On Rotational Steiner Triple Systems: Five, Seven and Eleven

Zhike Jiang
Department of Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario
Canada N2L 3G1

Abstract

The spectra of 5-, 7- and 11-rotational Steiner triple systems are determined. In the process, existence for a number of generalized Skolem sequences is settled.

1 Introduction

A *Steiner triple system of order* v, denoted by $S(v)$, is an ordered pair $(V, \mathcal{T})$, where V is a v-set and $\mathcal{T}$ a collection of 3-subsets of V, called *triples*, such that each pair of elements of V is contained in exactly one triple in $\mathcal{T}$. It is well-known that an $S(v)$ exists if and only if $v \equiv 1, 3 \pmod 6$. An *automorphism* of an $S(v)$ $(V, \mathcal{T})$ is a permutation on V which preserves $\mathcal{T}$.

A *k-rotational permutation of order* v is one which consists of a single fixed point and precisely k cycles of length $(v-1)/k$. An $S(v)$ $(V, \mathcal{T})$ is called *k-rotational* if it admits a k-rotational automorphism. The existence problem for a k-rotational $S(v)$ was first studied by Phelps and Rosa in [6] and it was completely solved there for $k = 1, 2, 6$. Then the problem was settled for $k = 3, 4$ by Cho [2]. The existence conditions of a reverse $S(v)$, namely an $S(v)$ admitting a k-rotational automorphism with $k = (v-1)/2$, was obtained in [7], [4], [10]. Recently, this problem was investigated by Colbourn and Jiang [3] and recursive constructions were developed to successfully determine the spectrum of a k-rotational $S(v)$ for any positive integer k. The recursion is on k, and its successful application depends on a complete solution for $k = 1, 2, 3, 4, 5, 7$ and 11. Hence the existence for k-rotational systems with $k = 5, 7$ and 11 is of interest. We completely settle these cases here.

Before we proceed to our main results on $k = 5, 7, 11$, we first look at the conditions that a k-rotational $S(v)$, where $1 \leq k \leq (v-1)/2$, has to satisfy. It is proved in [6] that $v \equiv 3 \pmod 6$ if $k = 1$. In general, the requirement on an order of an $S(v)$ and that on an order of a k-rotational $S(v)$ imply, separately, $v \equiv 1, 3 \pmod 6$ and $v \equiv 1 \pmod k$.

Now suppose π is a k-rotational automorphism of an $S(v)$ with $(v-1)/k$ being even. Then $\pi^{\frac{v-1}{2k}}$ is also an automorphism of $S(v)$. It is clear that $\pi^{\frac{v-1}{2k}}$ consists of a single fixed point and $(v-1)/2$ cycles of length 2 and therefore $S(v)$ is reverse. But a reverse $S(v)$ exists if and only if $v \equiv 1, 3, 9, 19 \pmod{24}$ [10]. So $v \not\equiv 7, 13, 15, 21 \pmod{24}$.

In summary we obtain

Lemma 1.1 *Let v, k be positive integers such that $1 \leq k \leq (v-1)/2$. Then a k-rotational $S(v)$ exists only if*

$$\begin{cases} v \equiv 1, 3 \pmod 6, & \\ v \equiv 3 \pmod 6 & \text{if } k = 1, \\ v \equiv 1 \pmod k, & \\ v \not\equiv 7, 13, 15, 21 \pmod{24} & \text{if } (v-1)/k \text{ is even.} \end{cases} \qquad (1)$$

The sufficiency of these conditions was established for $k = 1, 2, 3, 4, 6$ in [6], [2], and for $k = (v-1)/2$ in [7], [4], [10]. In particular, we mention these results for $k = 1, 2, 3$:

A 1-rotational $S(v)$ exists if and only if $v \equiv 3, 9 \pmod{24}$.

A 2-rotational $S(v)$ exists if and only if $v \equiv 1, 3, 7, 9, 15, 19 \pmod{24}$.

A 3-rotational $S(v)$ exists if and only if $v \equiv 1, 19 \pmod{24}$.

In this paper, we prove sufficiency for $k = 5, 7, 11$. To be precise, we establish

Theorem 1.2 *A 5-rotational $S(v)$ exists if and only if $v \equiv 1, 51^*, 81^*, 91 \pmod{120}$. A 7-rotational $S(v)$ exists if and only if $v \equiv 1, 43, 57^*, 99^* \pmod{168}$. An 11-rotational $S(v)$ exists if and only if $v \equiv 1, 67, 177^*, 243^* \pmod{264}$.*

The congruence classes marked with * satisfy $v \equiv 3, 9 \pmod{24}$. Now if a permutation π of order v is 1-rotational then π^k is k-rotational for an integer k with $k|(v-1)$. Moreover, an $S(v)$ with π as an automorphism certainly also has π^k as an automorphism. Therefore the existence of a 5-, 7- or 11-rotational $S(v)$ with v in these congruence classes is a direct consequence of that of a 1-rotational $S(v)$. The remainder of this paper is for establishing the existence of $S(v)$ for v in the remaining congruence classes, following some preliminary results on Skolem sequences in Section 2.

Throughout, $[i, j]$ denotes the set of integers ℓ such that $i \leq \ell \leq j$; Z_n denotes the residue class group of integers modulo n. We take $V = \{\infty\} \bigcup (Z_n \times \{1, 2, \ldots, k\})$ to be the ground set of a k-rotational $S(v)$ to be constructed, where $\infty \notin Z_n \times \{1, 2, \ldots, k\}$, $n = (v-1)/k$ and for brevity we denote by x_i an element $(x, i) \in Z_n \times \{1, 2, \ldots, k\}$. We then take

$$\pi = (\infty)(0_1 1_1 \ldots (n-1)_1) \ldots (0_k 1_k \ldots (n-1)_k)$$

to be the k-rotational automorphism of $S(v)$. The constructions we give are by difference methods. (Difference methods have been used extensively in previous work on k-rotational Steiner triple systems; see [2], [6] for more details.)

2 Skolem Sequences

In this section, we construct some Skolem sequences which are to be used in the constructions of the later sections. (For background on Skolem sequences the reader is referred to [1], [8]). We adopt the notation of extended near-Skolem sequences appearing in [1] and give the following definition.

Definition 2.1 *Let k be a positive integer, $M \subseteq [1, k]$, $N \subseteq [1, 2(k-|M|)+|N|]$. Then an N-extended M-near-Skolem sequence of order k, denoted by $(M, N) - ENS_k$, is a partition of $[1, 2(k-|M|)+|N|] \setminus N$ into a sequence of $k - |M|$ ordered pairs (a_r, b_r) ($r \in [1, k] \setminus M$) such that $b_r - a_r = r$ for each r.*

This notation generalizes many of the previously established Skolem sequences. For example, an (A, k)-system [9] (or a pure Skolem sequence of order k) is an $(\phi, \phi) - ENS_k$, a (B, k)–system [5] (or a hooked Skolem sequence of order k) an $(\phi, \{2k\}) - ENS_k$, an (F, k)-system [6] a $(\{1\}, \{2k\}) - ENS_{2k}$ when k is odd or a $(\{2\}, \{2k\}) - ENS_{2k}$ when k is even, an m-near Skolem sequence of order k [8] an $(\{m\}, \phi) - ENS_k$, and a hooked m-near Skolem sequence of order k [8] an $(\{m\}, \{2k-2\}) - ENS_k$. For clarity we first list the relevant Skolem sequences in Table 1. Then we prove the existence conditions in Theorem 2.2 - Theorem 2.9. The necessity of these conditions can be easily obtained by observing that

$$\sum_{r \in [1,k] \setminus M} a_r = \frac{1}{2}\Big(\sum_{r \in [1, 2(k-|M|)+|N|] \setminus N} r - \sum_{r \in [1,k] \setminus M} r \Big)$$

is an integer.

Theorem 2.2 *A $(\{4\}, \{k\}) - ENS_k$ exists if and only if $k \equiv 0, 3 \pmod 4$, $k \neq 3$.*

