On Distance Two Labellings of Graphs

Daphne D.-F. Liu*
Department of Mathematics and Computer Science
California State University
Los Angeles, USA
email: dliu@calstatela.edu

Roger K. Yeh†
Department of Applied Mathematics
Feng Chia University
Taiwan
email: rkyeh@math.fcu.edu.tw

ABSTRACT. A distance two labelling (or coloring) is a vertex labelling with constraints on vertices within distance two, while the regular vertex coloring only has constraints on adjacent vertices (*i.e.* distance one). In this article, we consider three different types of distance two labellings. For each type, the minimum span which is the minimum range of colors used, will be explored. Upper and lower bounds are obtained. Graphs that attain those bounds will be demonstrated. The relations among the minimum spans of these three types are studied.

1 Introduction

Distance two labellings arose from the channel assignment problem (also known as T-colorings) in which channels are assigned to a number of locations while the interference among close locations is avoided (cf. [3] [7] [8] [9]). Here, we consider the assignments that also avoid the interference among second-close (distance two) locations. A *distance two labelling* is a vertex labelling (using nonnegative integers) with constraints on the vertices within distance two. The *distance* between two vertices x and y in a

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graph G, denoted by $d_G(x, y)$, is defined by the length of a shortest path from x to y in G. Therefore, x and y have distance two (*i.e.* $d_G(x, y) = 2$) if x and y are not adjacent, and there is a path $x - u - y$ for some vertex u in G. Given two positive integers m and n, $m \geq n$, a distance two labelling, namely *$L(m, n)$-labelling*, of a simple graph $G = (V, E)$ is a function f defined on the vertex set of G, $f : V \to Z^+ \cup \{0\}$ such that

$$|f(u) - f(v)| \geq m \text{ whenever } \{u, v\} \in E(G),$$

and

$$|f(u) - f(v)| \geq n \text{ whenever } d_G(u, v) = 2.$$

A *consecutive $L(m, n)$-labelling* is an $L(m, n)$-labelling such that the colors used are consecutive. Three types of distance two labellings to be considered in this article are $L(1, 1)$-labelling, $L(2, 1)$-labelling, and consecutive (no-hole) $L(2, 1)$-labelling. Griggs and Yeh [4] first studied the $L(2d, d)$-labelling for any $d > 0$. They proved this labelling can be reduced to $L(2, 1)$-labellings. Sakai [10] obtained some results for consecutive $L(2, 1)$-labelling.

Since we are interested in the efficient use of colors, we define the *span* of a labelling f (also called the labelling number) as the difference of the largest and smallest colors used. The *$L(m, n)$-number* of G is the smallest number k such that G has an $L(m, n)$-labelling with no label greater than k. Note that the $L(m, n)$-number is also the minimum span among all possible $L(m, n)$-labellings of G. The $L(1, 1)$-number is denoted by $\lambda_0(G)$ or simply λ_0 when G is understood, and the $L(2, 1)$-number is denoted by λ. The *consecutive $L(m, n)$-number* of G is the minimum span among all possible consecutive $L(m, n)$-labellings of G if there exists one. If G does not have any consecutive $L(m, n)$-labelling, then the number is ∞. The consecutive $L(2, 1)$-number of G is denoted by $\lambda_c(G)$.

Section 2 will focus on the $L(1, 1)$-labelling and the number $\lambda_0(G)$. Bounds on $\lambda_0(G)$ in terms of $\Delta(G)$, which is the maximum degree of G, are obtained. The exact values of λ_0 for trees and cycles are demonstrated. Section 3 will study the relationships between $\lambda_0(G)$ and $\lambda(G)$. We will show that for any graph G, $\lambda_0 \leq \lambda \leq 2\lambda_0$. Furthermore, graphs that attain both equalities will be shown separately. Section 4 studies the consecutive $L(2, 1)$-labelling. The condition for the existence of such a labelling is obtained by using the Hamilton paths. The relations between these three numbers λ_0, λ_1 and λ_c will be explored. Especially, diameter two graphs and n-cube graphs will be focused on.

