On the Number of 6 × 7 Double Youden Rectangles

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ABSTRACT. Results concerning the enumeration and classification of 7 × 7 Latin squares are used to enumerate and classify all non-isomorphic Youden squares of order 6 × 7. We show that the number of non-isomorphic Youden squares obtainable from a species of Latin square δ, depends on the number of distinct adjugate sets and the order of the automorphism group of δ. Further, we use the results obtained for 6 × 7 Youden squares as a basis for the enumeration and classification of 6 × 7 DYRs.

1 Introduction

A $k \times n$ *Latin rectangle* l $(k < n)$, is a $k \times n$ rectangular array in which each of the symbols $l_1, l_2, \ldots, l_n$ occurs exactly once in each row and at most once in each column. It follows that each row of l is a permutation of order n. There is a large literature devoted to the enumeration of Latin rectangles. The impetus behind much of this work is due to the importance of Latin rectangles for enumerating Latin squares [3]. However, there are enumerations of Latin rectangles that are motivated by their usefulness as experimental designs.

Latin rectangles of order $k \times n$ that have each pair of symbols occurring together within columns of l a constant (λ) number of times are valuable as experimental designs and are "balanced" in the sense of a BIBD. Preece [9] enumerated and classified such balanced Latin rectangles for various values of k and n. In this paper we use the known enumerations of 7 × 7 Latin squares to enumerate and classify $(n-1) \times n$ Latin rectangles possessing "balance". Further, we use our results concerning such Latin rectangles to help enumerate and classify a form of experimental design that has a second set of $n-1$ symbols superimposed on a balanced Latin rectangle of order 6 × 7.

2 Definitions

Many of the following definitions were first given in [8, 9, 10] for Latin squares, Youden squares and double Youden rectangles respectively.

2.1 Latin squares

A Latin square δ is a Latin rectangle with $k = n$ and δ is said to be *reduced* if in the first row and first column, its symbols occur in natural order. Norton [8] defined an *intercalate* to be a 2×2 Latin square embedded in a larger square. The rows columns and symbols of δ are said to be the *constraints* of the square. If R, C, L represent the rows, columns and symbols of δ, then all permutations of R, C and L each having a serial order generate a group Θ, where $\Theta \equiv S_3$.

Let $\Theta = \{\theta_{\text{RCL}} = I, \theta_{\text{RLC}}, \theta_{\text{CRL}}, \theta_{\text{CLR}}, \theta_{\text{LRC}}\theta_{\text{LCR}}\}$; then $\forall\, \theta \in \Theta$, $\delta\theta = \delta^*$, where δ^* is a Latin square of the same order as δ and δ^* is said to be an *adjugate* of δ. We can obtain six adjugates of δ including the *trivial-adjugate* δ_I^*. Given an adjugate δ_x^* we can obtain an *adjugate set* Δ_x^* containing δ_x^* by permuting the rows $r_1, r_2, \ldots, r_n \in R$, permuting the columns, $c_1, c_2, \ldots, c_n \in C$, and permuting the symbols, $l_1, l_2, \ldots, l_n \in L$, or any combination of such permutations. Thus, with a single Latin square we can obtain six adjugate sets — $\{\Delta_I^*, \Delta_{\text{RLC}}^*, \Delta_{\text{CRL}}^*, \Delta_{\text{CLR}}^*, \Delta_{\text{LRC}}^*, \Delta_{\text{LCR}}^*\}$. Let δ_{RLC}^* be the adjugate when θ_{RLC} is applied to $\delta, \ldots, \delta_{\text{LCR}}^*$ be the adjugate of δ when θ_{LCR} is applied to δ; then $A^* = \{\delta_I^*, \delta_{\text{RLC}}^*, \delta_{\text{CRL}}^*, \delta_{\text{CLR}}^*, \delta_{\text{LRC}}^*, \delta_{\text{LCR}}^*\}$ is said to be an *adjugacy set* of δ where $\delta_I^* \in \Delta_I^*$, $\delta_{\text{RLC}}^* \in \Delta_{\text{RLC}}^*, \ldots, \delta_{\text{LCR}}^* \in \Delta_{\text{LCR}}^*$. Suppose that δ_x^* and δ_y^* are two Latin squares and let Δ_x^* and Δ_y^* be the adjugate sets of δ_x^* and δ_y^* respectively; then Δ_x^* and Δ_y^* will be said to be *distinct* if and only if $\nexists\, \theta \in \Theta$: $\delta_x^*\theta = \delta_y^*$.

2.2 Youden squares

Deleting a row from a Latin square leaves a Latin rectangle in which (a) every symbol occurs exactly once in each row and (b) the columns form a symmetric balanced incomplete design (SBIBD). Latin rectangles satisfying conditions (a) and (b) are known as Youden squares. A Youden square y of order $(n-1) \times n$ will be said to *originate* from a Latin square δ of order n if and only if a row may be added to y to form δ. A Youden square y is said to be in *standard form* if symbol j is absent from column j and if column 1 has symbol $i+1$ in row i $(i = 1, 2, \ldots, n-1)$. A Youden square has the same constraints as δ, but y has $|R| = |C| - 1 = |L| - 1$, whereas δ has $|R| = |C| = |L|$. Let Ψ represent the operation of interchanging the constraints C and L of y; then $y\Psi = y^*$ and y^* is said to be the *dual* of y. If $y\Psi = y$, then y is said to be *self-dual*.