Table 1: N-extended M-near Skolem sequences

M, N	$[1,k]\setminus M$, $[1,2(k-\|M\|)+\|N\|]\setminus N$	Existence (mod 4)	Name
ϕ, ϕ	$[1,k]$, $[1,2k]$	$k\equiv 0,1$ [9]	(A,k)- system
ϕ, $\{2k\}$	$[1,k]$, $[1,2k+1]\setminus\{2k\}$	$k\equiv 2,3$ [5]	(B,k)- system
$\begin{cases}\{1\}, & k \text{ odd};\\ \{2\}, & k \text{ even},\end{cases}$ $\{2k\}$	$\begin{cases}[1,2k]\setminus\{1\}, k \text{ odd};\\ [1,2k]\setminus\{2\}, k \text{ even},\end{cases}$ $[1,4k-1]\setminus\{2k\}$	$k\neq 2$ [6]	(F,k)- system
$\{m\}$, ϕ	$[1,k]\setminus\{m\}$, $[1,2k-2]$	$k\equiv 0,1$; m odd; $k\equiv 2,3$; m even [8]	m-near, order k
$\{m\}$, $\{2k-2\}$	$[1,k]\setminus\{m\}$, $[1,2k-1]\setminus\{2k-2\}$	$k\equiv 2,3$; m odd; $k\equiv 0,1$; m even [8]	hooked m-near, order k
$\{4\}$, $\{k\}$	$[1,k]\setminus\{4\}$, $[1,2k-1]\setminus\{k\}$	$k\equiv 0,3$ $k\neq 3$	
$\{5\}$, $\{k\}$	$[1,k]\setminus\{5\}$, $[1,2k-1]\setminus\{k\}$	$k\equiv 1,2$ $k\neq 1,2$	
$\{9\}$, $\{k\}$	$[1,k]\setminus\{9\}$, $[1,2k-1]\setminus\{k\}$	$k\equiv 1,2$ $k\geq 9$	
$\{2,4\}$, ϕ	$[1,k]\setminus\{2,4\}$, $[1,2k-4]$	$k\equiv 0,1$ $k\neq 1$	
$\{3,5\}$, ϕ	$[1,k]\setminus\{3,5\}$, $[1,2k-4]$	$k\equiv 0,1$ $k\neq 1,4$	
$\{2,3\}$, ϕ	$[1,k]\setminus\{2,3\}$, $[1,2k-4]$	$k\equiv 2,3$ $k\neq 2$	
$\{2,5\}$, ϕ	$[1,k]\setminus\{2,5\}$, $[1,2k-4]$	$k\equiv 2,3$ $k\neq 2,3$	
$\{2,3\}$, $\{k\}$	$[1,k]\setminus\{2,3\}$, $[1,2k-3]\setminus\{k\}$	$k\equiv 0,3$	
$\{3,5\}$, $\{k\}$	$[1,k]\setminus\{3,5\}$, $[1,2k-3]\setminus\{k\}$	$k\equiv 1,2$ $k\neq 1,2$	
$\{2,3\}$, $\{k,2k-3\}$	$[1,k]\setminus\{2,3\}$, $[1,2k-2]\setminus\{k,2k-3\}$	$k\equiv 1,2$ $k\neq 1,2$	
$\{3,4,7\}$, $\{k-2, 2k-5\}$	$[1,k]\setminus\{3,4,7\}$, $[1,2k-4]\setminus\{k-2, 2k-5\}$	$k\equiv 1,2$ $k\geq 9$	

Proof. For the sufficiency we take the pairs as follows:
$k = 4$: (1, 2), (5, 7), (3, 6);
$k \equiv 0 \pmod 4$, $k = 4s$ where $s \geq 2$:
$(r, 4s-1-r)$ $\quad r \in [1, s-1]$,
$(s-1+r, 3s-2-r)$ $\quad r \in [1, s-2]$ $(s \geq 3)$,
$(4s+r, 8s-2-r)$ $\quad r \in [1, 2s-4]$ $(s \geq 3)$,
$(3s-2, 3s-1)$, $(6s-1, 6s+1)$, $(6s, 8s-1)$, $(2s-1, 6s-3)$, $(4s-1, 8s-2)$, $(2s-2, 6s-2)$;
$k \equiv 3 \pmod 4$, $k = 4s+3$ where $s \geq 1$:
$(r, 4s+3-r)$ $\quad r \in [1, s-1]$ $(s \geq 2)$,
$(s+1+r, 3s+4-r)$ $\quad r \in [1, s]$,
$(4s+3+r, 8s+5-r)$ $\quad r \in [1, 2s-2]$ $(s \geq 2)$,
$(s, s+1)$, $(6s+3, 6s+5)$, $(6s+2, 8s+5)$, $(2s+2, 6s+4)$, $(2s+3, 6s+6)$.
□

Theorem 2.3 *Let $m = 5, 9$. Then an $(\{m\}, \{k\}) - ENS_k$ exists if and only if $k \equiv 1, 2 \pmod 4$, and $k \neq 1, 2$ when $m = 5$, and $k \neq 1, 2, 5, 6$ when $m = 9$.*

Proof. For the sufficiency we take the following pairs.
Suppose $m = 5$:
$k = 5$: (1, 2), (7, 9), (3, 6), (4, 8);
$k = 6$: (2, 3), (7, 9), (8, 11), (1, 5), (4, 10);
$k = 9$: (5, 6), (13, 15), (14, 17), (3, 7), (10, 16), (1, 8), (4, 12), (2, 11);
$k = 10$: (2, 3), (5, 7), (15, 18), (12, 16), (11, 17), (1, 8), (6, 14), (4, 13), (9, 19).
Suppose $m = 9$:
$k = 9$: (2, 3), (13, 15), (4, 7), (12, 16), (5, 10), (11, 17), (1, 8), (6, 14);
$k = 10$:
$(10+r, 20-r)$ $\quad r \in [1, 4]$,
(3, 4), (6, 9), (2, 7), (1, 8), (5, 15);
$k = 13$:
$(13+r, 25-r)$ $\quad r \in [1, 4]$,
(2, 3), (9, 11), (4, 7), (5, 10), (18, 25), (1, 12), (8, 20), (6, 19);
$k = 14$:
$(14+r, 26-r)$ $\quad r \in [1, 4]$,
(2, 3), (8, 10), (4, 7), (6, 11), (20, 27), (1, 12), (9, 21), (13, 26), (5, 19);
$k = 17$:
$(r, 17-r)$ $\quad r \in [1, 3]$,
$(17+r, 33-r)$ $\quad r \in [1, 6]$,
(12, 13), (4, 6), (8, 11), (5, 10), (26, 33), (9, 25), (7, 24);
$k = 18$:
$(r, 17-r)$ $\quad r \in [1, 3]$,
$(18+r, 34-r)$ $\quad r \in [1, 6]$,

(4, 5), (11, 13), (6, 9), (7, 12), (27, 34), (10, 26), (8, 25), (17, 35).

Suppose $k \equiv 1 \pmod 4$, $k = 4s+1$ where $s \geq 3$ when $m = 5$, or $s \geq 5$ when $m = 9$:

$$
\begin{array}{ll}
(r, 4s+1-r) & r \in [1, s-1], \\
(s-1+r, 3s-r) & \left\{ \begin{array}{ll} r \in [1, s-3]\ (s \geq 4), & \text{when } m = 5; \\ r \in [1, s-5]\ (s \geq 6), & \text{when } m = 9, \end{array} \right. \\
(4s+1+r, 8s+1-r) & r \in [1, 2s-2].
\end{array}
$$

The sequence is completed by adding

$(3s, 3s+1)$, $(2s-3, 2s-1)$, $(2s-2, 2s+1)$, $(6s, 8s+1)$, $(2s+2, 6s+2)$, $(2s, 6s+1)$, when $m = 5$; or, by adding

$(3s, 3s+1)$, $(2s-5, 2s-3)$, $(2s-2, 2s+1)$, $(2s-1, 2s+4)$, $(2s-4, 2s+3)$, $(6s, 8s+1)$, $(2s+2, 6s+2)$, $(2s, 6s+1)$, when $m = 9$.

Suppose $k \equiv 2 \pmod 4$, $k = 4s+2$ where $s \geq 3$ when $m = 5$, or $s \geq 5$ when $m = 9$:

$$
\begin{array}{ll}
(r, 4s+1-r) & r \in [1, s-1], \\
(s+1+r, 3s+2-r) & \left\{ \begin{array}{ll} r \in [1, s-3]\ (s \geq 4), & \text{when } m = 5; \\ r \in [1, s-5]\ (s \geq 6), & \text{when } m = 9, \end{array} \right. \\
(4s+2+r, 8s+2-r) & r \in [1, 2s-2].
\end{array}
$$

The sequence is completed by adding

$(s, s+1)$, $(2s, 2s+2)$, $(2s+1, 2s+4)$, $(6s+2, 8s+3)$, $(2s+3, 6s+3)$, $(4s+1, 8s+2)$, $(2s-1, 6s+1)$, when $m = 5$; or, by adding

$(s, s+1)$, $(2s-2, 2s)$, $(2s+2, 2s+5)$, $(2s+1, 2s+6)$, $(2s-3, 2s+4)$, $(6s+2, 8s+3)$, $(2s+3, 6s+3)$, $(4s+1, 8s+2)$, $(2s-1, 6s+1)$, when $m = 9$.

□

Theorem 2.4 *Let $M = \{2,4\}$ or $\{3,5\}$. Then there is an $(M, \phi) - ENS_k$ if and only if $k \equiv 0, 1 \pmod 4$, and $k \neq 1$ when $M = \{2,4\}$, and $k \neq 1, 4$ when $M = \{3,5\}$.*

Proof. For the sufficiency we take the following pairs.

Suppose $M = \{2,4\}$:

$k = 4$: (2, 3), (1, 4);

$k = 5$: (3, 4), (2, 5), (1, 6).

Suppose $M = \{3,5\}$:

$k = 5$: (2, 3), (4, 6), (1, 5);

$k = 8$: (7, 8), (1, 3), (6, 10), (5, 11), (2, 9), (4, 12);

$k = 9$: (2, 3), (6, 8), (9, 13), (5, 11), (7, 14), (4, 12), (1, 10).