2 $L(1, 1)$-labelling and λ_0

Given a graph G, it is not hard to learn the existence of a $L(1, 1)$-labelling of G by labelling each vertex by a different color. We are interested in finding the minimum span in an $L(1,1)$-labelling, that is the $L(1,1)$-number λ_0. This section will first show the lower and upper bounds of λ_0 in terms of Δ, the maximum degree of G. Then the exact values of λ_0 for cycles and trees will be demonstrated. Denote K_n the complete graph (also called a *clique*) with n vertices. It easily follows that $\lambda_0(K_n) = n - 1$. If G is a *diameter two graph* (*i.e.* any pair of vertices have distance less than or equal to 2), then $\lambda_0(G) = |V(G)| - 1$ since every two vertices are within distance two, they must receive different labels, and we can label all the vertices from 0 through $|V(G)| - 1$.

Given a graph G, construct a new graph G^2 from G by adding new edges $\{u, v\}$ if u and v are not adjacent, but have distance two in G. That is,

$$V(G^2) = V(G), E(G^2) = E(G) \cup E', \text{ where } E' = \{\{u,v\} \colon d_G(u,v) = 2\}.$$

Denote $\chi(G)$ the *chromatic number* of G which is the minimum number of colors used in a proper-coloring (adjacent vertices receive different colors) of G. One can learn that any vertex-coloring f is a proper coloring of G^2 if and only if f is an $L(1,1)$-labelling of G. Therefore, $\lambda_0(G) = \chi(G^2) - 1$. We have the difference 1 here because the chromatic number deals with the "number" of colors used, while the $L(1,1)$-labelling number deals with the "span" (range of the colors used.) Before showing the bounds on λ_0, we need to quote the Brooks' theorem on the chromatic number and the maximum degree.

Brooks' Theorem. [1, pp. 118 and 122]. *If G is simple and has maximum degree Δ, then $\chi(G) \leq \Delta + 1$. Furthermore, if G is not an odd cycle or a complete graph, then $\chi(G) \leq \Delta$.*

Theorem 2.1. *For any connected graph G with maximum degree Δ, $\Delta \leq \lambda_0 \leq \Delta^2$.*

Proof: Suppose $v \in V(G)$ has degree Δ, then any two vertices adjacent to v have distance two, so they need different colors. Since v is adjacent to all these vertices, v requires another color. Therefore, $\Delta \leq \lambda_0(G)$.

By the construction of G^2, the maximum degree of G^2 is less than or equals to Δ^2. According to the Brooks' theorem, $\chi(G^2) \leq \Delta^2 + 1$. Therefore, $\lambda_0(G) \leq \Delta^2$. □

Notice that the upper bound in the theorem above is attainable. For example, the Petersen graph drawn in Figure 1 has $\Delta = 3$, $G^2 = K_{10}$, so $\lambda_0(G) = \chi(G^2) = 9 = \Delta^2$.

It can be verified that for any G containing P_2, a path with 3 vertices, G^2 must contain K_3. Hence, if G is neither P_2 nor K_3, then G^2 is not an odd cycle. (Note for these degenerate cases that $\lambda_0(P_2) = 2 < \Delta 2 - 1$ and $\lambda_0(K_3) = 2 < \Delta^2 - 1$.) Thus we have the following result.

Theorem 2.2. *If G is not a diameter two graph, then $\lambda_0(G) \leq \Delta^2 - 1$.*

Proof: Since the diameter of G is not two, G^2 is not a complete graph. By the discussion above, G^2 is not an odd cycle. According to Brooks' theorem, $\lambda_0(G) = \chi(G^2) - 1 \leq \Delta^2 - 1$. □

With the following two results, we show the exact values of λ_0 for cycles and trees correspondingly. For $a \leq b$, denote $[a, b]$ as the set of integers $\{a, a+1, a+2, \ldots, b\}$.

