Local connectivity and cycle extension in claw-free graphs

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ABSTRACT. Let G be a connected claw-free graph, $M(G)$ the set of all vertices of G that have a connected neighborhood, and $\langle M(G)\rangle$ the induced subgraph of G on $M(G)$. We prove that

(i) if $M(G)$ dominates G and $\langle M(G)\rangle$ is connected, then G is vertex pancyclic orderable,

(ii) if $M(G)$ dominates G, $\langle M(G)\rangle$ is connected, and $G \setminus M(G)$ is triangle-free, then G is fully 2-chord extendible,

(iii) if $M(G)$ dominates G and the number of components of $\langle M(G)\rangle$ does not exceed the connectivity of G, then G is hamiltonian.

1 Introduction

We consider only finite undirected graphs without loops and multiple edges. For terminology and notation not defined here we refer to [2]. We say that a graph G is *claw-free* if it does not contain a copy of the claw $K_{1,3}$ as an induced subgraph. For $S \subset V(G)$, we denote by $N(S)$ the set of all vertices $x \in V(G) \setminus S$ having at least one neighbor in S and by $\langle S \rangle$ the induced subgraph on S. Let $M(G) = \{x \in V(G) | \langle N(x) \rangle \text{ is connected}\}$. If $M(G) = V(G)$ then we say that G is locally connected.

A graph G is *hamiltonian* if it contains a cycle of length $|V(G)|$. If G contains cycles of all lengths ℓ for $3 \leq \ell \leq |V(G)|$, then we say that G is *pancyclic*, and G is *vertex pancyclic* if every vertex of G is contained in cycles of all lengths ℓ for $3 \leq \ell \leq |V(G)|$. We say that G has a *pancyclic ordering* if the vertices of G can be ordered such that, for any j, $3 \leq j \leq |V(G)|$, the graph induced by the first j vertices is hamiltonian. A graph G is *vertex pancyclic orderable* if for every $x \in V(G)$ there is a pancyclic ordering of $V(G)$ such that x is the first vertex of the ordering.

Clearly, every vertex pancyclic orderable graph is vertex pancyclic. An easy example of a vertex pancyclic claw-free graph that is not vertex pancyclic orderable can be obtained by joining two copies of a complete graph by a perfect matching.

A cycle $C \subset G$ is *extendable* if there is a cycle $C' \subset G$ (called the *extension* of C) such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)| + 1$. If every nonhamiltonian cycle $C \subset G$ is extendible, then G is said to be *cycle extendible.* We say that G is *fully cycle extendable* if G is cycle extendible and each of its vertices is on a triangle.

If $C \subset G$ is a cycle, then every edge $xy \notin E(C)$ with $x, y \in V(C)$ is called a *chord* of C. A cycle $C' \subset G$ is a *k-chord extension* of a cycle $C \subset G$ (k being an integer) if C' is an extension of G and $E(C')$ contains at most k chords of C, and G is *k-chord extendable* if every nonhamiltonian cycle of G has a k-chord extension. Finally, G is *fully k-chord extendable* if G is k-chord extendible and fully cycle extendible.

Oberly and Sumner [6] proved that every connected locally connected claw-free graph with $|V(G)| \geq 3$ is hamiltonian. Clark [3] strengthened this result showing that, under the same conditions, G is vertex pancyclic and Hendry [5] observed that these assumptions imply that G is fully cycle extendible. Zhang [7] showed that if every vertex cut set of a claw-free graph G contains a vertex with a connected neighborhood, then G is pancyclic. Ainouche, Broersma and Veldman [1] observed that these assumptions imply that G is vertex pancyclic. In the present paper we proceed further with these considerations. Namely, we show that under the assumptions of [1], a claw-free graph G is vertex pancyclic orderable. We find conditions for G to be fully 2-chord extendible and we find weaker conditions than those in [1] which still imply hamiltonicity.

2 Results

Proposition 1. *Let G be a claw-free graph and $C \subset G$ a cycle. Suppose there is a vertex $v \in V(C)$ such that $N(v) \setminus V(C) \neq \emptyset$ and $\langle N(v) \rangle$ is connected. Then there is a cycle $C' \subset G$ such that $V(C') \subset V(C) \cup N(v)$ and C' is a 2-chord extension of C.*

Proof: Throughout the proof, whenever vertices of a claw are listed, its center is always the first vertex of the list. Let the cycle $C \subset G$ and the vertex $v \in V(C)$ satisfy the assumptions of Proposition 1 and suppose that there is no such cycle C'. For any fixed orientation of C and for any $u_1, u_2 \in V(C)$ denote by $u_1 C u_2$ the consecutive vertices on C from u_1 to u_2 in the direction specified by the orientation of C. The same vertices, in reverse order, will be denoted by $u_2 \overleftarrow{C} u_1$. For any $u \in V(C)$ denote by u^- and u^+ the predecessor and successor of u on C, respectively.

