On the Connection between the Undirected and the Acyclic Directed Two Disjoint Paths Problem

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ABSTRACT. Given an undirected graph G and four distinct *special vertices* s_1, s_2, t_1, t_2, the *Undirected Two Disjoint Paths Problem* consists in determining whether there are two disjoint paths connecting s_1 to t_1 and s_2 to t_2, respectively.

There is an analogous version of the problem for acyclic directed graphs, in which it is required that the two paths be directed, as well.

The well known characterizations for the nonexistence of solutions in both problems are, in some sense, the same, which indicates that under some weak conditions the edge orientations in the directed version are irrelevant. We present the first direct proof of the irrelevance of edge orientations.

1 Introduction

In this paper all graphs are *simple*, that is, free of loops and multiple edges.

Given an undirected graph G and four distinct *special vertices* s_1, s_2, t_1, t_2, the *Two Disjoint Paths Problem* consists in establishing that there exists

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in G a *disjoint* pair of paths, one connecting s_1 to t_1, the other connecting s_2 to t_2, or, if no such paths exist, producing a certificate of nonexistence.

This problem was solved independently by Seymour [4] and by Thomassen [6]. Both authors gave a structural characterization which leads to a polynomial algorithm for solving the problem. Polynomial algorithms were also given by Perl and Shiloach [3] and by Shiloach [5].

As pointed out by Seymour [4], if there exists a vertex set X of G such that $|X| \leq 3$ and $G - X$ has a connected component K free of special vertices, then the existence of solutions is preserved by the following *reduction*:

- Remove from G all vertices of K.
- Join each pair of vertices of X by an edge.

We say that G is *irreducible* if the reduction described above is not applicable.

It is also easy to see that if one adds four edges, joining each of s_1 and t_1 to each of s_2 and t_2, then the existence of solutions is also preserved: we call cycle s_1–s_2–t_1–t_2 thus obtained a *quadrilateral*.

The characterizations of the problem, mentioned above, are equivalent to the following.

Theorem 1 *If G is irreducible then the two disjoint paths problem has no solution if and only if addition of edges joining each of s_1 and t_1 to each of s_2 and t_2 yields a planar graph having quadrilateral s_1–s_2–t_1–t_2 as one of its faces.*

An analogous version of the problem, for acyclic directed graphs, called the *Acyclic Directed Two Disjoint Paths Problem*, requires that the two paths be directed, as well. This version of the problem was solved by Thomassen [7]. Again, the structural characterization given therein leads naturally to a polynomial algorithm. A very simple and elegant algorithm for solving this problem was given by Perl and Shiloach [3].

As pointed out by Thomassen [7], the aciclicity of the graph and the existence of solutions is preserved by the following *reductions*:

- Remove edges entering s_1 and s_2; remove edges leaving t_1 and t_2.
- Remove sources and sinks that are nonspecial vertices.
- Contract an edge if it is the only edge leaving (entering) a vertex, and at least one of its ends is non special.

After performing these reductions, Thomassen derives a characterization for the existence of solutions that is quite similar to that of Theorem 1. An equivalent assertion is given below (Corollary 4). A constructive generalization of this result was given by A. Metzler in her Ph. D. thesis [2]. It follows [7, Corollary 3.4] that if the graph is free of the reductions mentioned in the previous paragraph then there exists a solution for the directed version if and only if there exists a solution for the corresponding undirected version of the problem.

Thomassen [7] also indicates that it would be interesting to have a direct proof of this fact. That proof is given herein.

More specifically, we prove the following result.

Theorem 2 *Let G be an acyclic directed graph such that special vertices* s_1 *and* s_2 *are sources, special vertices* t_1 *and* t_2 *are sinks and every nonspecial vertex has indegree and outdegree at least 2. The Acyclic Directed Two Disjoint Paths Problem has a solution if and only if the corresponding undirected Two Disjoint Paths Problem has a solution.*

The following result will play a central role in the proof and follows from Menger's Theorem and the hypothesis of Theorem 2 [7].

Proposition 3 *Under the hypothesis of Theorem 2, for every nonspecial vertex* v *there exist directed paths from* s_1 *to* v*, from* s_2 *to* v*, from* v *to* t_1 *and from* v *to* t_2*, disjoint except at* v*.*

It follows that, under the hypothesis of Theorem 2, the underlying undirected graph is irreducible; we then have an alternate proof of the characterization of nonexistence of solution for the Acyclic Directed Disjoint Two Paths Problem:

Corollary 4 *Under the hypothesis of Theorem 2, the Acyclic Directed Two Disjoint Paths Problem has no solution if and only if addition of edges joining each of* s_1 *and* t_1 *to each of* s_2 *and* t_2 *yields a planar graph having quadrilateral* s_1–s_2–t_1–t_2 *as one of its faces.*

The results presented herein are a revision of an earlier version contained in the second author's M. Sc. Dissertation [1], written under the first author's supervision.