Suppose $k \equiv 0 \pmod 4$, $k = 4s$ with $s \geq 2$ when $M = \{2,4\}$, $s \geq 3$ when $M = \{3,5\}$:

$$(r, 4s-1-r) \quad \begin{cases} r \in [1, s-2]\ (s \geq 3), & \text{when } M = \{2,4\}; \\ r \in [1, s-2], & \text{when } M = \{3,5\}, \end{cases}$$
$$(s-2+r, 3s-1-r) \quad \begin{cases} r \in [1, s-2]\ (s \geq 3), & \text{when } M = \{2,4\}; \\ r \in [1, s-3]\ (s \geq 4), & \text{when } M = \{3,5\}, \end{cases}$$
$$(4s-1+r, 8s-3-r) \quad \begin{cases} r \in [1, 2s-4]\ (s \geq 3), & \text{when } M = \{2,4\}; \\ r \in [1, 2s-4], & \text{when } M = \{3,5\}. \end{cases}$$

The sequence is completed by adding

$(3s-1, 3s)$, $(6s-3, 6s)$, $(2s-2, 4s-1)$, $(2s, 6s-2)$, $(2s-3, 6s-4)$, $(2s-1, 6s-1)$, when $M = \{2,4\}$; or, by adding

$(3s-1, 3s)$, $(6s-2, 6s)$, $(2s-3, 2s+1)$, $(2s-2, 4s-1)$, $(2s-1, 6s-3)$, $(2s, 6s-1)$, $(2s-4, 6s-4)$, when $M = \{3,5\}$.

Suppose $k \equiv 1 \pmod 4$, $k = 4s+1$ with $s \geq 2$ when $M = \{2,4\}$, $s \geq 3$ when $M = \{3,5\}$:

$$(r, 4s-1-r) \quad \begin{cases} r \in [1, s-2]\ (s \geq 3), & \text{when } M = \{2,4\}; \\ r \in [1, s-2], & \text{when } M = \{3,5\}, \end{cases}$$
$$(s-2+r, 3s-1-r) \quad \begin{cases} r \in [1, s-2]\ (s \geq 3), & \text{when } M = \{2,4\}; \\ r \in [1, s-3]\ (s \geq 4), & \text{when } M = \{3,5\}, \end{cases}$$
$$(4s-1+r, 8s-1-r) \quad \begin{cases} r \in [1, 2s-3], & \text{when } M = \{2,4\}; \\ r \in [1, 2s-3], & \text{when } M = \{3,5\}. \end{cases}$$

The sequence is completed by adding

$(3s-1, 3s)$, $(6s-2, 6s+1)$, $(2s-2, 4s-1)$, $(2s, 6s-1)$, $(2s-3, 6s-3)$, $(2s-1, 6s)$, when $M = \{2,4\}$; or, by adding

$(3s-1, 3s)$, $(6s-1, 6s+1)$, $(2s-3, 2s+1)$, $(2s-2, 4s-1)$, $(2s-1, 6s-2)$, $(2s, 6s)$, $(2s-4, 6s-3)$, when $M = \{3,5\}$. □

Theorem 2.5 *Let $M = \{2,3\}$ or $\{2,5\}$. Then there is an $(M, \phi) - ENS_k$ if and only if $k \equiv 2, 3 \pmod 4$, and $k \neq 2$ when $M = \{2,3\}$, and $k \neq 2, 3$ when $M = \{2,5\}$.*

Proof. For the sufficiency we take the following pairs.

Suppose $M = \{2,3\}$:

$k = 3$: (1, 2);

$k = 6$: (4, 5), (2, 6), (3, 8), (1, 7);

$k = 7$: (6, 7), (1, 5), (3, 8), (4, 10), (2, 9).

Suppose $M = \{2,5\}$:

$k = 6$: (5, 6), (1, 4), (3, 7), (2, 8);

$k = 7$: (3, 4), (5, 8), (6, 10), (1, 7), (2, 9).

Suppose $k \equiv 2 \pmod 4$, $k = 4s+2$ with $s \geq 2$ for $M = \{2,3\}$ and $M = \{2,5\}$:

$$(r, 4s+1-r) \quad r \in [1, s-2]\ (s \geq 3),$$
$$(s-2+r, 3s+1-r) \quad \begin{cases} r \in [1, s-1], & \text{when } M = \{2,3\}; \\ r \in [1, s-2]\ (s \geq 3), & \text{when } M = \{2,5\}, \end{cases}$$
$$(4s+1+r, 8s+1-r) \quad \begin{cases} r \in [1, 2s-2], & \text{when } M = \{2,3\}; \\ r \in [1, 2s-3], & \text{when } M = \{2,5\}. \end{cases}$$

The sequence is completed by adding

$(3s+1,3s+2)$, $(2s-2,4s+1)$, $(2s+1,6s+1)$, $(2s-1,6s)$, $(2s,6s+2)$, when $M=\{2,3\}$; or, by adding

$(3s+1,3s+2)$, $(6s-1,6s+2)$, $(2s-3,2s+1)$, $(2s-2,4s+1)$, $(2s,6s)$, $(2s+2,6s+3)$, $(2s-1,6s+1)$, when $M=\{2,5\}$.

Suppose $k\equiv 3 \pmod 4$, $k=4s+3$ with $s\geq 2$ for $M=\{2,3\}$ and $M=\{2,5\}$:

$(r,4s+1-r)$ $\quad r\in[1,s-2]$ $(s\geq 3)$,

$$(s-2+r,3s+1-r)\quad \begin{cases} r\in[1,s-1], & \text{when } M=\{2,3\};\\ r\in[1,s-2]\ (s\geq 3), & \text{when } M=\{2,5\},\end{cases}$$

$$(4s+1+r,8s+3-r)\quad \begin{cases} r\in[1,2s-1], & \text{when } M=\{2,3\};\\ r\in[1,2s-2], & \text{when } M=\{2,5\}.\end{cases}$$

The sequence is completed by adding

$(3s+1,3s+2)$, $(2s-2,4s+1)$, $(2s,6s+1)$, $(2s+1,6s+3)$, $(2s-1,6s+2)$, when $M=\{2,3\}$; or, by adding

$(3s+1,3s+2)$, $(6s,6s+3)$, $(2s-3,2s+1)$, $(2s-2,4s+1)$, $(2s,6s+1)$, $(2s+2,6s+4)$, $(2s-1,6s+2)$, when $M=\{2,5\}$. □

Theorem 2.6 *A* $(\{2,3\},\{k\})-ENS_k$ *exists if and only if* $k\equiv 0,3 \pmod 4$.

Proof. For the sufficiency we take the pairs as follows:

$k=3$: (1, 2);

$k=4$: (2, 3), (1, 5);

$k=7$: (2, 3), (5, 9), (6, 11), (4, 10), (1, 8);

$k=8$: (11, 12), (2, 6), (4, 9), (1, 7), (3, 10), (5, 13);

$k\equiv 0 \pmod 4$, $k=4s$ where $s\geq 3$:

$(r,4s-r)$ $\quad r\in[1,s]$,

$(s+2+r,3s-r)$ $\quad r\in[1,s-3]$ $(s\geq 4)$,

$(4s+1+r,8s-2-r)$ $\quad r\in[1,2s-4]$,

$(s+1,s+2)$, $(4s+1,6s-1)$, $(2s+1,6s-2)$, $(2s+2,6s+1)$, $(2s,6s)$;

$k\equiv 3 \pmod 4$, $k=4s+3$ where $s\geq 2$:

$(r,4s+3-r)$ $\quad r\in[1,2s-1]$,

$(4s+4+r,8s+4-r)$ $\quad r\in[1,s-2]$ $(s\geq 3)$,

$(5s+2+r,7s+4-r)$ $\quad r\in[1,s-1]$,

$(7s+4,7s+5)$, $(2s+2,4s+4)$, $(2s+3,6s+3)$, $(2s,6s+2)$, $(2s+1,6s+4)$.

□

Theorem 2.7 *A* $(\{3,5\},\{k\})-ENS_k$ *exists if and only if* $k\equiv 1,2 \pmod 4$, $k\neq 1,2$.

Proof. For the sufficiency we take the pairs as follows:

$k=5$: (1, 2), (4, 6), (3, 7);

$k=6$: (3, 4), (7, 9), (1, 5), (2, 8);

$k\equiv 1 \pmod 4$, $k=4s+1$ where $s\geq 2$:

$(r, 4s+1-r)$ $r \in [1, s-2]$ $(s \geq 3)$,
$(s+r, 3s+3-r)$ $r \in [1, s-2]$ $(s \geq 3)$,
$(4s+2+r, 8s-r)$ $r \in [1, 2s-3]$,
$(s-1, s)$, $(2s+2, 2s+4)$, $(2s-1, 4s+2)$, $(2s+3, 6s+1)$, $(2s, 6s)$, $(2s+1, 6s+2)$;
$k \equiv 2 \pmod 4$, $k = 4s+2$ where $s \geq 2$:
$(r, 4s+1-r)$ $r \in [1, s-2]$ $(s \geq 3)$,
$(s-2+r, 3s+1-r)$ $r \in [1, s-2]$ $(s \geq 3)$,
$(4s+2+r, 8s+2-r)$ $r \in [1, 2s-2]$,
$(3s+1, 3s+2)$, $(2s-3, 2s-1)$, $(2s-2, 4s+1)$, $(2s+1, 6s+1)$, $(2s+2, 6s+3)$, $(2s, 6s+2)$. □

Theorem 2.8 *A* $(\{2,3\}, \{k, 2k-3\}) - ENS_k$ *exists if and only if* $k \equiv 1, 2 \pmod 4$, $k \neq 1, 2$.

