A NOTE ON THE P_3-SEQUENCEABILITY OF FINITE GROUPS

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ABSTRACT. Applying Glauberman's Z^*-theorem, it is shown that every finite group G is strongly P_3-sequenceable, i.e. there exists a sequencing $(x_1, \dots, x_N)$ of the elements of $G \setminus \{1\}$, such that all products $x_i x_{i+1} x_{i+2}$ $(1 \leq i \leq N-2)$, $x_{N-1}x_N x_1$ and $x_N x_1 x_2$ are nontrivially rewritable. This was conjectured by J.Nielsen in [N].

Let G be a finite group, $|G| = N+1$. J. Nielsen [N] defined G to be strongly P_n-sequenceable if there exists a sequencing $(x_1, \dots, x_N)$ of all the elements of $G^* = G \setminus \{1\}$, such that for all i the product $x_i x_{i+1} \cdots x_{i+n-1}$ (indices modulo N) is rewritable, i.e. there is a permutation $\sigma \neq 1$ of the indices $\{i, i+1, \dots, i+n-1\}$ with $x_i x_{i+1} \cdots x_{i+n-1} = x_{\sigma(i)} x_{\sigma(i+1)} \cdots x_{\sigma(i+n-1)}$. Obviously, strong P_n-sequenceability implies strong P_{n+1}-sequenceability.

In the paper of Nielsen [N] it is shown that every finite group is strongly P_5-sequenceable and it is conjectured that every finite group is strongly P_3-sequenceable. Meanwhile P. Longobardi and M. Maj [LM] showed that every group is P_4-sequenceable and that every countably infinite group is P_3-sequenceable. The aim of this note is to supply a proof of Nielsen's conjecture.

Theorem. *Every finite group G is strongly P_3-sequenceable.*

As in [N] we will build up our sequencing for G^* from subsequences obtained from a suitable partition of G^*.

First of all we collect all non-involutions of G^* in pairs $\{g, g^{-1}\}$. Next, if an involution x is a non-trivial power of some element in G, choose any such element, say g, and extend the set $\{g, g^{-1}\}$ by x. Observe that x is the unique involution of $< g >$. Hence, if $\mathcal{I} = \{x \in G^* \mid x^2 = 1, x \neq g^k$ for all $g \in G \setminus \{x\}, k > 1\}$, we may partition the set $G^* \setminus \mathcal{I}$ into subsets X_i $(1 \leq i \leq n_1)$ with $|X_i| \in \{2, 3\}$, such that $[g, h] = 1$ whenever $g, h \in X_i$. The crucial step in the proof of the theorem is to partition the involutions in $\mathcal{I}$. For this purpose, we distinguish between those involutions in $\mathcal{I}$ which are weakly closed in their centralizer and those which are not. Hence let

$$\mathcal{I}_1 = \{x \in \mathcal{I} \mid C_G(x) \cap x^G \neq \{x\}\}, \qquad \mathcal{I}_2 = \mathcal{I} \setminus \mathcal{I}_1.$$

Lemma. *$\mathcal{I}_1$ is the disjoint union of subsets Y_i $(1 \le i \le n_2)$ with $|Y_i| \in \{2,3\}$, such that there exists $y_i \in Y_i$ with $[y_i, Y_i] = 1$ for every $i \le n_2$.*

Proof. Obviously $\mathcal{I}_1$ is a union of conjugacy classes of G, and it suffices to find a partition of any such conjugacy class x^G according to the Lemma. Consider x^G as the set of vertices of a graph $\mathcal{G}$, where $y, z \in x^G$ are joined by an edge iff $[y,z] = 1$. (For the terminoloy used in connection with graphs see any book on graph theory or combinatorics, e.g. [C].) As G acts transitively on the vertices of $\mathcal{G}$, $\mathcal{G}$ is a regular graph of degree ≥ 1. Hence the lemma follows from the following claim.

(∗) A finite regular graph $\mathcal{G}$ of degree $d \ge 1$ with vertices $V = V(\mathcal{G})$ and edges $E(\mathcal{G})$ admits a decomposition of the set of vertices $V = \dot{\bigcup}_{i=1}^{l} Y_i$, such that $Y_i \in E(\mathcal{G})$ or Y_i is a set of order 3, $Y_i = \{x_i, y_i, z_i\}$, with $\{x_i, y_i\}, \{y_i, z_i\} \in E(\mathcal{G})$ $(1 \le i \le l)$.