2 Terminology

A *path* is a sequence $P := (u_0, \alpha_1, u_1, \cdots, \alpha_m, u_m)$ in a directed graph, where the u_i are pairwise distinct vertices, the α_i are edges, and u_{i-1} and u_i are the ends of α_i, for each i such that $1 \leq i \leq m$. The *reversal* of P is path

$(u_m, \alpha_m, \cdots, u_1, \alpha_1, u_0)$. We allow $m = 0$, in which case P is *degenerate*. Vertices u_0 and u_m are, respectively, the *origin* and the *terminus* of P. We denote by VP vertex set $\{u_0, u_1, \cdots, u_m\}$.

Let $Q := (v_0, \beta_1, v_1, \cdots, \beta_n, v_n)$ be a path such that u_m and v_0 are identical. The *product* $P \circ Q$ of P and Q is defined to be the sequence $(u_0, \alpha_1, u_1, \cdots, \alpha_m, u_m, \beta_1, v_1, \cdots, \beta_n, v_n)$; such sequence is a path if and only if $VP \cap VQ = \{u_m\}$.

For $0 \le i \le j \le m$, we denote by $P[u_i, u_j]$ the subpath of P with origin u_i and terminus u_j, that is, $P[u_i, u_j] := (u_i, \alpha_{i+1}, u_{i+1}, \cdots, \alpha_j, u_j)$; we denote by $P[u_j, u_i]$ the reversal of $P[u_i, u_j]$, that is, $P[u_j, u_i] := (u_j, \alpha_j, \cdots, u_{i+1}, \alpha_{i+1}, u_i)$. It is very important to realize that for vertices u and v in VP, $P[u, v]$ may not be a subpath of P, but in that case it certainly is a subpath of the reversal of P.

An edge α_i $(1 \le i \le m)$ is *forward* in P if it is directed away from u_{i-1} into u_i, otherwise it is a *reverse* edge. Vertex u_i $(0 \le i \le m)$ is a *switch* in P if

either (i) $i = 0 < m$ and α_1 is a reverse edge in P,
or (ii) $0 < i < m$, one of α_i and α_{i+1} is a forward edge, the other a reverse edge in P,
or (iii) $0 < i = m$ and α_m is a reverse edge in P.

In other words, a switch in P usually is a vertex where a change of direction occurs, but it is important to notice that we also consider the origin of P a switch if its first edge is reverse; likewise, the terminus of P is a switch if its last edge is reverse.

We denote by SP the set of switches of P.

Path P is *directed* if α_i is directed away from u_{i-1} into u_i for each i such that $1 \le i \le m$. Thus P is directed if and only if SP is the null set.

Proposition 5 *Let $P := A \circ B$ be a path, let v denote the origin of B (and the terminus of A).*

(a) If v is not a switch of A, then $SP \cap VB = SB$.

(b) If v is not a switch of B, then $SP \cap VA = SA$.

Corollary 6 *Let $P := A \circ B \circ C$ be a path. If A and C are both directed, then $SP = SB$.*

3 Proof of Theorem 2

Clearly, any solution for the directed version is also a solution to the undirected version. To prove the converse, assume that there exist two disjoint (not necessarily directed) paths in G, joining s_1 to t_1 and s_2 to t_2, respectively. Among such pairs of paths, choose one, (P_1, P_2), say, such that the corresponding set of switches $S := SP_1 \cup SP_2$ is minimal. We now prove that each of P_1 and P_2 is a directed path, thereby proving Theorem 2.

Assume, to the contrary, that at least one of P_1 and P_2 is not directed. That is, assume that S is nonnull.

Define relation $\leq$ on the vertex set of G by $u \leq v$ if and only if there exists a directed path from u to v in G. Since G is acyclic, relation $\leq$ is a partial order.

Let u_0 be a minimal element of S, with respect to partial order $\leq$. That is, $u_0 \in S$ and $\forall x \in S, x \not\leq u_0$. Let v_0 be a maximal element of $\{v : v \in S, u_0 \leq v\}$. Clearly, v_0 is maximal in S.

By Proposition 3, there exist two directed paths, Q_1 and Q_2, respectively from s_1 and s_2 to u_0, disjoint except at u_0. Similarly, there exist two directed paths, R_1 and R_2, from v_0 to respectively t_1 and t_2, disjoint except at v_0.