Proof. For the sufficiency we take the pairs as follows:
$k = 5$: (2, 3), (4, 8), (1, 6);
$k = 6$: (2, 3), (4, 8), (5, 10), (1, 7);
$k \equiv 1 \pmod 4$, $k = 4s+1$ where $s \geq 2$:
$(r, 4s-r)$ $r \in [1, s-1]$,
$(s+1+r, 3s+1-r)$ $r \in [1, s-2]$ $(s \geq 3)$,
$(4s+1+r, 8s-2-r)$ $r \in [1, 2s-4]$ $(s \geq 3)$,
$(s, s+1)$, $(6s-2, 8s-2)$, $(2s+2, 6s-1)$, $(2s+1, 6s)$, $(4s, 8s)$, $(2s, 6s+1)$;
$k \equiv 2 \pmod 4$, $k = 4s+2$ where $s \geq 2$:
$(r, 4s+1-r)$ $r \in [1, s-2]$ $(s \geq 3)$,
$(s+r, 3s+3-r)$ $r \in [1, s-1]$,
$(4s+3+r, 8s+1-r)$ $r \in [1, 2s-3]$,
$(s-1, s)$, $(2s, 4s+3)$, $(2s+3, 6s+1)$, $(2s+2, 6s+2)$, $(4s+1, 8s+2)$, $(2s+1, 6s+3)$. □

Theorem 2.9 *A* $(\{3,4,7\}, \{k-2, 2k-5\}) - ENS_k$ *exists if and only if* $k \equiv 1, 2 \pmod 4$, $k \neq 1, 2, 5, 6$.

Proof. For the sufficiency we take the pairs as follows:
$k = 9$: (9, 10), (1, 3), (6, 11), (2, 8), (4, 12), (5, 14);
$k = 10$: (4, 5), (11, 13), (9, 14), (1, 7), (2, 10), (3, 12), (6, 16);
$k \equiv 1 \pmod 4$, $k = 4s+1$ where $s \geq 3$:
$(r, 4s-2-r)$ $r \in [1, s-2]$,
$(s-2+r, 3s-2-r)$ $r \in [1, s-3]$ $(s \geq 4)$,
$(4s+r, 8s-3-r)$ $r \in [1, 2s-6]$ $(s \geq 4)$,
$(3s-2, 3s-1)$, $(6s-1, 6s+1)$, $(6s-3, 6s+2)$, $(2s-2, 4s-2)$, $(2s-1, 6s-4)$, $(4s, 8s-2)$, $(2s-4, 6s-5)$, $(2s, 6s)$, $(2s-3, 6s-2)$;
$k \equiv 2 \pmod 4$, $k = 4s+2$ where $s \geq 3$:

$(r, 4s-r) \quad r \in [1, s-2],$
$(s+r, 3s+2-r) \quad r \in [1, s-2],$
$(4s+2+r, 8s-1-r) \quad r \in [1, 2s-6] \ (s \geq 4),$

$(s-1, s)$, $(6s-1, 6s+1)$, $(6s-3, 6s+2)$, $(2s, 4s+2)$, $(2s+1, 6s-2)$, $(4s+1, 8s)$, $(2s+3, 6s+3)$, $(2s-1, 6s)$, $(2s+2, 6s+4)$. □

3 5-rotational $S(v)$ for $v \equiv 1, 91 \pmod{120}$

When difference methods are employed, constructing a k-rotational $S(v)$ on $V = \{\infty\} \bigcup (Z_n \times \{1, 2, \ldots, k\})$ with

$$\pi = (\infty)(0_1 1_1 \ldots (n-1)_1) \ldots (0_k 1_k \ldots (n-1)_k)$$

as an automorphism requires a partition of all the pure differences of types x_{ii} $(1 \leq i \leq k)$ and mixed differences of types x_{ij} $(1 \leq i \neq j \leq k)$ into base triples. When k is small the number of the types of these differences is small. Therefore the partition may be rather easily obtained by exhaustively looking at all the differences.

Lemma 3.1 *There is a 5-rotational $S(v)$ whenever $v \equiv 1, 91 \pmod{120}$.*

Proof. Case 1. $v \equiv 1 \pmod{120}$, $v = 120t + 1$: $n = 24t$, $t \geq 1$.
$V = \{\infty\} \bigcup (Z_{24t} \times \{1, 2, 3, 4, 5\})$.
$\pi = (\infty)(0_1 1_1 \ldots (24t-1)_1) \ldots (0_5 1_5 \ldots (24t-1)_5)$.
Skolem sequences:
(a_r, b_r) $(r \in [1, 4t] \setminus \{2, 3\})$, a $(\{2, 3\}, \{4t\}) - ENS_{4t}$,
(c_r, d_r) $(r \in [1, 12t-1])$, a $(B, 12t-1)$-system.
Construction: This construction, as well as that in Case 2, is based on the partition of the edge set of a complete graph K_5 on $\{1, 2, 3, 4, 5\}$ into two triangles $(1, 2, 4)$, $(1, 3, 5)$ and a 4-circuit $(2, 3, 4, 5)$. The base triples in this case are listed as follows:

1. $\{\infty, 0_i, (12t)_i\} \quad i \in [1, 5]$;
2. $\{0_1, (8t)_1, (16t)_1\}$, $\{0_1, r_1, (b_r + 4t)_1\} \quad r \in [1, 4t] \setminus \{2, 3\}$;
3. $\{0_2, r_2, (d_r + 2)_3\}$, $\{0_3, r_3, (d_r + 3)_4\}$, $\{0_4, r_4, (d_r + 1)_5\}$, $\{0_5, r_5, (d_r - 1)_2\}$,
   where, for all, $r \in [1, 12t-1]$;
4. $\{0_4, r_1, (2r-1)_2\} \quad r \in [1, 12t] \setminus \{12t-2\}$,
   $\{0_4, r_1, (2r)_2\} \quad r \in [12t+1, 24t-1]$,
   $\{0_5, r_1, (2r-1)_3\} \quad r \in [1, 12t] \setminus \{12t-1\}$,
   $\{0_5, r_1, (2r)_3\} \quad r \in [12t+1, 24t-1]$;

5. $\{0_1, 3_1, (12t)_2\}$, $\{0_1, 2_1, (12t)_3\}$, $\{0_1, (12t-2)_1, 0_4\}$,
$\{0_1, (12t-1)_1, 0_5\}$, $\{0_2, 2_3, 5_4\}$, $\{0_3, 1_4, 0_5\}$,
$\{0_4, 1_5, 0_2\}$, $\{0_5, (-3)_2, (-3)_3\}$.

Case 2. $v \equiv 91 \pmod{120}$, $v = 120t + 91$: $n = 24t + 18$, $t \geq 0$.
$V = \{\infty\} \bigcup (Z_{24t+18} \times \{1, 2, 3, 4, 5\})$.
$\pi = (\infty)(0_1 1_1 \ldots (24t+17)_1) \ldots (0_5 1_5 \ldots (24t+17)_5)$.
Skolem sequences:
(a_r, b_r) $(r \in [1, 4t+3] \setminus \{2, 3\})$, a $(\{2,3\}, \{4t+3\}) - ENS_{4t+3}$,
(c_r, d_r) $(r \in [1, 12t+8])$, an $(A, 12t+8)$-system.
Construction: The base triples are listed as follows:

1. $\{\infty, 0_i, (12t+9)_i\}$ $i \in [1, 5]$;

2. $\{0_1, (8t+6)_1, (16t+12)_1\}$,
$\{0_1, r_1, (b_r + 4t + 3)_1\}$ $r \in [1, 4t+3] \setminus \{2, 3\}$;

3. $\{0_2, r_2, (d_r + 3)_3\}$, $\{0_3, r_3, (d_r + 2)_4\}$, $\{0_4, r_4, (d_r + 2)_5\}$,
$\{0_5, r_5, (d_r - 1)_2\}$,
where, for all, $r \in [1, 12t+8]$;

4. $\{0_4, r_1, (2r-1)_2\}$ $r \in [1, 12t+9] \setminus \{12t+7\}$,
$\{0_4, r_1, (2r)_2\}$ $r \in [12t+10, 24t+17]$,
$\{0_5, r_1, (2r-1)_3\}$ $r \in [1, 12t+9] \setminus \{12t+8\}$,
$\{0_5, r_1, (2r)_3\}$ $r \in [12t+10, 24t+17]$;

5. $\{0_1, 3_1, (12t+9)_2\}$, $\{0_1, 2_1, (12t+9)_3\}$, $\{0_1, (12t+7)_1, 0_4\}$,
$\{0_1, (12t+8)_1, 0_5\}$, $\{0_2, 3_3, 5_4\}$, $\{0_3, 1_4, 3_5\}$,
$\{0_4, 1_5, 0_2\}$, $\{0_5, (-2)_2, 0_3\}$.

