_2, \ldots, v_n, v_1$ is a Hamiltonian cycle in D. Then v_i and v_{i+1} compete at least p times, namely for $v_{i+2 \bmod n}, v_{i+3 \bmod n}, \ldots, v_{i+(p+1) \bmod n}$. Since v_i does not have an arc to v_i or $v_{i+1 \bmod n}$, the nonconsecutive vertices compete at most $p - 1$ times, i.e., $C_p(D) = G$ where D is Hamiltonian. □

We now establish a result for two special classes of trees. A *caterpillar* is a tree such that the removal of all pendant vertices results in a path (the *spine*).

Theorem 5.5. *If G is a caterpillar on $n \geq p + 3$ vertices, then G is the p-competition graph of a loopless Hamiltonian digraph.*

Proof: We need only verify this result for $n = p + 3$ by Corollary 4.2. Let $v_1, v_2, \ldots, v_q$ denote the consecutively labeled vertices of the spine of G. Observe $q \geq 3$. Since a path on 3 vertices is the competition graph of a loopless Hamiltonian digraph D, if $q = 3$ create D' by successively adding all but one of the remaining pendant vertices as in Lemma 4.3. Then add the final pendant vertex as in Lemma 4.1. Then D' is Hamiltonian and $C_p(D') = G$. If $q > 3$, then the spine of G is the r-competition graph of a loopless Hamiltonian digraph D by Lemma 5.4, where $r = q - 3$. Create D' by successively adding the remaining pendant vertices to D' as in Lemma 4.3. Then $D'$is Hamiltonian and $C_p(D') = G$. □

Theorem 5.6. *Let $p \geq 2$. If T is a tree on $n \geq 2p$ vertices, then T is the p-competition graph of a loopless Hamiltonian digraph.*

Proof: By Corollary 4.2 we need only consider the case $n = 2p$. The case $p = 2$ can be verified by examination of all possible trees so $p > 2$. Let q be the length of the longest path P in T. If $q \geq p + 3$, T is the p-competition graph of a loopless Hamiltonian digraph by Lemma 5.4 and Corollary 4.2. If $q \leq 4$, T is a caterpillar and we are done by Theorem 5.5.

Saving the case $q = 5$, consider the case that $q \geq 6$. Then $n = 2p \geq 2p - q + 6$. Thus, if we can show that a tree on $2p - q + 6$ vertices is the p-competition graph of a loopless Hamiltonian digraph, we are done by Corollary 4.2. Assume T is such a tree with maximum path P.

By Lemma 5.4, P is the $(q-3)$-competition graph of a loopless Hamiltonian digraph. Construct a sequence of subtrees $T_0, T_1, \ldots, T_k$ where T_0 is a path, T_k is T and T_i is constructed from T_{i-1} by the addition of a maximal branch. Let n_i be the number of vertices added to T_{i-1} to get T_i. Lemmas 4.1, 4.3 and 4.4 guarantee that T_i is a $(q-3+\sum_{l=1}^{i} k_l)$-competition graph where $k_i = \max\{n_i - 1, 1\}$. The worst case occurs when each additional branch adds two new vertices. In this case, T is a $(q-3+j)$-competition graph, where j is half the number of vertices added to T_0. Since there are q vertices in T_0, we add $2p-q+6-q = 2(p-(q-3))$ vertices to obtain T, and conclude T is a $(q-3)+(p-(q-3)) = p$ competition graph.

If $q=5$, T must have a maximal branch with one vertex; otherwise all branches of T are of length 2, i.e., T has an odd number of vertices, a contradiction since $n = 2p$. Remove this branch. The resulting tree has $2p-1 = 2p-2-q+6$ vertices and is therefore, by the previous case, a $(p-1)$-competition graph of a loopless Hamiltonian digraph. Using Lemma 4.3 we conclude T is the p-competition graph of a loopless Hamiltonian digraph. □

Corollary 5.7. *Let $p \geq 2$. If G is a forest and all maximal subtrees of G have $n \geq 2p$ vertices, then G is the p-competition graph of a loopless Hamiltonian digraph.*

Proof: This follows from Theorem 5.6 and Lemma 4.5.

6 Classes of 2-Competition Graphs

Using Lemmas 5.2 and 5.3 of the previous section, we prove a result for chordal (hence interval) graphs.