Proposition 7 *For each vertex q in $VQ_1 \cup VQ_2$ and each vertex r in $VR_1 \cup VR_2$, inequality $q \leq u_0 \leq v_0 \leq r$ holds.*

Proof. Each of Q_1, Q_2, R_1 and R_2 is a directed path. Vertex u_0 is the terminus of Q_1 and Q_2, whence $q \leq u_0$. Vertex v_0 is the origin of R_1 and R_2, whence $v_0 \leq r$.

By definition of v_0, $u_0 \leq v_0$.

We conclude that $q \leq u_0 \leq v_0 \leq r$, and the assertion follows.

Vertex s_1, the origin of P_1, lies in Q_1. Let q_1 be the last vertex of P_1 in $VQ_1 \cup VQ_2$: that is, no vertex of $P_1[q_1, t_1]$ except q_1 lies in $VQ_1 \cup VQ_2$. Likewise, define r_1 to be the first vertex of P_1 in $VR_1 \cup VR_2$. Define q_2 and r_2, vertices of P_2, similarly.

Proposition 8 *For $i, j, k \in \{1, 2\}$, for each vertex q in $VP_i \cap VQ_j$ and for each vertex r in $VP_i \cap VR_k$, the following properties hold:*

(a) Neither q nor r is a switch in $P_i[q, r]$.

(b) If $q \neq u_0$ then path $P_i[s_i, q]$ is directed.

(c) If $r \neq v_0$ then path $P_i[r, t_i]$ is directed.

(d) Vertex u_0 lies in $\{q_1, q_2\}$ and vertex v_0 lies in $\{r_1, r_2\}$.

(e) If r precedes q in P_i then $q = u_0$ and $r = v_0$.

(f) Vertex q_i is the only vertex of $P_i[q_i, r]$ in $VQ_1 \cup VQ_2$.

(g) *Vertex r_i is the only vertex of $P_i[q, r_i]$ in $VR_1 \cup VR_2$.*

Proof.
(a) Assume, to the contrary, that q is a switch in $P_i[q, r]$. By definition of switch, $q \neq r$ and the first edge of $P_i[q, r]$ is reverse. Or, equivalently, the last edge of $P_i[r, q]$ is forward. Let thus T be a maximal (nondegenerate) directed subpath of $P_i[r, q]$ having terminus q; let t be the origin of T. Path T is directed, whence $t < q$. Since q lies in $VQ_1 \cup VQ_2$, $q \leq u_0$, whence

$$t < u_0. \tag{1}$$

From this, by Proposition 7, t does not lie in $VR_1 \cup VR_2$. In particular, $t \neq r$. From this, by the maximality of T, we conclude that t is a switch of $P_i[r, q]$. Since t does not lie in $\{q, r\}$, it follows that t is a switch of P_i, regardless of the order in which q and r occur in P_i. In that case, (1) is a contradiction to the minimality of u_0. As asserted, q is not a switch in $P_i[q, r]$.

A similar argument may be used to prove that r is not a switch in $P_i[q, r]$, either.

(b) Let T be a maximal directed subpath of $P_i[s_i, q]$ having terminus q, let t be its origin. Assume, to the contrary, that $s_i \neq t$.

By the maximality of T, vertex t is a switch in $P_i[s_i, q]$. By part (a), q is not a switch in $P_i[q, t_i]$. By Proposition 5(b), t is a switch in P_i.

Since T is directed, $t \leq q$. By hypothesis, $q \neq u_0$. From these, by Proposition 7, $t \leq q < u_0$, in contradiction to the minimality of u_0 in S.

(c) Analogous to (b).

(d) Let i be such that u_0 lies in P_i. We assert that $u_0 = q_i$. For this, assume the contrary. By definition of q_i, vertex u_0 lies in $P_i[s_i, q_i]$. By part (b), $P_i[s_i, q_i]$ is directed, whence $u_0 < q_i$. This inequality contradicts Proposition 7.

We conclude that $u_0 \in \{q_1, q_2\}$. By symmetry, v_0 lies in $\{r_1, r_2\}$.

(e) Assume that r precedes q in P_i. Assume, to the contrary, that $q \neq u_0$ or $r \neq v_0$. If $q \neq u_0$ then, by part (b), $P_i[s_i, q]$ is directed. If $r \neq v_0$, then, by part (c), $P_i[r, t_i]$ is directed. In both cases, $P_i[r, q]$ is directed. Thus, $r \leq q$. By Proposition 7, $q = u_0 = v_0 = r$.

(f) If q_i precedes r in P_i, the assertion follows immediately, by definition of q_i. Assume thus that r precedes q_i in P_i. By part (e), $q_i = u_0$; also by part (e), q_i is the only vertex of $VQ_1 \cup VQ_2$ that vertex r precedes in P_i. The assertion follows.

