A poset-based method for counting partitions and Ferrers diagrams*

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Abstract

The counting of partitions of a natural number, when they have to safisfy certain restrictions, is done traditionally by using generating functions. We develop a polynomial time algorithm for counting the weighted ideals of partially ordered sets of dimension 2. This allows the use of the *same* algorithm for counting partitions under all sorts of different constraints. In *contrast* with this, the method is very flexible, and can be used for an extremely large variety of different partitions.

1 Introduction

The problem of counting partitions of a natural number n has received a lot of attention, going back all the way to Euler [3]. A comprehensive treatment can be found in Andrews [1]. Traditionally, counting various types of partitions of n is done through the use of their generating functions. For some special cases more efficient recursions have been found, exploiting the relationship between different generating functions [2]. In this paper we discuss a new, efficient method for counting partitions, which is based on counting the "ideals" of certain partial orders. This represents a very flexible tool, and enables us to use the same program and algorithm for counting an extremely large variety of different kinds of partitions.

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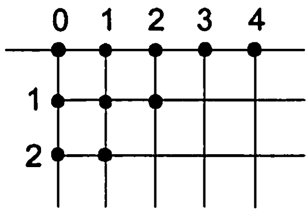

Figure 1:

Ferrers diagrams are well known representations for the partitions of numbers: If the natural number n is partitioned into $n = g_1 + g_2 + \ldots + g_k$, with $g_1 \geq g_2 \geq \ldots \geq g_k \geq 0$ integer, then the partition $\mathbf{g} = (g_1, g_2, \ldots, g_k)$ can be represented by a row of g_1 dots, followed by a row of g_2 dots, and so on. For example, the partition $(5,3,2)$ corresponds to the Ferrers diagram shown in Figure 1. Since we will want to refer to specific dots in a diagram, we embed it into the set **Square** (or **Sq**) $= N\text{x}N$, where N $= \{0, 1, 2, \ldots\}$, and will refer to the individual dots by their "coordinates", using the horizontal coordinate as the first one. (The numbers shown in Figure 1 represent the horizontal and vertical coordinates of the points, respectively.) Define the partial ordering $\preceq$ on **Sq** by $(a,b) \preceq (c,d)$ if and only if $a \leq c$ and $b \leq d$ as numbers. Thus $(\mathbf{Sq}, \preceq)$ is a partially ordered set (poset), where moving to the right or down means moving to greater elements.

If $(P, \leq_P)$ is a poset then $I \subseteq P$ is an *order ideal* (or *ideal* in short) if $b \in I$ and $a \leq_P b$ imply $a \in I$. It can be easily seen that the Ferrers diagrams are in a one-to-one correspondence with the ideals of the poset $(\mathbf{Sq}, \preceq)$. A *linear extension* of a poset $(P, \leq_P)$ is a linear (total) order L with $a \leq_P b$ implying $a \leq_L b$ for $a, b \in P$. Every poset P can be defined as the intersection of its linear extensions (as binary relations). The minimum number of linear extensions defining P in this way is the *dimension of P*, denoted by $dimP$. It is well known that $dimP = k$ if and only if P can be embedded into the k-dimensional Euclidean space $\mathbf{R}^k$, with

$a \leq b$ for $a, b \in P$ exactly if a is less than or equal to b in every coordinate in $\mathbf{R}^k$. Based on this, we clearly have $dim(\mathbf{Sq}, \preceq) = 2$.

We will be particularly interested in subsets of $(\mathbf{Sq}, \preceq)$. The first family of these subsets is $\mathbf{Sq}_n = \{(i,j) \in \mathbf{Sq},\ i+j \leq n-1\}$. It follows from the definition of dimension that if $dimP = k$ and P' is a subposet of P then $dimP' \leq k$. Thus we have $dim\mathbf{Sq}_n \leq 2$ for every n, and the same is true for any other subposet of $\mathbf{Sq}$. A defining pair of linear extensions for $\mathbf{Sq}_n$ is

$$L_1 = (0,0), (1,0), ..., (n-1,0), ..., (0,i), (1,i), ..., (n-1-i,i), ..., (0,n-1),$$
$$L_2 = (0,0), (0,1), ..., (0,n-1), ..., (j,0), (j,1), ..., (j,n-1-j), ..., (n-1,0).$$

Let us consider a poset $P = (\{v_1, v_2, ..., v_m\}, \leq_P)$, and define

$$a_t(P) = |\{I : I \subseteq P \ ideal,\ |I| = t\}| \quad for\ t = 0, 1, 2, ..., m.$$

The ideals of P form a *lattice* $\mathcal{J}(\mathcal{P})$, whose *rank-generating function* is defined by $f(P,x) = \sum_{t=0}^{m} a_t(P)x^t$ [6]. Since the Ferrers diagrams for a given n correspond to the ideals of cardinality n of the poset $(\mathbf{Sq}_n, \preceq)$, counting the partitions of n is equivalent to obtaining the coefficient of x^n in the polynomial $f(\mathbf{Sq}_n, x)$. In fact, the first $n+1$ terms of $f(\mathbf{Sq}_n, x)$ are identical to the the first $n+1$ terms of the well known generating function for partitions, $F(x) = \prod_{i=1}^{\infty}(1-x^i)^{-1}$.