Lemma 6.1. *If G is chordal on $n \geq 5$ vertices and not complete, then G is the 2-competition graph of a loopless Hamiltonian digraph D, in which every maximal clique of G is contained in at least one inset of D.*

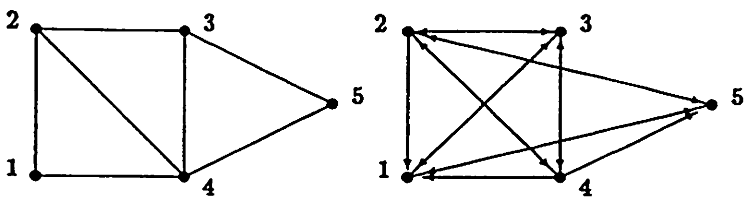

Figure 3.
An example of a chordal graph, G, on five vertices and loopless Hamiltonian digraph D such that $C_2(D) = G$. Note that the insets of the vertices in the digraph form a 2-ECC for the graph.

Proof: (by induction on n) If $n = 5$, we verify the result by considering all possible graphs (see Figure 3). Assume the statement is true for chordal graphs which are not complete on $n = k \geq 5$ vertices and let G be such a chordal graph on $n = k + 1$ vertices. Let x be a simplicial vertex in G (a vertex is simplicial if its neighborhood induces a complete subgraph; every chordal graph has at least two simplicial vertices [5]). Consider $G' = G - \{x\}$.

Case 1: G' is complete. Suppose there is exactly one vertex in G' that is not adjacent to x in G. Then G is K_n minus one edge and by Lemma 5.3, is the 2-competition graph of a loopless Hamiltonian digraph. Suppose there are at least two vertices, x_i and x_j of G' that are not adjacent to x in G. Let C be the maximal clique in G containing x. Create D as follows. Let $\text{In}(x) = V(D) - \{x\}$. Let $\text{In}(x_i) = \text{In}(x_j) = C$. For all x_0, $x_0 \neq x$, $x_0 \neq x_i$, $x_0 \neq x_j$, let $\text{In}(x_0) = V(D) - \{x_0, x\}$. Then G is the 2-competition graph of D. Since the digraph $D - \{x_i, x_j\}$ has all possible arcs, x has an arc to x_i, x_i has an arc to some vertex x_0 of $V(G) - \{x, x_i, x_j\}$, x_0 has an arc to x_j, and x_j has an arc to some vertex x_1 of $V(G) - \{x, x_i, x_j, x_0\}$, we conclude that D is Hamiltonian. Furthermore $C \subseteq \text{In}(x_i)$ and $V(G') \subseteq \text{In}(x)$, so every maximal clique is contained in at least one inset of D.

Case 2: G' is not complete. By induction hypothesis, G' is the 2-competition graph of a loopless Hamiltonian digraph D' such that every maximal clique of G' is contained in at least one inset of D'. Let $x_1, x_2, \ldots, x_{n-1}$ denote the Hamiltonian cycle of D'. Let C be the maximal clique containing x in G. Let $C' = C \cap V(G')$. By the inductive hypothesis there is a vertex x_j such that C' is contained in $\text{In}(x_j)$ in D'. Create digraph D as follows. Add arc (x, x_j) to D'. Observe that since D' is loopless, x_j is not adjacent to x in G.

Suppose there is a vertex $x_i \neq x_j$ that is not adjacent to x in G. Then let $\text{In}_D(x) = \text{In}_{D'}(x_i)$ and $\text{In}_{D'}(x_i) = C$. Observe that x competes with the other vertices of C in D at x_i and x_j, while x competes at most once with any other vertex in D, namely at x_j. Since no other competitions have changed, $C(D) = G$ and $x_1, \ldots, x_{i-1}, x, x_i, \ldots, x_{n-1}$ is a Hamiltonian cycle in D.

Suppose x_j is the only vertex that is not adjacent to x in G. Then $G' - \{x_j\}$ is a clique and $G - \{x_j\}$ is a clique. Since this implies x_j is simplicial in G and $G - x_j$ is complete, we are in case one, completing the proof. □

From Lemma 6.1 and Lemma 5.2 we have the following results.

Theorem 6.2. *If G is a chordal graph on $n \geq 5$ vertices, then G is the 2-competition graph of a loopless Hamiltonian digraph.*

Corollary 6.3. *If G is interval on $n \geq 5$ vertices, then G is the 2-competition graph of a loopless Hamiltonian digraph.*

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