Proof of (∗): Take a disjoint copy $\tilde{V} = \{\tilde{v} \mid v \in V\}$ of V and define a bipartite graph $\mathcal{G}_b$ with $V(\mathcal{G}_b) = V \dot{\cup} \tilde{V}$ and $E(\mathcal{G}_b) = \{\{v, \tilde{w}\} \mid \{v,w\} \in E(\mathcal{G})\}$. Obviously, $\mathcal{G}_b$ is a regular bipartite graph of degree d and therefore, by a theorem of König [K], it possesses a complete matching M. But now the set of edges $\{\{v,w\} \mid \{v,\tilde{w}\} \in M\} \subseteq E(\mathcal{G})$ gives a decomposition of V into cycles in $\mathcal{G}$. (Here, a 2-cycle is simply an edge of $\mathcal{G}$.) Finally, any such cycle can be partitioned into paths of length two and three and the vertices of these paths form sets Y_i, as asserted in (∗).

<u>Proof of the theorem:</u>

<u>case I</u> $\quad \mathcal{I}_2 = \emptyset$

We have $G^* = (\dot{\bigcup}_{i=1}^{n_1} X_i) \dot{\cup} (\dot{\bigcup}_{i=1}^{n_2} Y_i)$ and concatenating sequences for the X_i's and Y_i's gives a sequencing s for G^*. Provided that the subsequence for any Y_i with $|Y_i| = 3$, say (x,y,z) is chosen in such an order that $[x,y] = [y,z] = 1$, every product of three consecutive elements of the resulting sequence s is easily seen to be rewritable.

<u>case II</u> $\quad \mathcal{I}_2 \ne \emptyset$

If $Z(G) \ne 1$, than G is easily seen to be strongly P_3-sequenceable [N, Proposition 1]. Hence we may assume $Z(G) = 1$. Let $O(G)$ denote the largest normal $2'$-subgroup of G. We now employ Glauberman's Z^*-theorem [Gl] to conclude that for every $x \in \mathcal{I}_2$ we have $xO(G) \in Z(G/O(G))$. As $x \in \mathcal{I}_2$ is not a non-trivial power of any element in G, we infer that $C_G(x)$ is a 2-group. In particular $C_G(x) \cap O(G) = 1$ and x acts fixed-point-freely on $O(G)$. It follows that $O(G)$ is abelian and x inverts every element of $O(G)$ [Go, 10.1.4]. Hence $x^{O(G)} = xO(G)$ and $x^G = xO(G)$ as well.

As $|G| = |C_G(x)||x^G|$ we have $G = O(G)C_G(x)$. Now, $G/O(G) \cong C_G(x)$ acts faithfully on $O(G)$: otherwise $1 \ne C_{C_G(x)}(O(G)) \cap Z(C_G(x))$ (as $C_G(x)$ is a 2-group) in contradiction to our assumption $Z(G) = 1$. If y is any element in $\mathcal{I}_2$, then y inverts every element of $O(G)$ as well, and we have $xy \in C_G(O(G)) \leq O(G)$, i.e. $y \in xO(G) = x^G$.

Hence $\mathcal{I}_2 = x^G$. Now consider any sequencing $s = (x_1, \ldots, x_m)$ for the elements of $\mathcal{I}_2$. The product of any three consecutive elements of this sequence is an involution (element of $xO(G)$) and therefore rewritable: $x_i x_{i+1} x_{i+2} = x_{i+2} x_{i+1} x_i$. As $G^* = (\dot{\bigcup}_{i=1}^{n_1} X_i) \dot{\cup} (\dot{\bigcup}_{i=1}^{n_2} Y_i) \dot{\cup} \mathcal{I}_2$ we may get a sequencing for G^* by concatenating s with sequences for the X_i's and Y_i's. If the subsequences for the Y_i's with $|Y_i| = 3$ are chosen as in case I, then every product of three consecutive elements of the resulting sequence is rewritable if $x_0 x_1 x_2$ and $x_{m-1} x_m x_{m+1}$ are rewritable, where x_0 is the last element of the subsequence s^- preceeding s and x_{m+1} is the first element of the subsequence s^+ following s. As $x_1 x_2, x_{m-1} x_m \in O(G)$ and $O(G)$ is abelian, this can easily be accomplished if there are two sets $X_{i_1}, X_{i_2} (1 \leq i_1 < i_2 \leq n_1)$ with $X_{i_1} \cap O(G) \neq \emptyset \neq X_{i_2} \cap O(G)$. If there are no two such sets, $|O(G)| = 3$. Now, the faithfull action of $G/O(G)$ on $O(G)$ forces G to be isomorphic to S_3 and $G^* = X_1 \dot{\cup} \mathcal{I}_2$ is strongly P_3-sequenceable in this case as well.

References

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