□

4 7-rotational $S(v)$ for $v \equiv 1, 43 \pmod{168}$

Applying difference methods, we can also establish

Lemma 4.1 *There is a 7-rotational $S(v)$ whenever $v \equiv 1, 43 \pmod{168}$.*

Proof. Case 1. $v \equiv 1 \pmod{168}$, $v = 168t + 1$: $n = 24t$, $t \geq 1$.
$V = \{\infty\} \bigcup (Z_{24t} \times \{1, 2, \ldots, 7\})$.
$\pi = (\infty)(0_1 1_1 \ldots (24t-1)_1) \ldots (0_7 1_7 \ldots (24t-1)_7)$.
Skolem sequences:
(a_r, b_r) $(r \in [1, 4t] \setminus \{4\})$, a $(\{4\}, \{4t\}) - ENS_{4t}$,
(c_r, d_r) $(r \in [1, 4t] \setminus \{2\})$, a $(\{2\}, \{8t-2\}) - ENS_{4t}$ (a hooked 2-near Skolem sequence of order $4t$),

(e_r, f_r) ($r \in [1,4t] \setminus \{4\}$), a $(\{4\},\{8t-2\}) - ENS_{4t}$ (a hooked 4-near Skolem sequence of order $4t$),

(g_r, h_r) ($r \in [1,4t+1] \setminus \{2,4\}$), a $(\{2,4\},\phi\}) - ENS_{4t+1}$,

(p_r, q_r) ($r \in [1,4t] \setminus \{2\}$), a $(\{2\},\{4t\}) - ENS_{4t}$ (an $(F,2t)$-system) when $t \geq 2$.

Construction: This construction, as well as those in Case 2, is based on the partition of the edge set of a complete graph K_7 on $\{1,2,\ldots,7\}$ into the 7 triangles $(1,2,4)$, $(2,3,5)$, $(3,4,6)$, $(4,5,7)$, $(5,6,1)$, $(6,7,2)$, $(7,1,3)$. The base triples in this case are listed as follows:

1. $\{\infty, 0_i, (12t)_i\}$ $\quad i \in [1,7]$;

2. $\{0_1, (8t)_1, (16t)_1\}$, $\{0_6, (8t)_6, (16t)_6\}$,

$$\begin{array}{ll}
\{0_1, r_1, (b_r+4t)_1\} & r \in [1,4t] \setminus \{4\}, \\
\{0_2, r_2, (d_r+4t)_2\} & r \in [1,4t] \setminus \{2\}, \\
\{0_3, r_3, (f_r+4t)_3\} & r \in [1,4t] \setminus \{4\}, \\
\{0_4, r_4, (h_r+4t+1)_4\} & r \in [1,4t+1] \setminus \{2,4\}, \\
\{0_5, r_5, (h_r+4t+1)_5\} & r \in [1,4t+1] \setminus \{2,4\}, \\
\{0_7, r_7, (d_r+4t)_7\} & r \in [1,4t] \setminus \{2\},
\end{array}$$

and, when $t = 1$,

$\{0_6, 1_6, 7_6\}$, $\{0_6, 3_6, 13_6\}$, $\{0_6, 4_6, 9_6\}$,

or, when $t \geq 2$,

$\{0_6, r_6, (q_r+4t)_6\}$ $\quad r \in [1,4t] \setminus \{2\}$;

3.
$$\begin{array}{ll}
\{0_1, r_2, (2r+1)_4\} & r \in [1,12t-1] \setminus \{2\}, \\
\{0_1, r_2, (2r)_4\} & r \in [12t+1,24t], \\
\{0_2, r_3, (2r+1)_5\} & r \in [1,12t-1] \setminus \{2\}, \\
\{0_2, r_3, (2r)_5\} & r \in [12t+1,24t], \\
\{0_3, r_4, (2r+1)_6\} & r \in [1,12t-1] \setminus \{2\}, \\
\{0_3, r_4, (2r)_6\} & r \in [12t+1,24t], \\
\{0_4, r_5, (2r+1)_7\} & r \in [0,12t-1], \\
\{0_4, r_5, (2r)_7\} & r \in [12t+1,24t-1], \\
\{0_5, r_6, (2r-2)_1\} & r \in [4,12t+2] \setminus \{5\}, \\
\{0_5, r_6, (2r-3)_1\} & r \in [12t+4,24t] \bigcup [1,3], \\
\{0_7, r_2, (2r+1)_6\} & r \in [1,12t-1] \setminus \{2\}, \\
\{0_7, r_2, (2r)_6\} & r \in [12t+1,24t], \\
\{0_1, r_7, (2r)_3\} & r \in [1,12t-1] \setminus \{2\}, \\
\{0_1, r_7, (2r-1)_3\} & r \in [12t+1,24t];
\end{array}$$

$\{0_5, 4_1, 8_1\}$,
$\{0_2, 2_2, 3_6\}$, $\{0_7, 2_2, (12t)_2\}$,
$\{0_3, (12t-2)_3, (12t)_4\}$, $\{0_3, 4_3, 5_6\}$,

4. $\{0_1, 1_4, 5_4\}$, $\{0_2, 1_4, 3_4\}$,
$\{0_2, 1_5, 5_5\}$, $\{0_3, 1_5, 3_5\}$,
$\{0_6, 2_6, 3_1\}$,
$\{0_1, 2_7, (12t)_7\}$, $\{0_7, 2_7, 2_3\}$;

5. $\{0_1, (12t)_2, 0_3\}$, $\{0_1, 2_2, 4_3\}$;

6. $\{0_4, (12t)_5, 3_6\}$, $\{0_5, 0_7, 5_6\}$, $\{0_4, 0_7, 1_6\}$.

Case 2. $v \equiv 43 \pmod{168}$, $v = 168t + 43$: $n = 24t + 6$, $t \geq 0$.
$V = \{\infty\} \bigcup (Z_{24t+6} \times \{1, 2, \ldots, 7\})$.
$\pi = (\infty)(0_1 1_1 \ldots (24t+5)_1) \ldots (0_7 1_7 \ldots (24t+5)_7)$.
Skolem sequences:
(a_r, b_r) $(r \in [1, 4t+1] \setminus \{9\})$ $(t \geq 2)$, a $(\{9\}, \{4t+1\}) - ENS_{4t+1}$,
(c_r, d_r) $(r \in [1, 4t+1] \setminus \{3\})$ $(t \geq 1)$, a $(\{3\}, \phi) - ENS_{4t+1}$ (a 3-near Skolem sequence of order $4t+1$),
(e_r, f_r) $(r \in [1, 4t+2] \setminus \{2, 3\})$ $(t \geq 1)$, a $(\{2, 3\}, \phi) - ENS_{4t+2}$,
(g_r, h_r) $(r \in [1, 4t+1] \setminus \{5\})$ $(t \geq 1)$, a $(\{5\}, \{4t+1\}) - ENS_{4t+1}$,
Constructions: First suppose $t \geq 1$. The base triples of a required $S(168t + 43)$ are listed as follows:

1. $\{\infty, 0_i, (12t+3)_i\}$ $i \in [1, 7]$;

2. $\{0_1, (8t+2)_1, (16t+4)_1\}$, $\{0_6, (8t+2)_6, (16t+4)_6\}$,
$\{0_2, r_2, (d_r + 4t + 1)_2\}$ $r \in [1, 4t+1] \setminus \{3\}$,
$\{0_3, r_3, (d_r + 4t + 1)_3\}$ $r \in [1, 4t+1] \setminus \{3\}$,
$\{0_4, r_4, (f_r + 4t + 2)_4\}$ $r \in [1, 4t+2] \setminus \{2, 3\}$,
$\{0_5, r_5, (f_r + 4t + 2)_5\}$ $r \in [1, 4t+2] \setminus \{2, 3\}$,
$\{0_6, r_6, (h_r + 4t + 1)_6\}$ $r \in [1, 4t+1] \setminus \{5\}$,
$\{0_7, r_7, (d_r + 4t + 1)_7\}$ $r \in [1, 4t+1] \setminus \{3\}$,
and, when $t = 1$,
$\{0_1, 1_1, 7_1\}$, $\{0_1, 2_1, 13_1\}$, $\{0_1, 3_1, 8_1\}$, $\{0_1, 4_1, 16_1\}$,
or, when $t \geq 2$,
$\{0_1, r_1, (b_r + 4t + 1)_1\}$ $r \in [1, 4t+1] \setminus \{9\}$;