Theorem 2.3. *Let C_n be a cycle of length n, $n \geq 3$. Then*

$$\lambda_0(C_n) = \begin{cases} 2, & \text{if } n \equiv 0 \pmod 3 \\ 4, & \text{if } n = 5 \\ 3, & \text{otherwise.} \end{cases}$$

Proof: Suppose $V(C_n) = \{v_1, v_2, \ldots, v_n\}$, we consider the following three cases:

(1) $n \equiv 0 \pmod 3$.

Define f on $V(G)$ as follows:

$$f(v_i) = \begin{cases} 0, & \text{if } i \equiv 1 \pmod 3 \\ 1, & \text{if } i \equiv 2 \pmod 3 \\ 2 & \text{if } i \equiv 3 \pmod 3. \end{cases}$$

It is easy to learn that f is an $L(1,1)$-labelling, hence $\lambda_0(C_n) \leq 2$. By definitions, for any n, $n \geq 3$, we must use at least 2 colors to label $V(C_n)$. Thus $\lambda_0(C_n) = 2$.

(2) $n \equiv 1 \pmod 3$.

Define f as follows:

$$f(v_i) = \begin{cases} 3, & \text{if } i = 1 \\ 0, & \text{if } i \equiv 2 \pmod 3 \\ 1, & \text{if } i \equiv 0 \pmod 3 \\ 2, & \text{if } i \equiv 1 \pmod 3, i > 1. \end{cases}$$

It is not difficult to verify that f is an $L(1,1)$-labelling, so $\lambda_0(C_n) = 3$.

(3) $n \equiv 2 \pmod 3$.

If $n = 5$ then obviously $\lambda_0(C_n) = 4$. Suppose $n > 5$, define f as follows.

$$f(v_i) = \begin{cases} 0, & \text{if } i = 1, 4 \\ 3, & \text{if } i = 2 \text{ or } i \equiv 0 \pmod 3 \text{ but } i \neq 3 \\ 1, & \text{if } i = 3 \text{ or } i \equiv 1 \pmod 3 \text{ but } i \neq 1, 4 \\ 2, & \text{if } i \equiv 2 \pmod 3 \text{ but } i \neq 2. \end{cases}$$

By a similar argument as in (2), we have $\lambda_0(C_n) = 3$. □

Theorem 2.4. *Let T be a tree with maximum degree Δ. Then $\lambda_0(T) = \Delta$.*

Proof: By Theorem 2.1, $\lambda_0(T) \geq \Delta$. Next we obtain the upper bound by a first-fit (greedy) labelling. First, order $V(T)$ so that $V(T) = \{v_1, v_2, \ldots, v_n\}$, where, for all $i > 1$, v_i is attached just once to $\{v_1, v_2, \ldots, v_{i-1}\}$. This can be done since T is a tree. Now we describe an $L(1,1)$-labelling of T: label v_1 as 0; then successively label $v_2, v_3, \ldots, v_n$ by the smallest available number in $[0, \Delta]$. Since each v_i, $2 \leq i \leq n$, is adjacent to only one v_j, $j < i$ and is distance two away from at most $\Delta - 1$ v_j's with $j < i$, there are at most Δ labels that cannot be used for v_i. Hence at least one label in $[0, \Delta]$ is available to v_i when its turn comes to be labeled. Thus the labelling number is at most Δ, and the theorem follows. □

3 Relations between λ_0 and λ

In the previous section, we learned the bounds of $\lambda_0(G)$ in terms of $\Delta(G)$. This section will first demonstrate upper and lower bounds of $\lambda(G)$ in terms of $\lambda_0(G)$. We will show for any graph G,

$$\lambda_0 \leq \lambda \leq 2\lambda_0.$$

Two classes of graphs that attain the lower bounds are discovered. With regard to *d-regular graphs* (all vertices have the same degree d), a family of graphs that attain the upper bound will be demonstrated.

Theorem 3.1. *For any graph G, $\lambda_0 \leq \lambda \leq 2\lambda_0$.*

Proof: According to the definition, an $L(2,1)$-labelling is also an $L(1,1)$-labelling for the same graph G, thus $\lambda_0(G) \leq \lambda(G)$.