2.3 Double Youden rectangles

A double Youden rectangle (DYR) of size $k \times n$ was defined by Bailey [1] to be an arrangement of kv ordered pairs x, y in k rows and n columns $(k < n)$ such that

(i) each value x is drawn from a set S of n elements;

(ii) each value y is drawn from a set T of k elements;

(iii) each element from S occurs exactly once in each row and no more than once per column;

(iv) each element from T occurs exactly once in each column and either b or $b+1$ times in each row, where b is the integral part of v/k;

(v) each element from S is paired exactly once with each element from T;

(vi) each pair of elements from S occurs together in exactly λ columns, where $\lambda = k\ (k-1)/(n-1)$, i.e. the sets of elements of S in the columns are the blocks of a symmetric balanced incomplete block design (SBIBD or a symmetric 2-design) with parameter (n, k, λ);

(vii) if b occurrences of each element from T are removed from each row, leaving $a = n - bk$ elements from T in each row, then (a) the remaining sets of elements of T in the rows are the blocks of a SBIBD with parameters $\{k, a, \mu\}$ where $\mu = a\ (a-1)/(k-1)$, or else (b) $a = 1$. DYRs discussed in this paper are of size $(n-1) \times n$. Thus $b = 1$ in (iv) and the SBIBD in (vi) is trivial, with each block having every symbol except one, and in (vii) $a = 1$. Removing the elements of T from a DYR leaves a Youden square. To emphasize the relationship between a DYR ω and y we will say that a DYR of order $(n-1) \times n$ is *built* on a Youden square of order $(n-1) \times n$. If the sets of rows and columns of a DYR are denoted by P and Q respectively, then P, Q, S and T are the four constraints of ω.

Let ν represent the operation of interchanging the roles of the constraints S and Q, let ϕ represent the operation of interchanging the roles of constraints P and T and let $(\nu\phi)$, $(= (\phi\nu))$ represent the operation of ν followed by that of ϕ. Then the adjugates ω^*, ${}^*\omega$ and ${}^*\omega^*$ of ω can be obtained by operating on ω, where $\nu(\omega) = \omega^*$, $\phi(\omega) = {}^*\omega$, and $\nu\phi(\omega) = {}^*\omega^*$.

2.4 Automorphisms, isomorphisms, transformation sets and species of Latin squares, Youden squares and DYRs

Let Δ be a set of Latin squares, Y be a set of Youden squares and Ω be a set of DYRs where Δ, Y and Ω are finite and where $\delta_1, \delta_2 \in \Delta$, $y_1, y_2 \in Y$, and

$\omega_1, \omega_2 \in \Omega$. Let $\Pi_\alpha = (\pi_r, \pi_c, \pi_l)$ be a triple where π_r is a permutation of the rows of δ, π_c is a permutation of the columns of δ, and π_l is a permutation of the symbols of δ. Similarly let $\Pi_\beta = (\pi_r, \pi_c, \pi_l)$ be a permutation of the rows, columns and symbols respectively of y and $\Pi_\gamma = (\pi_p, \pi_q, \pi_s, \pi_t)$ be a permutation of the elements of P, Q, S and T respectively. If there exists $\Pi_\alpha\colon \delta_1 \to \delta_2$, then δ_1 and δ_2 are said to be *isomorphic* and Π_α is said to be an *isomorphism* from δ_1 to δ_2. Similarly, y_1 and y_2 are said to be *isomorphic* if $\exists\, \Pi_\beta\colon y_1 \to y_2$ and ω_1 and ω_2 are said to be *isomorphic* if $\exists\, \Pi_\gamma\colon \omega_1 \to \omega_2$. The set Δ is said to be a *transformation set* if and only if $\forall\, \delta_1, \delta_2 \in \Delta\, \exists\, \Pi_\alpha\colon \delta_1 \to \delta_2$; similarly Y is said to be a transformation set if and only if $\forall\, y_1, y_2 \in Y\, \exists\, \Pi_\beta\colon y_1 \to y_2$, and Ω is said to be a transformation set if and only if $\forall\, \omega_1, \omega_2 \in \Omega\, \exists\, \Pi_\gamma\colon \omega_1 \to \omega_2$. Thus two adjugates, whether Latin squares, Youden squares or DYRs may or may not be isomorphic to one another. The set Δ is said to be a *species* if and only if $\forall\, \delta_1, \delta_2 \in \Delta$ $\exists\, \Pi_\alpha\colon \delta_1 \to \delta_2$ and $\forall\, \delta_i \in \Delta$, $\theta \in \Theta, \delta\theta = \delta^*$, where $\delta^* \in \Delta$. The set Y is said to be a *species* $\forall\, y_1, y_2 \in Y$ if $\exists\, \Pi_\beta\colon y_1 \to y_2$ and $\forall\, y_i \in Y$, $y\Psi = y^*$, where $y^* \in Y$. The set Ω is said to be a species if and only if $\forall\, \omega_1, \omega_2 \in \Omega\, \exists\, \Pi_\gamma\colon \omega_1 \to \omega_2$ and $\forall\, \omega_i \in \Omega$, $\omega_i^*, \omega_i^*, \omega_i^* \in \Omega$. Hence a species of Latin squares can contain 1, 2, 3 or 6 transformation sets, a species of Youden squares can contain 1 or 2 transformation sets and a species of DYRs can contain 1, 2 or 4 transformation sets. For Latin squares the terms *isotropy class* and *main class* are often used instead of transformation set and species respectively [3, 5]. However, as this paper concerns Youden squares and DYRs we would be wise to adhere to the terms already used for the enumeration and classification of Youden squares [9] and DYRs [2, 10].