2 Ideals in weighted 2-dimensional posets and partitions

Let us assume now that each element v_k in the poset P has a *nonnegative integer weight* $w(v_k)$. For any subset (ideal) $I \subseteq P$ define its *weight* by $w(I) = \sum_{v_k \in I} w(v_k)$, and define the *weighted rank-generating function* of P by $f_w(P,x) = \sum_{k=0}^{m} \sum_{|I|=k} x^{w(I)}$. Note that if $w(v_k) \equiv 1$ for $v_k \in P$, then this definition reduces to that of the original rank-generating function of P. Computing $f_w(P,x)$, or even $f(P,x)$, is a difficult (#P-complete) problem for general posets [5], and so it would require excessive computer time for large posets.

Consider now a poset P with $dimP = 2$ and assume, without loss of generality, that the elements of P have been numbered so that one of its defining linear extensions is $L_1 = v_1, v_2, ..., v_m$. Define the *principal ideals* of P by $I_k = \{v_i : v_i \leq_P v_k\}$ and their *"complements"* by $\bar{I}_k = \{v_i : i <$

$k,\ v_i \not\leq_P v_k\}$ for $k = 1, 2, ..., m$. The following theorem presents a recursion for the weighted rank-generating function of 2-dimensional posets. This will lead to a polynomial time computation of $f_w(P, x)$ if $dim P = 2$.

Theorem 1 *If P is a 2-dimensional poset with ground set* $\{v_1, v_2, ..., v_m\}$ *and* $L_1 = v_1, v_2, ..., v_m$ *is a defining linear extension for P, then*

$$f_w(P, x) = 1 + \sum_{k=1}^{m} [1 + \sum_{j||k} f_w(\bar{I}_j, x)\ x^{w(C_{kj})}]\ x^{w(I_k)}, \tag{1}$$

where $j||k$ is short notation for $j < k$ and $v_j \not\leq_P v_k$, and

$$C_{kj} = \{v_i : v_i \in \bar{I}_k,\ v_i \leq_P v_j\}\ for\ j||k,\ k = 1, 2, ..., m.$$

Proof: Partition the set of ideals by their highest numbered element. If I is an ideal with v_k its highest numbered element, then the principal ideal I_k must be contained in I and $I - I_k$ is an ideal in the subposet $\bar{I}_k$. Thus every q element ideal I of P, with weight $w(I)$, is in a one-to-one correspondence with a $q - |I_k|$ element ideal of $\bar{I}_k$, with weight $w(I) - w(I_k)$. Therefore,

$$f_w(P, x) = 1 + \sum_{k=1}^{m} f_w(\bar{I}_k, x)\ x^{w(I_k)}. \tag{2}$$

Let

$$\bar{C}_{kj} = \{v_i : v_i \in \bar{I}_k, i < j, v_i \not\leq_P v_j\}\ for\ k = 1, 2, ..., m\ and\ j||k.$$

Apply (2) to the posets $\bar{I}_k$ to yield

$$f_w(\bar{I}_k, x) = 1 + \sum_{j||k} f_w(\bar{C}_{kj}, x)\ x^{w(C_{kj})}. \tag{3}$$

The repeated application of recursion (2)— to the new "complements" $\bar{C}_{kj}$, the new "complements" within these and so on— would eventually yield $f_w(P, x)$, but this is not an efficient procedure for general posets P. On the other hand, we show that when $\dim P = 2$, then each subposet $\bar{C}_{kj}$ is *identical* to a complement set considered in the first application of recursion (2).

We claim that $dim P = 2$ and $j||k$ imply $\bar{C}_{kj} = \bar{I}_j$: If $v_i \in \bar{C}_{kj}$, then it can be easily seen that $v_i \in \bar{I}_j$. Conversely, if $v_i \in \bar{I}_j$, then $i < j$ and

$v_i \not\leq_P v_j$, i.e., $v_j <_{L_2} v_i$ if L_2 is the second defining linear extension for P. Furthermore, $j||k$ implies $j < k$ and $v_j \not\leq_P v_k$, i.e., $v_k <_{L_2} v_j$ too, so that $v_k <_{L_2} v_i$ also holds, implying $v_i \in \bar{I}_k$, and therefore $v_i \in \bar{C}_{kj}$.