To show the upper bound of $\lambda(G)$, let f be an $L(1,1)$-labelling of G. Define $g = 2f$, *i.e.*, $g(v) = 2f(v)$ for any $v \in G$. Define $\|f(G)\| = \max\{f(v) \colon v \in V(G)\}$. Hence $\|g(G)\| = 2\|f(G)\|$

We claim that g is an $L(2,1)$-labelling of G. Since f is an $L(1,1)$-labelling, if $\{u,v\} \in E(G)$, then $|g(u) - g(v)| = |2f(u) - 2f(v)| = 2|f(u) - f(v)| \geq 2$. Suppose $d_G(u,v) = 2$, then $|g(u) - g(v)| = 2|(u) - f(v)| \geq 2 \geq 1$. Thus g is an $L(2,1)$-labelling. Therefore $\lambda \leq \|g(G)\| = 2\|f(G)\|$. Since this is true for any $L(1,1)$-labelling, $\lambda \leq 2\lambda_0$. □

With the following two theorems, we show classes of graphs G such that $\lambda_0(G) = \lambda(G)$. A graph G is an *incidence graph of a projective plane* $\Pi(n)$ of order n, if $G = (A, B, E)$ is a bipartite graph such that

1. $|A| = |B| = n^2 + n + 1$,
2. each $a \in A$ corresponds to a point p_a in $\Pi(n)$ and each $b \in B$ corresponds to a line ℓ_b in $\Pi(n)$, and

3. $E = \{\{a, b\} : a \in A, b \in B$ such that $p_a \in \ell_b$ in $\Pi(n)\}$.

By the definition of $\Pi(n)$, we know that such G is $(n+1)$-regular, for every $x, y \in A$, $d_G(x, y) = 2$, and for every $u, v \in B$, $d_G(u, v) = 2$. Also, if $a \in A$, $b \in B$ such that a is not adjacent to b, then $d_G(a, b) = 3$. Therefore, we have the following result.

Theorem 3.2. *If G is an incidence graph of a protective plane of order n, then $\lambda(G) = \lambda_0(G) = n^2 + n = \Delta^2 - \Delta$, where $\Delta = n + 1$ the maximum degree of G.*

Proof: Let $G = (A, B, E)$, by Theorem 8.1 in [4], $\lambda(G) = n^2 + n$. Hence it is sufficient to show that $\lambda_0(G) = n^2 + n$.

Let $A = \{a_0, a_1, \ldots, a_t\}$and $B = \{b_0, b_1, \ldots, b_t\}$ where $t = n^2 + n$. Since $|A| = t+1$ and $d_G(a_i, a_j) = 2$, for every a_i, a_j in A, for any $L(1,1)$-labelling f of G, we have $\|(A)\| \geq t$. Thus $\lambda_0(G) \geq t$.

Define a labelling f by $f(a_i) = i$, for every a_i in A, $i = 0, 1, \ldots, t$. Let $G' = K_{t+1,t+1} \setminus G$, *i.e.*, a_i and b_j are adjacent in G if and only if a_i and b_j are not adjacent in G'. Since G is Δ-regular, G' is $(t - \Delta)$-regular. Therefore there is a matching M in G' by Hall's theorem.

Let b_i be adjacent to $m(b_i)$, $i = 0, \ldots, t$, in M where $\{m(b_i) : i = 0, \ldots, t\} = \{a_i : i = 0, \ldots, t\}$. Define $f(b_i) = f(m(b_i))$, for each i. Since $m(b_i)$ is adjacent to b_i in G', $m(b_i)$ is not adjacent to b_i in G, for each i, so we have just used $\{0, 1, 2, \ldots, t\}$ to label the graph G. Therefore $\|(G)\| = t$. This implies $\lambda_0(G) \leq t$. The result then follows. □

Next we define another class of graphs from projective planes as well. Denoting a Galois plane (over the coordinate field $GF(n)$) by $PG_2(n)$ (cf. [2][6]), we construct a class of graphs H by the following: $V(H)$ is the set of points of $PG_2(n)$ and $E(H)$ is formed by joining the points (x, y, z) and (x', y', z') for which $xx' + yy' + zz' = 0$, *i.e.*, for which (x', y', z') lies on the line $[x, y, z]$. We call such a graph H the *polarity graph of* $PG_2(n)$. Then by the properties of $PG_2(n)$, we know that $|V(H)| = n^2 + n + 1$, $\Delta(H) = n + 1$, the minimum degree $\delta(G) = n$ and the diameter is 2 (cf. [2]). We have the following result.