(g) Analogous to (f).

The proof of the Theorem is divided into 3 cases. In each case a new pair of paths is defined and it is shown that the corresponding set of switches is a proper subset of S, a contradiction.

Case 1 *$r_1 \in VR_1$ and $r_2 \in VR_2$.*

Define

$$\begin{aligned} P_1' &:= P_1[s_1, r_1] \circ R_1[r_1, t_1] \\ P_2' &:= P_2[s_2, r_2] \circ R_2[r_2, t_2]. \end{aligned}$$

We begin the analysis of Case 1 by showing that P_1' and P_2' are disjoint paths.

By definition, r_1 is the only vertex of $P_1[s_1, r_1]$ in R_1. Thus P_1' is a path. Similarly, P_2' is a path.

Paths P_1 and P_2 are disjoint, whence so too are $P_1[s_1, r_1]$ and $P_2[s_2, r_2]$. In particular, r_1 and r_2 are distinct.

Paths $R_1[r_1, t_1]$ and $R_2[r_2, t_2]$ are subpaths of R_1 and R_2, respectively. Paths R_1 and R_2 are disjoint except at their common origin v_0. Since r_1 and r_2 are distinct, paths $R_1[r_1, t_1]$ and $R_2[r_2, t_2]$ are disjoint.

By definition of r_1, no vertex of $P_1[s_1, r_1]$, except possibly r_1, lies in R_2. Vertex r_1 lies in $R_1[r_1, t_1]$, in turn disjoint with $R_2[r_2, t_2]$. It follows that paths $P_1[s_1, r_1]$ and $R_2[r_2, t_2]$ are disjoint. Likewise, $P_2[s_2, r_2]$ and $R_1[r_1, t_1]$ are disjoint. It follows that P_1' and P_2' are disjoint.

We now conclude the analysis of Case 1 by showing that

$$SP_1' \cup SP_2' \subseteq (SP_1 \cup SP_2) \setminus \{v_0\},$$

thereby contradicting the choice of (P_1, P_2).

Path R_1 is directed and $R_1[r_1, t_1]$ is a subpath of R_1. Thus, $R_1[r_1, t_1]$ is directed. By Corollary 6,

$$SP_1' = SP_1[s_1, r_1].$$

By Proposition 8(a), vertex r_1 is not a switch of $P_1[s_1, r_1]$. Moreover, $P_1[s_1, r_1]$ is a subpath of P_1. Thus

$$SP_1[s_1, r_1] \subseteq SP_1 \setminus \{r_1\}.$$

By Proposition 8(d), either v_0 does not lie in P_1 or it is equal to r_1. We conclude that

$$SP_1' \subseteq SP_1 \setminus \{v_0\}.$$

Similarly,

$$SP_2' \subseteq SP_2 \setminus \{v_0\}.$$

It follows that the new set of switches is thus a proper subset of S, a contradiction. This concludes the analysis of Case 1.

Case 2 *$q_1 \in VQ_1$ and $q_2 \in VQ_2$.*

Define

$$\begin{aligned} P_1' &:= Q_1[s_1, q_1] \circ P_1[q_1, t_1] \\ P_2' &:= Q_2[s_2, q_2] \circ P_2[q_2, t_2]. \end{aligned}$$

The proof in this Case is the directional dual of that of Case 1.

Case 3 *The hypotheses of Cases 1 and 2 are both false.*

We begin the analysis of this case by showing that

$$r_1 \in VR_2,\ r_2 \in VR_1,\ q_1 \in VQ_2 \text{ and } q_2 \in VQ_1.$$

Since the hypothesis of Case 1 does not apply, either $r_1 \notin VR_1$ or $r_2 \notin VR_2$. Assume that $r_1 \notin VR_1$. Thus $r_1 \in VR_2 \setminus VR_1$. Since v_0 is the common origin of R_1 and R_2, $r_1 \neq v_0$. By Proposition 8(d), $v_0 \in \{r_1, r_2\}$, whence $r_2 = v_0 \in VR_1$. It follows that if $r_1 \notin VR_1$ then $r_1 \in VR_2$ and $r_2 \in VR_1$. The same conclusion holds if $r_2 \notin VR_2$. We conclude that $r_1 \in VR_2$ and $r_2 \in VR_1$. Likewise, the hypothesis of Case 2 does not apply, whence $q_1 \in VQ_2$ and $q_2 \in VQ_1$.