Replacing $f_w(\bar{C}_{kj}, x)$ with $f_w(\bar{I}_j, x)$ in (3), and substituting into (2) completes the proof. □

Corollary 2 *There is an algorithm with $O(m^3)$ time and $O(m^2)$ space complexity to compute $f_w(P, x)$ for a 2-dimensional poset P on m elements.*

Proof: We observe that, in the 2-dimensional case, to compute $f_w(\bar{I}_k, x)$ one needs only the previously computed functions $f_w(\bar{I}_j, x)$ (for $j||k$) if the summation in (1) is done in the order $k = 1, 2, ..., m$. Thus, computing $f_w(\bar{I}_k, x)$ for one k requires the addition of at most $k \leq m$ previously computed polynomials of degree at most m, and so it needs at most $O(m^2)$ time. We have to repeat this for $k = 1, 2, ..., m$, so the whole computation of $f_w(P, x)$ can be done in $O(m^3)$ time. We need to store the coefficients of at most m polynomials of degree at most m during this process, so the claimed space complexity follows too. □

As an example, consider the posets $\frac{1}{2}\mathbf{Sq}_n = \{(i,j) \in \mathbf{Sq},\ i \geq j,\ i \leq n-1\}$ with the following weights:

$$w(i,j) = \begin{cases} 1 & if \quad i = j \\ 3 & if \quad (i+j)\ is\ odd \\ 5 & if \quad i \neq j,\ (i+j)\ is\ even \end{cases}$$

Figure 2 shows $\frac{1}{2}\mathbf{Sq}_4$ with these weights and Table 1 shows the calculations for $f_w(\frac{1}{2}\mathbf{Sq}_4, x)$, which is the sum of the polynomials in the last column plus 1.

Propp [4] has studied a variety of 2-dimensional posets in which the ideals correspond to various restricted partitions, and identified the generating function for many of them. Recursion (1) represents a polynomial time counting method for *all* of these cases. In addition to these, Theorem 1 gives us a powerful tool to count efficiently a very large variety of restricted partitions, including many cases where the generating function may *not* be known in the form of an identity. Here we mention only a few applications.

It can be easily seen that the coefficient of x^n in $f_w(\frac{1}{2}\mathbf{Sq}_n, x)$ counts the number of partitions of n in which each part is equal to 1 or 4 (*mod* 8), *and* each part is distinct.

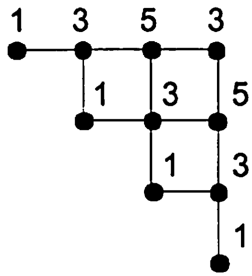

Figure 2: The weighted poset $\frac{1}{2}\mathbf{Sq}_4$

Propp [4] has defined the poset $\frac{1}{2}\mathbf{Tilt}$, whose diagram corresponded to partitions of n in which no part may occur more than twice. Let us consider the poset $\frac{1}{2}\mathbf{Sq}_n\mathrm{x}k$ in which each row of the poset $\frac{1}{2}\mathbf{Sq}_n$ is repeated k times and each element has the same weights as before. It is clear that $dim\ \frac{1}{2}\mathbf{Sq}_n\mathrm{x}k = 2$. Thus, $\frac{1}{2}\mathbf{Sq}_n\mathrm{x}k$ allows us to count efficiently the Ferrers diagrams in generalizations of $\frac{1}{2}\mathbf{Tilt}$ where the number of repetitions is limited by k instead of 2. Furthermore, the use of nonunit weights leads to incorporating the additional restriction for the modularity of parts. Thus the coefficient of x^n in $f_w(\frac{1}{2}\mathbf{Sq}_n\mathrm{x}k, x)$ counts the partitions of n in which each part is equal to 1 or 4 (mod 8), *and* no part may occur more than k times.

Since every subset of a 2-dimensional poset is 2-dimensional again, we can also count the Ferrers diagrams with *arbitrary* geometric boundaries. For example, we can prescribe minimum and maximum values that each part has to fall between. Our method could also be used for counting the partitions (Ferrers diagrams) in *all* the problems mentioned as *open* by Propp [4] at the end of his article, since omitting an arbitrary subset of elements from a 2- dimensional poset does not increase its dimension. It seems quite likely that the method could be useful for counting partitions with many new types of restrictions too. Although it is possible to design alternative recursive procedures for counting partitions with a certain *fixed*

(i,j)	k	$w(I_k)$	$\bar{I}_k$	$f_w(\bar{I}_k,x)$	$f_w(\bar{I}_k,x)x^{w(I_k)}$
(0,0)	1	1	ϕ	1	x
(1,0)	2	4	ϕ	1	x^4
(1,1)	3	5	ϕ	1	x^5
(2,0)	4	9	$\{v_3\}$	$1+x$	x^9+x^{10}
(2,1)	5	13	ϕ	1	x^{13}
(2,2)	6	14	ϕ	1	x^{14}
(3,0)	7	12	$\{v_3,v_5,v_6\}$	$1+x+x^4+x^5$	$x^{12}+x^{13}+x^{16}+x^{17}$
(3,1)	8	21	$\{v_6\}$	$1+x$	$x^{21}+x^{22}$
(3,2)	9	25	ϕ	1	x^{25}
(3,3)	10	26	ϕ	1	x^{26}

Table 1:

set of restrictions, each new set of constraints would require developing a *new recursion*. A major advantage of our method is that, by simply changing the input poset, the *same* algorithm can be used to count a large variety of partitions with all sorts of different constraints.

In conclusion, we note that the rank-generating functions computed above could be viewed as series of functions approximating the classical generating function for the partitions studied. This is particularly interesting for those cases where the generating function has *no easy-to-handle closed* form, because our method represents an effective (polynomial time) way to compute its power series representation up to the first n terms, for any n. In fact, one could argue that finding the generating function in this form, as opposed to an identity, is more useful for counting restricted partitions.

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