3. $\{0_1, r_2, (2r+1)_4\}$ $r \in [0, 12t+2] \setminus \{1\}$,
$\{0_1, r_2, (2r)_4\}$ $r \in [12t+4, 24t+5]$,
$\{0_2, r_3, (2r+1)_5\}$ $r \in [0, 12t+2] \setminus \{1\}$,
$\{0_2, r_3, (2r)_5\}$ $r \in [12t+4, 24t+5]$,
$\{0_3, r_4, (2r+1)_6\}$ $r \in [0, 12t+2] \setminus \{1\}$,
$\{0_3, r_4, (2r)_6\}$ $r \in [12t+4, 24t+5]$,
$\{0_4, r_5, (2r+1)_7\}$ $r \in [2, 12t+4]$,
$\{0_4, r_5, (2r)_7\}$ $r \in [12t+6, 24t+6] \bigcup \{1\}$,
$\{0_5, r_6, (2r+1)_1\}$ $r \in [24t+4, 24t+6] \bigcup [1, 12t] \setminus \{2\}$,
$\{0_5, r_6, (2r)_1\}$ $r \in [12t+2, 24t+3]$,
$\{0_2, r_6, (2r+1)_7\}$ $r \in [0, 12t+2] \setminus \{1\}$,
$\{0_2, r_6, (2r)_7\}$ $r \in [12t+4, 24t+5]$,
$\{0_7, r_1, (2r+1)_3\}$ $r \in [0, 12t+2] \setminus \{1\}$,
$\{0_7, r_1, (2r)_3\}$ $r \in [12t+4, 24t+5]$;

4. $\{0_5, (24t+2)_1, 5_1\}$,
$\{0_2, (12t+2)_2, (12t+3)_6\}$, $\{0_2, 3_2, 3_7\}$,
$\{0_3, (12t+2)_3, (12t+3)_4\}$, $\{0_3, 3_3, 3_6\}$,
$\{0_1, 0_4, 3_4\}$, $\{0_2, 0_4, 2_4\}$,
$\{0_2, 0_5, 3_5\}$, $\{0_3, 0_5, 2_5\}$,
$\{0_6, 5_6, 3_1\}$,
$\{0_7, (12t+2)_7, (12t+3)_1\}$, $\{0_7, 3_7, 3_3\}$;

5. $\{0_1, (12t+3)_2, 0_3\}$, $\{0_1, 1_2, 2_3\}$;

6. $\{0_4, (12t+5)_5, 0_6\}$, $\{0_5, 2_6, 2_7\}$, $\{0_4, 2_6, 4_7\}$.

For $t = 0$, a 7-rotational STS(43) is given by the following base triples:

1. $\{\infty, 0_i, 3_i\}$ $i \in [1, 7]$;

2. $\{0_1, 2_1, 4_1\}$, $\{0_6, 2_6, 4_6\}$;

3. $\{0_1, r_2, (2r+1)_4\}$ $r = 0, 1, 2$, $\{0_1, 4_2, 2_4\}$,
$\{0_2, r_3, (2r+1)_5\}$ $r = 0, 1, 2$, $\{0_2, 4_3, 2_5\}$,
$\{0_4, r_3, (2r+1)_6\}$ $r = 0, 1, 2$, $\{0_4, 5_3, 4_6\}$,
$\{0_4, r_5, (2r)_7\}$ $r = 0, 1, 2$, $\{0_4, r_5, (2r-1)_7\}$ $r = 4, 5$,
$\{0_5, r_6, (2r+1)_1\}$ $r = 1, 2$, $\{0_5, r_6, (2r)_1\}$ $r = 4, 5$,
$\{0_6, r_7, (2r+1)_2\}$ $r = 0, 1, 2$, $\{0_6, 4_7, 2_2\}$,
$\{0_7, r_1, (2r+1)_3\}$ $r = 0, 1, 2$, $\{0_7, 5_1, 4_3\}$;

$\{0_5, 0_1, 1_1\}$,
$\{0_6, 0_2, 4_2\}$, $\{0_7, 0_2, 5_2\}$,
$\{0_4, 3_3, 4_3\}$, $\{0_3, 2_3, 0_6\}$,
4. $\{0_1, 4_4, 0_4\}$, $\{0_2, 5_4, 0_4\}$,
$\{0_2, 4_5, 0_5\}$, $\{0_3, 5_5, 0_5\}$,
$\{0_6, 1_6, 1_1\}$,
$\{0_7, 1_7, 4_1\}$, $\{0_7, 2_7, 2_3\}$;

5. $\{0_1, 3_2, 0_3\}$, $\{0_1, 5_2, 4_3\}$;

6. $\{0_4, 3_5, 0_6\}$, $\{0_5, 0_6, 5_7\}$, $\{0_4, 2_6, 5_7\}$.

□

5 11-rotational $S(v)$ for $v \equiv 1, 67 \pmod{264}$

As the number k of cycles in a rotational automorphism increases, the structure of rotational Steiner triple systems becomes more and more complicated. This causes real difficulty for constructing such a system using difference methods. There is almost no way to handle the large number of the possible pure and mixed differences when k becomes large. However, as the entire structure becomes complicated, the dependency of a part of the structure on another is also weakened. This makes it possible to deal with the structure part by part. For example, such a system may be more easily obtained if some subsystems can be embedded. This is essentially the idea for the recursive constructions appearing in [3]. Although these recursive constructions are still not applicable to the case $k = 11$, a modification turns out to be successful, as we see next.

Lemma 5.1 *Let $n = 6s$ with $s \geq 1$. Then there is a set $\mathcal{A}$ of (base) triples over $Z_n \times \{1,2,3,4\}$ which covers each of the mixed differences exactly once.*

Proof. Take $\mathcal{A} = \mathcal{A}_1 \bigcup \mathcal{A}_2 \bigcup \mathcal{A}_3 \bigcup \mathcal{A}_4$, where $\mathcal{A}_i$ are as follows:

$$\begin{array}{lll} \mathcal{A}_1: & \{0_1, r_2, (2r)_3\} & r \in [0, 3s-1]; \\ \mathcal{A}_2: & \{0_2, r_3, (2r+1)_4\} & r \in [3s, 6s-1]; \\ \mathcal{A}_3: & \{0_3, r_4, (2r-1)_1\} & r \in [1, 3s]; \\ \mathcal{A}_4: & \{0_4, r_1, (2r)_2\} & r \in [3s, 6s-1]. \end{array}$$

□

Lemma 5.2 *Let $n = 6s$ with $s \geq 1$. Then there is a set $\mathcal{B}$ of (base) triples over $Z_n \times \{1,2,3,4\}$ such that $\mathcal{B}$ covers each of the mixed differences exactly once, except for*

$$\pm 2_{14}, \pm(6s-1)_{14}, \pm 4_{24}, \pm(6s-1)_{24}, \pm 0_{34}, \pm 2_{34}$$

which are not covered by $\mathcal{B}$.

Proof. Let $\mathcal{A} = \mathcal{A}_1 \bigcup \mathcal{A}_2 \bigcup \mathcal{A}_3 \bigcup \mathcal{A}_4$ be the set of the triples constructed in Lemma 5.1. Destroy the triples $\{0_2, (6s-1)_3, (6s-1)_4\}$ in $\mathcal{A}_2$, $\{0_3, 2_4, 3_1\}$ in $\mathcal{A}_3$, $\{0_4, (6s-2)_1, (6s-4)_2\}$ in $\mathcal{A}_4$. Reconstruct the triple $\{0_1, (6s-2)_2, (6s-3)_3\}$ and adjoin it to $\mathcal{A}_1$. To be precise, take $\mathcal{B} = \mathcal{B}_1 \bigcup \mathcal{B}_2 \bigcup \mathcal{B}_3 \bigcup \mathcal{B}_4$, where

$$\begin{aligned}
\mathcal{B}_1 &= \mathcal{A}_1 \bigcup \{\{0_1, (6s-2)_2, (6s-3)_3\}\};\\
\mathcal{B}_2 &= \mathcal{A}_2 \setminus \{\{0_2, (6s-1)_3, (6s-1)_4\}\};\\
\mathcal{B}_3 &= \mathcal{A}_3 \setminus \{\{0_3, 2_4, 3_1\}\};\\
\mathcal{B}_4 &= \mathcal{A}_4 \setminus \{\{0_4, (6s-2)_1, (6s-4)_2\}\}.
\end{aligned}$$

□

Lemma 5.3 *There is a set $\mathcal{C}$ of (base) triples over $Z_6 \times \{1,2,3,4\}$ such that $\mathcal{C}$ covers each of the mixed differences exactly once, except for*

$$\pm 3_{14}, \pm 4_{14}, \pm 1_{24}, \pm 4_{24}, \pm 2_{34}, \pm 4_{34}$$

which are not covered by $\mathcal{C}$.

Proof. Take $\mathcal{C} = \mathcal{C}_1 \bigcup \mathcal{C}_2 \bigcup \mathcal{C}_3 \bigcup \mathcal{C}_4$, where $\mathcal{C}_i$ are as follows:

$$\begin{array}{lllll}
\mathcal{C}_1: & \{0_2, 3_1, 0_3\}, & \{0_2, 4_1, 2_3\}, & \{0_2, 5_1, 4_3\}, & \{0_2, 1_1, 3_3\};\\
\mathcal{C}_2: & \{0_3, 3_4, 1_2\}, & \{0_3, 5_4, 5_2\}; & & \\
\mathcal{C}_3: & \{0_1, 0_3, 0_4\}, & \{0_1, 1_3, 2_4\}; & & \\
\mathcal{C}_4: & \{0_4, 1_2, 1_1\}, & \{0_4, 3_2, 5_1\}. & &
\end{array}$$

□

Lemma 5.4 *There is an 11-rotational $S(v)$ whenever $v \equiv 1, 67 \pmod{264}$.*

Proof. Case 1. $v \equiv 1 \pmod{264}$, $v = 264t + 1$: $n = 24t$, $t \geq 1$.
$V = \{\infty\} \bigcup (Z_{24t} \times \{1, 2, \ldots, 11\})$.
$\pi = (\infty)(0_1 1_1 \ldots (24t-1)_1) \ldots (0_{11} 1_{11} \ldots (24t-1)_{11})$.
Skolem sequences:

(a_r, b_r) $(r \in [1, 4t+1] \setminus \{3, 5\})$, a $(\{3,5\}, \phi) - ENS_{4t+1}$,

(c_r, d_r) $(r \in [1, 4t] \setminus \{3\})$, a $(\{3\}, \phi) - ENS_{4t}$ (a 3-near Skolem sequence of order $4t$),

(e_r, f_r) $(r \in [1, 4t] \setminus \{2, 3\})$, a $(\{2,3\}, \{4t\}) - ENS_{4t}$,

(g_r, h_r) $(r \in [1, 4t] \setminus \{2\})$ $(t \geq 2)$, a $(\{2\}, \{4t\}) - ENS_{4t}$ (an $(F, 2t)$-system).