Define

$$\begin{aligned} P_1' &:= Q_1[s_1, q_2] \circ P_2[q_2, r_2] \circ R_1[r_2, t_1] \\ P_2' &:= Q_2[s_2, q_1] \circ P_1[q_1, r_1] \circ R_2[r_1, t_2]. \end{aligned}$$

We proceed in the analysis of Case 3 by showing that P_1' and P_2' are disjoint paths.

By Proposition 8(f), q_2 is the only vertex of $P_2[q_2, r_2]$ in $Q_1[s_1, q_2]$; similarly, r_2 is the only vertex of $P_2[q_2, r_2]$ in $R_1[r_2, t_1]$.

Paths $Q_1[s_1, q_2]$ and $R_1[r_2, t_1]$ are subpaths of Q_1 and R_1, respectively. The terminus of Q_1 is u_0, the origin of R_1 is v_0. By Proposition 7, $VQ_1 \cap VR_1 = \{u_0\} \cap \{v_0\}$. It follows that

$$VQ_1[s_1, q_2] \cap VR_1[r_2, t_1] = \{q_2\} \cap \{r_2\}. \tag{2}$$

Thus P_1' is a path. Similarly, P_2' is a path.

Paths P_1 and P_2 are disjoint by hypothesis, whence

$$VP_1[q_1, r_1] \cap VP_2[q_2, r_2] = \emptyset. \tag{3}$$

In particular, each of q_1 and r_1 is distinct from each of q_2 and r_2.

Paths $Q_1[s_1, q_2]$ and $Q_2[s_2, q_1]$ are subpaths of Q_1 and Q_2, respectively. Paths Q_1 and Q_2 are disjoint except at their common terminus u_0. Since q_1 and q_2 are distinct,

$$VQ_1[s_1, q_2] \cap VQ_2[s_2, q_1] = \emptyset. \tag{4}$$

In a way similar to the proof of (2), we obtain $VQ_1[s_1, q_2] \cap VR_2[r_1, t_2] = \{q_2\} \cap \{r_1\}$. But q_2 and r_1 are distinct. Thus,

$$VQ_1[s_1, q_2] \cap VR_2[r_1, t_2] = \emptyset. \tag{5}$$

By Proposition 8(f), no vertex of $P_2[q_2, r_2]$, except possibly q_2, lies in Q_2. But q_2 lies in $Q_1[s_1, q_2]$, in turn disjoint with $Q_2[s_2, q_1]$. It follows that

$$VP_2[q_2, r_2] \cap VQ_2[s_2, q_1] = \emptyset. \tag{6}$$

By symmetry, from (4), (5) and (6), respectively, we obtain (7), (8) and (9)–(11), below.

$$\begin{aligned}
VR_1[r_2, t_1] \cap VR_2[r_1, t_2] &= \emptyset. && (7)\\
VQ_2[s_2, q_1] \cap VR_1[r_2, t_1] &= \emptyset. && (8)\\
VP_2[q_2, r_2] \cap VR_2[r_1, t_2] &= \emptyset, && (9)\\
VP_1[q_1, r_1] \cap VQ_1[s_1, q_2] &= \emptyset, && (10)\\
VP_1[q_1, r_1] \cap VR_1[r_2, t_1] &= \emptyset. && (11)
\end{aligned}$$

From (3)–(11), it follows that P_1' and P_2' are disjoint.

We now conclude the analysis of Case 3 by showing that

$$SP_1' \cup SP_2' \subseteq (SP_1 \cup SP_2) \setminus \{u_0, v_0\},$$

thereby contradicting the choice of (P_1, P_2).

Path Q_1 is directed and $Q_1[s_1, q_2]$ is a subpath of Q_1. Thus, $Q_1[s_1, q_2]$ is directed. Similarly, $R_1[r_2, t_1]$ is directed. By Corollary 6,

$$SP_1' = SP_2[q_2, r_2].$$

By Proposition 8(a), neither q_2 nor r_2 is a switch of $P_2[q_2, r_2]$. Moreover, $P_2[q_2, r_2]$ is a subpath of either P_2 or the reversal of P_2. Thus

$$SP_2[q_2, r_2] \subseteq SP_2 \setminus \{q_2, r_2\}.$$

By Proposition 8(d), either u_0 does not lie in P_2 or $q_2 = u_0$; similarly, either v_0 does not lie in P_2 or $r_2 = v_0$. We conclude that

$$SP_1' \subseteq SP_2 \setminus \{u_0, v_0\}.$$

Similarly,

$$SP'_2 \subseteq SP_1 \setminus \{u_0, v_0\}.$$

From the last two inclusions we conclude that the new set of switches is a proper subset of S, a contradiction.

The conclusion of the analysis of Case 3 completes the proof of Theorem 2.

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