Theorem 3.3. *If H is the polarity graph of the Galois plane $PG_2(n)$, then $\lambda(H) = \lambda_0(H) = n^2 + n = \Delta^2 - \Delta$.*

Proof: It is known [4] that $\lambda(H) = n^2 + n = \Delta^2 - \Delta$. Since H is a diameter two graph, $\lambda_0(H) = |V(H)| - 1 = n^2 + n$. □

We now turn to the upper bound of $\lambda(G)$, $\lambda(G) \leq 2\lambda_0(G)$. A family of regular graphs G will be shown that $\lambda(G) = 2\lambda_0(G)$. Before presenting this, we introduce the following simple result.

Proposition 3.4. *If G is a connected d-regular graph with n vertices, then $\lambda_0(G) = 2$ if and only if $d = 2$ and n is a multiple of 3.*

Proof: ($\Leftarrow$) If $d = 2$ and $n \equiv 0 \pmod 3$, then G is a cycle; by Theorem 2.3, $\lambda_0 = 2$.

($\Rightarrow$) Suppose $\lambda_0(G) = 2$; by Theorem 2.1, $d = \Delta(G) \leq 2$. If $d \leq 1$, then $G = P_0$ or P_1 in which cases $\lambda_0(P_0) = 0$ and $\lambda_0(P_1) = 1$, respectively. Hence $d = 2$, so G is a cycle; by Theorem 2.3, n must be a multiple of 3 which finishes the proof. □

Theorem 3.5. *If G is a Δ-regular bipartite graph with $2(\Delta + 1)$ vertices and $\Delta \geq 2$, then $\lambda_0(G) = \Delta$ and $\lambda(G) = 2\Delta$.*

Proof: Let $G = (A, B, E)$ and $H = K_{\Delta+1,\Delta+1} \setminus G$, then H is a matching with $|H| = \Delta+1$ edges. Let $E(H) = \{(a_0, b_0), (a_1, b_1), \ldots, (a_\Delta, b_\Delta)\}$ where $a_i \in A$ and $b_i \in B$ for each i. We then label a_i and b_i with i. It is easy to see that this labelling is an $L(1,1)$-labelling of G, so $\lambda_0(G) \leq \Delta$. By Theorem 2.1, we have $\lambda_0(G) = \Delta$.

Now, we claim that $\lambda(G) = 2\Delta$ by induction on Δ. The indexes of a_i and b_i defined above will be kept in the proof.

Initial step: $\Delta = 2$. Then G is a cycle of length 6, C_6. By a previous result in [3], we have $\lambda(C_6) = 4$.

Inductive step: For any $0 \leq i \leq \Delta$ with $\Delta \geq 3$, define the reduced graph G_i by deleting vertices a_i and b_i and the edges adjacent to them from G, then the reduced graph is a $(\Delta - 1)$-regular bipartite graph. By the inductive hypothesis, $\lambda(G_i) = 2\Delta - 2$ for any i.