Construction: Our construction employs the blocks of a $\{3,4\}-GDD$ of type $3^3 2^1$, obtained by deleting one point from a $TD(4,3)$. Suppose $(W, \mathcal{G}, \mathcal{D})$ is such a GDD on $W = \{1, 2, \ldots, 11\}$ with

$$\mathcal{G} = \{\{1,2,3\},\{4,5,6\},\{7,8,9\},\{10,11\}\}.$$

Without loss of generality, assume that $\{1,4,7\}, \{2,5,8\}, \{3,6,9\}$ are the three blocks of size 3 in $\mathcal{D}$. The base triples of an 11-rotational $S(v)$ in this case are listed as follows:

1. base triples of a 3-rotational $S(3 \times 24t + 1)$ on $V_1 = \{\infty\} \bigcup (Z_{24t} \times \{4,5,6\})$, with $\pi|_{V_1}$ as the automorphism,

   base triples of a 2-rotational $S(2 \times 24t + 1)$ on $V_2 = \{\infty\} \bigcup (Z_{24t} \times \{10,11\})$, with $\pi|_{V_2}$ as the automorphism;

2. a copy of $\mathcal{A} = \mathcal{A}_1 \bigcup \mathcal{A}_2 \bigcup \mathcal{A}_3 \bigcup \mathcal{A}_4$ as constructed in Lemma 5.1, taking $s = 4t$, on $Z_{24t} \times D$, for each block $D \in \mathcal{D}$ of size 4;

3. $\{\infty, 0_i, (12t)_i\} \quad i \in [1,3] \bigcup [7,9]$;

4. $\{0_3, (8t)_3, (16t)_3\}, \quad \{0_7, (8t)_7, (16t)_7\}, \quad \{0_8, (8t)_8, (16t)_8\}$,

   $\{0_1, r_1, (b_r + 4t + 1)_1\} \quad r \in [1, 4t+1] \setminus \{3,5\}$,
   $\{0_2, r_2, (d_r + 4t)_2\} \quad r \in [1, 4t] \setminus \{3\}$,
   $\{0_3, r_3, (f_r + 4t)_3\} \quad r \in [1, 4t] \setminus \{2,3\}$,
   $\{0_7, r_7, (f_r + 4t)_7\} \quad r \in [1, 4t] \setminus \{2,3\}$,
   $\{0_9, r_9, (b_r + 4t + 1)_9\} \quad r \in [1, 4t+1] \setminus \{3,5\}$,

   and, when $t = 1$,

   $\{0_8, 1_8, 7_8\}, \quad \{0_8, 3_8, 13_8\}, \quad \{0_8, 4_8, 9_8\}$,

   or, when $t \geq 2$,

   $\{0_8, r_8, (h_r + 4t)_8\} \quad r \in [1, 4t] \setminus \{2\}$;

5. $\{0_1, r_3, (2r+1)_2\} \quad r \in [0, 12t-1] \setminus \{1\}$,
   $\{0_1, r_3, (2r)_2\} \quad r \in [12t+1, 24t-1]$,
   $\{0_9, r_7, (2r+1)_8\} \quad r \in [0, 12t-1] \setminus \{1\}$,
   $\{0_9, r_7, (2r)_8\} \quad r \in [12t+1, 24t-1]$,
   $\{0_4, r_7, (2r+1)_1\} \quad r \in [0, 12t-1] \setminus \{2\}$,
   $\{0_4, r_7, (2r)_1\} \quad r \in [12t+1, 24t-1]$,
   $\{0_2, r_5, (2r+1)_8\} \quad r \in [0, 12t-1] \setminus \{1\}$,
   $\{0_2, r_5, (2r)_8\} \quad r \in [12t+1, 24t-1]$,
   $\{0_6, r_3, (2r+1)_9\} \quad r \in [0, 12t-1] \setminus \{2\}$,
   $\{0_6, r_3, (2r)_9\} \quad r \in [12t+1, 24t-1]$;

	$\{0_1, 3_1, 3_2\}$,	$\{0_4, 0_1, 5_1\}$,	$\{0_2, (12t-1)_2, (12t)_5\}$,
	$\{0_2, 3_2, 3_8\}$,	$\{0_1, 1_3, (12t)_3\}$,	$\{0_3, 2_3, 2_2\}$,
6.	$\{0_6, 2_3, (12t)_3\}$,	$\{0_3, 3_3, 3_9\}$,	$\{0_7, 3_7, 3_1\}$,
	$\{0_4, 2_7, (12t)_7\}$,	$\{0_7, 2_7, 2_8\}$,	$\{0_9, 1_7, (12t)_7\}$,
	$\{0_5, 0_8, 2_8\}$,	$\{0_6, 0_9, 5_9\}$,	$\{0_9, 3_9, 3_8\}$.

Case 2. $v \equiv 67 \pmod{264}$, $v = 264t + 67$: $n = 24t + 6$, $t \geq 0$.
$V = \{\infty\} \bigcup (Z_{24t+6} \times \{1, 2, \ldots, 11\})$.
$\pi = (\infty)(0_1 1_1 \ldots (24t+5)_1) \ldots (0_{11} 1_{11} \ldots (24t+5)_{11})$.
Skolem sequences:

(a_r, b_r) ($r \in [1, 4t+1] \setminus \{2, 3\}$) ($t \geq 1$), a $(\{2,3\}, \{4t+1, 8t-1\}) - ENS_{4t+1}$,

(c_r, d_r) ($r \in [1, 4t+1] \setminus \{5\}$) ($t \geq 1$), a $(\{5\}, \{4t+1\}) - ENS_{4t+1}$,

(e_r, f_r) ($r \in [1, 4t+2] \setminus \{3, 4, 7\}$) ($t \geq 2$), a $(\{3,4,7\}, \{4t, 8t-1\}) - ENS_{4t+2}$,

(g_r, h_r) ($r \in [1, 4t+2] \setminus \{2, 3\}$) ($t \geq 1$), a $(\{2,3\}, \phi) - ENS_{4t+2}$,

(p_r, q_r) ($r \in [1, 4t+1] \setminus \{3, 5\}$) ($t \geq 1$), a $(\{3,5\}, \{4t+1\}) - ENS_{4t+1}$,

(u_r, w_r) ($r \in [1, 4t+2] \setminus \{2, 5\}$) ($t \geq 1$), a $(\{2,5\}, \phi) - ENS_{4t+2}$,

(x_r, y_r) ($r \in [1, 12t+2]$) ($t \geq 0$), an $(\phi, \{24t+4\}) - ENS_{12t+2}$ (a $(B, 12t+2)$-system).

Constructions: The constructions are still based on the structure of a $\{3,4\} - GDD$ of type $3^3 2^1$ as in the above case. Since there is no 2-rotational subsystem which can be embedded in the system in this case, the constructions are a bit more complicated. Suppose $(W, \mathcal{G}, \mathcal{D})$ is a GDD of this type as in Case 1 on $W = \{1, 2, \ldots, 11\}$ with $\mathcal{G}$ the same as there, having $\{1,4,7\}, \{2,5,8\}, \{3,6,9\}$ to be the three blocks of size 3 in $\mathcal{D}$. Furthermore, we assume that $D_0 = \{1, 6, 8, 10\}$ is one of the blocks in $\mathcal{D}$ of size 4. We first construct the following base triples, for each $t \geq 0$:

1. base triples of a 3-rotational $S(3 \times (24t+6)+1)$ on $V_1 = \{\infty\} \bigcup (Z_{24t+6} \times \{4, 5, 6\})$, with $\pi|_{V_1}$ as the automorphism;
2. a copy of $\mathcal{A} = \mathcal{A}_1 \bigcup \mathcal{A}_2 \bigcup \mathcal{A}_3 \bigcup \mathcal{A}_4$ as constructed in Lemma 5.1, taking $s = 4t + 1$, on $Z_{24t+6} \times D$, for each block $D \neq D_0$, $D \in \mathcal{D}$ of size 4;
3. $\{\infty, 0_i, (12t+3)_i\}$ $\quad i \in [1,3] \bigcup [7, 11]$;
4. $\{0_{11}, r_{11}, (y_r)_{10}\}$ $\quad r \in [1, 12t+2]$.