Choose a vertex $x \in N(v) \setminus V(C)$. As obviously $xv^- \notin E(G)$, $xv^+ \notin E(G)$, and $\langle v, x, v^+, v^- \rangle$ cannot be a claw, we have $v^-v^+ \in E(G)$. Since $\langle N(v) \rangle$ is connected, there is a path P in $\langle N(v) \rangle$ joining x to at least one of v^-, v^+. Suppose that x and P are chosen such that P is shortest possible. Let the orientation of C be chosen such that P is an x, v^+-path and let $x = x_0, x_1, \ldots, x_\ell = v^+$ be the vertices of P. Since P is a shortest path, necessarily $x_i x_j \notin E(G)$ for $|i-j| \geq 2$. Hence we have $\ell \leq 3$ (since otherwise $\langle v, x, x_2, x_4 \rangle$ is a claw). As $xv^+ \notin E(G)$, we have $2 \leq \ell \leq 3$. By the choice of x and P, $x_i \in V(C)$ for $1 \leq i \leq \ell$. Since obviously $xx_1^- \notin E(G)$ and $xx_1^+ \notin E(G)$, from $\langle x_1, x_1^-, x_1^+, x \rangle$ we have $x_1^- x_1^+ \in E(G)$.

Suppose first that $\ell = 2$. If x_1 and v^+ are consecutive on C, then the cycle $xx_1Cv^-v^+vx$ is a 1-chord extension of C. Thus $x_1^- \neq v^+$, but then the cycle $xx_1v^+Cx_1^-x_1^+Cvx$ is a 2-chord extension of C. Hence we have $\ell = 3$.

We consider $\langle v, x, x_2, v^- \rangle$. Obviously $xv^- \notin E(G)$ and since, by the choice of P, also $xx_2 \notin E(G)$, we have $x_2v^- \in E(G)$. Thus, by symmetry, we can assume without loss of generality that $x_2 \in v^+Cx_1^-$.

Since $xx_1^+ \notin E(G)$ and $xx_2 \notin E(G)$, from $\langle x_1, x, x_2, x_1^+ \rangle$ we have $x_2x_1^+ \in E(G)$. We show that x_2 cannot be consecutive on C with any of x_1, x_1^- and v^+. Indeed, if x_2 and x_1 are consecutive on C (i.e., $x_2 = x_1^-$), then the cycle $xvCx_2v^-\overleftarrow{C}x_1x$ is a 1-chord extension of C, if x_2 and x_1^- are consecutive on C (i.e., $x_2^+ = x_1^-$), then the cycle $xvCx_2v^-\overleftarrow{C}x_1^+x_1^-x_1x$ is a 2-chord extension of C and if x_2 and v^+ are consecutive on C (i.e., $x_2^- = v^+$), then the cycle $xvv^+v^-\overleftarrow{C}x_1^+x_2Cx_1x$ is a 2-chord extension of C.

We now consider $\langle x_2, x_2^+, x_1^+, v^+ \rangle$. Obviously $x_1^+v^+ \notin E(G)$ (otherwise $xv\overleftarrow{C}x_1^+v^+Cx_1x$ is a 1-chord extension of C). If $x_2^+v^+ \in E(G)$, then the cycle $xx_1\overleftarrow{C}x_2^+v^+Cx_2x_1^+Cvx$ is a 2-chord extension of C and if $x_2^+x_1^+ \in E(G)$, then the cycle $xvCx_2v^-\overleftarrow{C}x_1^+x_2^+Cx_1x$ is a 2-chord extension of C. Hence $\langle x_2, x_2^+, x_1^+, v^+\rangle$ is an induced claw. This contradiction proves Proposition 1. □

An immediate consequence of Proposition 1 is the following corollary.

Corollary 2. *Let G be a claw-free graph, $C \subset G$ a cycle, and $v \in V(C)$ a vertex of C such that $N(v) \setminus V(C) \neq \emptyset$ and $\langle N(v)\rangle$ is connected. Then, there is a sequence of cycles $C_1, \ldots, C_t$ such that $C_1 = C$, C_{i+1} is a 2-chord extension of C_i, $1 \leq i \leq t-1$, and $V(C_t) = V(C) \cup N(v)$.*

Theorem 3. *Let G be a claw-free graph on $n \geq 3$ vertices and put $M(G) = \{x \in V(G) | \langle N(x)\rangle$ is connected$\}$.*

(i) *If $M(G)$ is a dominating set of G and $\langle M(G)\rangle$ is connected, then G is vertex pancyclic orderable.*

(ii) *If, moreover, $G \setminus M(G)$ is triangle-free, then G is fully 2-chord extendible.*

Proof:

(i) Let $x \in V(G)$ and suppose first that x has degree 1 in G. Let y be the neighbor of x. Then $x \in M(G)$ and, as $|V(G)| \geq 3$, $y \notin M(G)$. Since $M(G)$ is dominating, there is $z \in M(G)$, $z \neq x$. But then every x,z-path in G contains y which contradicts the fact that $\langle M(G)\rangle$ is connected. Hence, we have $\delta(G) \geq 2$. Consequently, every $x \in M(G)$ is on a triangle.