Suppose $\lambda(G) < 2\Delta$, then let f be an $L(2,1)$-labelling such that f has span less then 2Δ, that is,

$$f\colon V(G) \to \{0, 1, 2, 3, \ldots, 2\Delta - 1\}.$$

According to the definition of an $L(2,1)$-labelling, only a_i and b_i can possibly use the same color, and $f(a_i) \neq f(a_j)$, $f(b_i) \neq f(b_j)$, if $i \neq j$. Without loss of generality, we may assume $f(a_0) = 0$. If $f(b_0) = 1$, $f|_{G_0}$ is an $L(2,1)$-labelling and then $\lambda(G_0) \leq 2\Delta - 3$, a contradiction. Hence, $f(b_0) \neq 1$. Furthermore, we then know that $f(b_i) \neq 1$ for all i. If $f(a_i) \neq 1$ for all $i \neq 0$, we again have the contradiction $\lambda(G_0) \leq 2\Delta - 3$. Thus, without loss of generality, assume $f(a_1) = 1$. Continuing this process, we finally have $f(a_i) = i$ for all i. This implies for each i, $f(b_i) \geq \Delta + 2$, and no b_i and b_j can receive the same label. Therefore, we need $\Delta + 1$ new colors starting from $\Delta + 2$ to color all the vertices in B, a contradiction, since $|[0, 2\Delta - 1] \setminus [0, \Delta + 1]| < \Delta + 1$. Therefore, $\lambda(G) = 2\Delta$. □

4 Consecutive $L(2,1)$-labelling

This section studies the consecutive $L(2,1)$-labelling and the consecutive $L(2,1)$-number $\lambda_c(G)$. We will first characterize the existence of a consecutive $L(2,1)$-labelling by using the concept of Hamilton paths. Then the relations among λ_0, λ and λ_c will be discussed. The values of $\lambda_c(G)$ for a group of graphs G will be calculated. Notice that if G has a consecutive $L(2,1)$-labelling, then $\lambda_c \le |V(G)| - 1$ and vice versa. Denote G^c the complement graph of G.

Proposition 4.1. *There is a Hanlilton path in G^c if and only if $\lambda_c(G) \le |V(G)| - 1 < \infty$.*

Proof: ($\Rightarrow$) Suppose there is a Hamilton path $P = \{v_0, v_1, \ldots, v_{n-1}\}$ in G^c ($n = |V(G)|$). Then we can label v_i as i for each i. It is easy to check this labelling is a consecutive $L(2,1)$-labelling of G. Hence $\lambda_c(G) \le n-1 = |V(G)| - 1 < \infty$.

($\Leftarrow$) Suppose $\lambda_c(G) = m < \infty$. Let $\{v_0^i, v_i^i, \ldots, v_{n_i}^i\}$ be the set of vertices with label i, $0 \le i \le m$. Then $v_0^0, v_0^0, \ldots, v_{n_0}^0, v_1^1, \ldots, v_{n_1}^1, \ldots, v_0^i, v_1^i, \ldots, v_{n_i}^i, \ldots, v_0^m, v_1^m, \ldots, v_{n_m}^m$ form a Hamilton path in G^c. □

If f is a consecutive $L(2,1)$-labelling of G, then f is also an $L(2,1)$-labelling of G. If G does not have any consecutive $L(2,1)$-labelling, then $\lambda_c(G) = \infty$. Therefore, for any graph G, we have

$$\lambda(G) \le \lambda_c(G).$$

Considering the cases for which $\lambda = \lambda_c$, we have the following result.

Theorem 4.2. *Given a graph G, if $\lambda_0 = \lambda$, then $\lambda = \lambda_c$.*

Proof: Suppose $\lambda = k < \lambda_c$, and let f be an $L(2,1)$-labelling obtaining the optimal span k. Without loss of generality, we may assume

$$f\colon V \to [0, k].$$

Because $k < \lambda_c$, f is not a consecutive coloring. Hence, there exists a number m, $0 < m < k$, such that there is no $v \in V$ with $f(v) = m$. Define a coloring $g\colon V \to [0, k-1]$ by,

$$g(v) = \begin{cases} f(v), & \text{if } f(v) < m, \\ f(v) - 1, & \text{if } f(v) > m. \end{cases}$$

One can learn that g is an $L(1,1)$-labelling. This implies $\lambda_0 < k = \lambda$, a contradiction. Therefore, $\lambda = \lambda_c$. □

According to the theorem above, graphs in theorems 3.2 and 3.3 satisfy the equality $\lambda = \lambda_c$. Applying previous results of λ on paths P_n, cycles C_n, and trees T, in [4] and [10], we have the following proposition that provides another three classes of graphs such that $\lambda = \lambda_c$. Inspection of Theorems 2.3 and 2.4 shows that these are also examples indicating that the converse of Theorem 4.2 is not always true.