   Now suppose $t \geq 1$ first. The set of the base triples of an 11-rotational $S(v)$ in this case consists of the above triples in 1 through 4 and those in the following 5 through 8:

5. $\{0_i, (8t+2)_i, (16t+4)_i\}$ $\quad i \in \{1,2,3,8\}$,

$$\begin{array}{ll}
\{0_1, r_1, (b_r+4t+1)_1\} & r \in [1, 4t+1] \setminus \{2,3\}, \\
\{0_2, r_2, (d_r+4t+1)_2\} & r \in [1, 4t+1] \setminus \{5\}, \\
\{0_7, r_7, (h_r+4t+2)_7\} & r \in [1, 4t+2] \setminus \{2,3\}, \\
\{0_8, r_8, (q_r+4t+1)_8\} & r \in [1, 4t+1] \setminus \{3,5\}, \\
\{0_9, r_9, (w_r+4t+2)_9\} & r \in [1, 4t+2] \setminus \{2,5\}, \\
\{0_{10}, r_{10}, (h_r+4t+2)_{10}\} & r \in [1, 4t+2] \setminus \{2,3\},
\end{array}$$

and, when $t = 1$,

$$\{0_3, 1_3, 12_3\}, \quad \{0_3, 2_3, 8_3\}, \quad \{0_3, 5_3, 14_3\},$$

or, when $t \geq 2$,

$$\{0_3, r_3, (f_r+4t+2)_3\} \quad r \in [1, 4t+2] \setminus \{3,4,7\};$$

6. a copy of $\mathcal{B} = \mathcal{B}_1 \bigcup \mathcal{B}_2 \bigcup \mathcal{B}_3 \bigcup \mathcal{B}_4$ as constructed in Lemma 5.2, taking $s = 4t+1$, on $Z_{24t+6} \times D_0$ such that $\mathcal{B}$ misses exactly the differences

$$\pm 2_{10,8}, \pm(24t+5)_{10,8}, \pm 4_{68}, \pm(24t+5)_{68}, \pm 0_{18}, \pm 2_{18}$$

by identifying the subscripts of 10, 6, 1, 8 here with those of 1, 2, 3, 4 in Lemma 5.2, separately;

7.
$$\begin{array}{ll}
\{0_2, r_1, (2r+1)_3\} & r \in [0, 12t+2] \setminus \{3\}, \\
\{0_2, r_1, (2r)_3\} & r \in [12t+4, 24t+5], \\
\{0_8, r_9, (2r+1)_7\} & r \in [0, 12t+2] \setminus \{1\}, \\
\{0_8, r_9, (2r)_7\} & r \in [12t+4, 24t+5], \\
\{0_1, r_7, (2r+1)_4\} & r \in [0, 12t+2] \setminus \{1\}, \\
\{0_1, r_7, (2r)_4\} & r \in [12t+4, 24t+5], \\
\{0_2, r_8, (2r+1)_5\} & r \in [0, 12t+2] \setminus \{2\}, \\
\{0_2, r_8, (2r)_5\} & r \in [12t+4, 24t+5], \\
\{0_9, r_3, (2r+1)_6\} & r \in [0, 12t+2] \setminus \{2\}, \\
\{0_9, r_3, (2r)_6\} & r \in [12t+4, 24t+5];
\end{array}$$

8.
$$\begin{array}{ll}
\{0_2, 3_1, (12t+3)_1\}, & \{0_1, 3_1, 3_4\}, \\
\{0_1, (12t+2)_1, (12t+3)_7\}, & \{0_1, 2_1, 2_8\}, \\
\{0_2, 5_2, 5_5\}, & \\
\{0_1, 0_3, 4_3\}, & \{0_2, 0_3, 7_3\}, \\
\{0_3, 3_3, 3_6\}, & \{0_9, 2_3, (12t+3)_3\}, \\
\{0_7, 2_7, 2_4\}, & \{0_8, 0_7, 3_7\}, \\
\{0_2, 2_8, (12t+3)_8\}, & \{0_8, 3_8, 3_5\}, \\
\{0_6, (24t+5)_8, 4_8\}, & \{0_8, (12t+2)_8, (12t+3)_9\}, \\
\{0_9, 5_9, 5_6\}, & \{0_9, 2_9, 2_7\}, \\
\{0_{10}, 3_{10}, 2_8\}, & \{0_{11}, (24t+4)_{10}, 0_{10}\}.
\end{array}$$

Secondly suppose $t = 0$. The set of the base triples of an 11-rotational $S(67)$ is given by the triples in the above 1 through 4 and the following 9 through 13:

9. $\{0_7, 2_7, 4_7\}$;

10. a copy of the triples $\mathcal{C} = \mathcal{C}_1 \bigcup \mathcal{C}_2 \bigcup \mathcal{C}_3 \bigcup \mathcal{C}_4$ as constructed in Lemma 5.3 on $Z_6 \times D_0$ such that $\mathcal{C}$ misses exactly the differences

$$\pm 3_{10,8}, \pm 4_{10,8}, \pm 1_{68}, \pm 4_{68}, \pm 2_{18}, \pm 4_{18}$$

by identifying the subscripts of 10, 6, 1, 8 here with those of 1, 2, 3, 4 in Lemma 5.3, separately;

11.
$\{0_1, 0_2, 0_3\}$, $\{0_1, 1_2, 2_3\}$,
$\{0_1, 2_2, 1_3\}$, $\{0_1, 5_2, 3_3\}$,
$\{0_7, 0_8, 0_9\}$, $\{0_7, 1_8, 2_9\}$,
$\{0_7, 2_8, 4_9\}$, $\{0_7, 4_8, 1_9\}$,
$\{0_4, 0_7, 0_1\}$, $\{0_4, 1_7, 2_1\}$,
$\{0_4, 2_7, 4_1\}$, $\{0_4, 3_7, 1_1\}$,
$\{0_5, 0_8, 0_2\}$, $\{0_5, 1_8, 2_2\}$,
$\{0_5, 2_8, 4_2\}$, $\{0_5, 3_8, 1_2\}$,
$\{0_6, 0_9, 0_3\}$, $\{0_6, 1_9, 2_3\}$,
$\{0_6, 2_9, 4_3\}$, $\{0_6, 4_9, 1_3\}$;

12.
$\{0_1, 1_1, 5_3\}$, $\{0_4, 3_1, 5_1\}$,
$\{0_1, 3_2, 4_2\}$, $\{0_5, 3_2, 5_2\}$,
$\{0_2, 2_3, 3_3\}$, $\{0_6, 3_3, 5_3\}$,
$\{0_4, 4_7, 5_7\}$,
$\{0_8, 2_8, 5_2\}$, $\{0_5, 4_8, 5_8\}$,
$\{0_9, 1_9, 5_3\}$, $\{0_7, 3_9, 5_9\}$,
$\{0_{10}, 1_{10}, 4_8\}$, $\{0_{11}, 4_{10}, 0_{10}\}$;

13.
$\{0_7, 3_1, 5_8\}$, $\{0_7, 5_1, 3_8\}$,
$\{0_6, 1_8, 5_9\}$, $\{0_6, 4_8, 3_9\}$.

□

6 Concluding Remarks

The results obtained in Sections 3, 4 and 5 established the existence of a 5-, 7- and 11-rotational $S(v)$ for the nontrivial cases and then proved the validity of Theorem 1.2. The constructions given here are technically detailed, and do not appear to lead to a general technique for settling existence of k-rotational Steiner triple systems with arbitrary k. However, while the specific cases of $k = 5, 7$ and 11 are quite involved, their solution turns out to provide the necessary "building blocks" for some recursive constructions [3] which are successful in determining existence in general.

Acknowledgments

I am appreciated of the helpful discussions and valuable suggestions from my supervisor, Dr. Charles J. Colbourn, which have contributed very much to the success of this work. This paper forms part of the author's Ph.D. thesis at the University of Waterloo.

References

[1] C. A. Baker, Extended Skolem sequences, *J. Combin. Designs* **Vol. 3, No. 5** (1995), 363-379.

[2] C. J. Cho, Rotational Steiner triple systems, *Discrete Math.* **42** (1982), 153-159.

[3] C. J. Colbourn and Z. Jiang, The spectrum for rotational Steiner triple systems, submitted for publication.

[4] J. Doyen, A note on reverse Steiner triple systems, *Discrete Math.* **1** (1971-72), 315-319.

[5] E. S. O'Keefe, Verification of a conjecture of Th. Skolem, *Math. Scand.* **9** (1961), 80-82.

[6] K. T. Phelps and A. Rosa, Steiner triple systems with rotational automorphisms, *Discrete Math.* **33** (1981), 57-66.

[7] A. Rosa, On reverse Steiner triple systems, *Discrete Math.* **2** (1972), 61-71.

[8] N. Shalaby, The existence of near-Skolem sequences and hooked near-Skolem sequences, *Discrete Math.* **135** (1994), 303-319.

[9] Th. Skolem, On certain distributions of integers in pairs with given differences, *Math. Scand.* **5** (1957), 57-68.

[10] L. Teirlinck, The existence of reverse Steiner triple systems, *Discrete Math.* **6** (1973), 301-302.