Let now $x \notin M(G)$. Since $M(G)$ is dominating, there is $y \in M(G)$ such that $xy \in E(G)$. Since $\delta(G) \geq 2$, there is $z \in V(G)$ such that $z \neq x$ and $\{x, z\} \subset N(y)$. As $\langle N(y)\rangle$ is connected, there is a triangle containing both x and y.

Thus, for every $x \in V(G)$ there is a triangle $C \subset G$ such that $x \in V(G)$ and $V(C) \cap M(G) \neq \emptyset$. The rest of the proof follows immediately from Corollary 2.

(ii) It remains to prove that every nonhamiltonian cycle $C \subset G$ is 2-chord extendible. If $V(C) \cap M(G) \neq \emptyset$, then C is 2-chord extendible by Corollary 2. Thus suppose that $V(C) \subset V(G) \setminus M(G)$. Let $x \in V(C)$. Denote by x', x'' the vertices consecutive to x on C and choose a vertex $y \in M(G)$ such that $xy \in E(G)$ (which exists since $M(G)$ is dominating). Consider $\langle x, x', x'', y\rangle$. Since $G \setminus M(G)$ is triangle-free, we have $x'x'' \notin E(G)$. This implies that $yx' \in E(G)$ or $yx'' \in E(G)$, but in both of these cases we obtain a cycle C' which is a 0-chord extension of C. □

Remarks:

1. It is easy to observe that G satisfies the assumptions of Theorem 3(i) if and only if every cutset of G contains a vertex $x \in M(G)$. Indeed, if there is a cutset S with $S \cap M(G) = \emptyset$, then either $\langle M(G) \rangle$ is disconnected or $M(G)$ is not dominating; conversely, if $x \notin M(G)$ and $N(x) \cap M(G) = \emptyset$, then $N(x)$ is a cutset and if M_1 is one of the components of $\langle M(G) \rangle$, then also $N(V(M_1))$ is a cutset with $N(V(M_1)) \cap M(G) = \emptyset$. Thus, the assumptions of Theorem 3(i) are equivalent to those of [7] and [1], but they are easier to verify.

   Moreover, from the proof of Theorem 3(i) we easily see that, under the same assumptions, for each $x \in V(G)$, G has a pancyclic ordering such that x is the first vertex and every extension is a 2-chord extension.

2. Let $k \geq 3$ be an integer and let G be a graph on $n = 3k$ vertices which is obtained by joining every vertex of a copy of K_k to two different vertices of a copy of K_{2k}, where the pairs in the copy of K_{2k} are chosen to be disjoint. Then G is vertex pancyclic orderable but is not fully cycle extendible since every cycle of length k in the copy of K_k is nonextendable. Thus, the assumption that $G \setminus M(G)$ is triangle-free is essential in Theorem 3(ii).

In the case when $\langle M(G) \rangle$ is disconnected we can prove the following.

Theorem 4. *Let G be a claw-free graph of connectivity $\kappa(G) \geq 2$ and $M(G) = \{x \in V(G) | \langle N(x) \rangle$ is connected$\}$. Suppose that $M(G)$ is a dominating set of G and $\langle M(G) \rangle$ has r components. If $r \leq \kappa(G)$, then G is hamiltonian.*

Proof: Let $H_1, \ldots, H_r$ be the components of $\langle M(G) \rangle$ and for every i, $1 \leq i \leq r$, choose a vertex $a_i \in V(H_i)$. We use the following theorem by Dirac (see, e.g. [4]).

Theorem. *If G is a graph of connectivity $\kappa(G) \geq 2$ and $\{x_1, \ldots, x_k\} \subset V(G)$ is a set of $k \leq \kappa(G)$ vertices, then there is a cycle $C \subset V(G)$ such that $\{x_1, \ldots, x_k\} \subset V(C)$.*

By this theorem, there is a cycle $C \subset G$ containing all vertices $a_1, \ldots, a_r$. By Corollary 2, C can be extended to a hamiltonian cycle of G. □

Remarks:

1. Let H_1, H_2, H_3 be locally connected claw-free graphs on at least 3 vertices and $a_i, b_i \in V(H_i)$ such that $\langle N(a_i) \rangle$ and $\langle N(b_i) \rangle$ are complete graphs ($i = 1, 2, 3$). Construct a graph G by adding the edges $a_i a_j$ and $b_i b_j$ for $i, j = 1, 2, 3$, $i \neq j$. Then G is a claw-free graph with connectivity $\kappa(G) = 2$, $M(G)$ is dominating, $\langle M(G) \rangle$ has 3 components, and G is not hamiltonian.

2. The graph in Figure 1 shows that the assumptions of Theorem 4 do not imply pancyclicity.

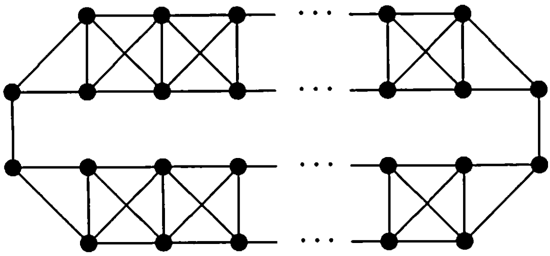

Figure 1

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