Proposition 4.3.

(a) $\lambda(P_4) = \lambda_c(P_4) = 3$, *and* $\lambda(P_n) = \lambda_c(P_n) = 4$ *if* $n \geq 5$.

(b) $\lambda(C_n) = \lambda_c(C_n) = 4$, *if* $n = 5$ *or* $n \geq 7$.

(c) $\lambda(T) = \lambda_c(T) = \Delta + 1$ *or* $\Delta + 2$, *if* T *is not* $K_{1,\Delta}$. □

We have learned that if G is a diameter two graph, $\lambda_0 = |V| - 1 \leq \lambda_c$. If in addition G has a consecutive $L(2,1)$-labelling, then $\lambda_c = |V| - 1$. Thus, by Proposition 4.1, we have the following result for diameter two graphs.

Corollary 4.4. *Let G be a diameter two graph. The following are equivalent:*

(a) *G has a consecutive $L(2,1)$-labelling.*

(b) *There is a Hamilton path in G^c.*

(c) $\lambda = \lambda_c = |V| - 1$. □

Notice that there are no consecutive $L(2,1)$-labellings for complete graphs with more than one vertex. For diameter two graphs, refereeing to Figure 1, the Petersen graph is an example to the corollary above which has $\lambda_c = |V| - 1$. (The vertices $0, 1, 2, \ldots, 9$ form a Hamilton path in G^c. Notice that these labels also form a valid consecutive $L(2,1)$-labelling.)

Now we consider the *n-cube graph* Q_n which has 2^n vertices where each vertex v can be written as $v = (v_1, v_2, \ldots, v_n)$, where each v_i is either 0 or 1, and where edges join vertices v, w when there exists a unique i such that $v_i \neq w_i$. It has been proved [4] that $\lambda(Q_n) \leq 2n+1$, for all $n \geq 3$ by defining the labelling f on $V(Q_n)$ by $f(v) = \sum_{i:\, v_i=1}(i+1) \pmod{(2n+2)}$, where all labels are chosen from $[0, 2n+1]$. Furthermore, by checking the labels used, we find that this labelling in fact is a consecutive $L(2,1)$-labelling as well. Therefore, we have

Theorem 4.5. $\lambda_c(Q_n) \leq 2n + 1$, *for all* $n \geq 3$. □

Notice that the inequality is not tight for some n. For example, Sakai [10] showed that $\lambda_c(Q_4) = 7$ whereas $\lambda_c(Q_3) = 7$. It would be interesting to determine when equality holds.

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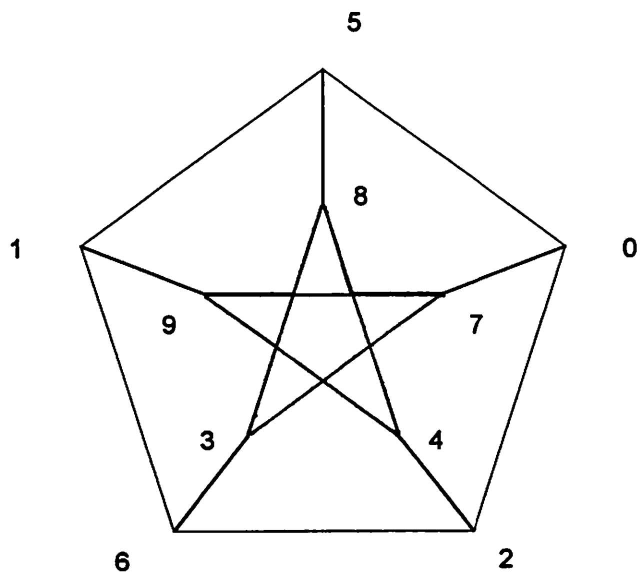

Figure 1. The Petersen Graph