If there exists $\Pi_\alpha\colon \delta_i \to \delta_i$ then Π_α is said to be an *automorphism* of δ_i. Similarly, Π_β is an *automorphism* of y_i if there exists $\Pi_\beta\colon y_i \to y_i$ and Π_γ is an *automorphism* of ω if $\exists\, \Pi_\gamma\colon \omega_i \to \omega_i$. An automorphism Π_α or Π_β will be said to be *trivial* if and only if $\pi_r = \pi_c = \pi_l = I$ and Π_γ will be said to be *trivial* if $\pi_p = \pi_q = \pi_s = \pi_t = I$. The collection of all automorphisms of design form a group under composition called the *full automorphism* group. We will denote the full automorphism group of δ, y and ω by $G(\delta)$, $H(y)$ and $J(\omega)$ respectively. A full automorphism group is said to be trivial if it has order 1.

3 The 7×7 Latin squares

In a reduced Latin square, any permutation of all the symbols $l_1, l_2, \ldots, l_n$ other than l_1 may be made, and the rows and columns (excluding the first) then rearranged to give another reduced Latin square. Such a transformation is said to be an *intramutation*. The basic approach used by Norton [8] was to make a preliminary classification of Latin squares of order 7 according to the nature of their leading diagonal, the 'type' of diagonal being invariant under intramutation. As two Latin squares related by intramutation are different if their diagonals differ, attention was confined to those intramutations that do not affect the diagonal, for these alone can leave the square unaltered. Using the process of intramutation, he classified species of 7×7 Latin squares according to the number of intercalates, the number of transformation sets and the number of distinct adjugate sets contained in each. To obtain the number of distinct adjugate sets in each species, he used the notion of an intercalate to show that a species could contain 1, 2, 3 or 6 distinct adjugate sets. For example, suppose δ has constraints columns and symbols equivalent then $C \equiv L \Rightarrow \Delta^*_{\mathrm{LRC}} \equiv \Delta^*_{\mathrm{LCR}}$, $\Delta^*_{\mathrm{CLR}} \equiv \Delta^*_{\mathrm{CRL}}$ and $\Delta^*_{\mathrm{RCL}} \equiv \Delta^*_{\mathrm{RLC}}$ and the species containing δ contains 3 distinct adjugate sets. For a detailed discussion of how to test for equivalence between constraints see [8]. It is sufficient for the purpose of enumerating all non-isomorphic 6×7 Youden squares that we utilize the information given in [8].

Norton obtained 146 species of 7×7 Latin squares and a total of 16,927,968 reduced squares. However, Sade [11] found a species that was overlooked by Norton, and concluded that with this additional square included the list of 146 species given by Norton is complete and that there are 16,942,080 standard 7×7 Latin squares. Therefore there are 147 species of 7×7 Latin squares and these 147 species form the starting point in our enumeration and classification of all non-isomorphic Youden squares of order 6×7.

4 Theoretical basis for the enumeration and classification of 6×7 Youden squares

Our approach to enumerate the non-isomorphic 6×7 Youden squares consists of two main steps:

(i) Obtain the number of non-isomorphic 6×7 Youden squares from each of the 147 squares.

(ii) Sum over all 147 species to obtain the total number of non-isomorphic Youden squares. To carry out step (i), we use the number of distinct adjugate sets and tic automorphism group of each square.

Let $\delta_{i,j}$ represent the 7×7 Latin square of species i containing j distinct adjugate sets as given in [8]. By omitting any one of the 7 rows from $\delta_{i,j}$ we can obtain 7 Youden squares of order 6×7. However, if $\delta_{i,j}$ has more than 1 distinct adjugate sat we can obtain further 6×7 Youden squares. Let $Y_{i,j}$ represent the set of 6×7 Youden squares obtainable from $\delta_{i,j}$. Then: (i) $j = 1 \Rightarrow |Y_{i,j}| = 7$, (ii) $j = 2 \Rightarrow |Y_{i,j}| = 14$, (iii) $j = 3 \Rightarrow |Y_{i,j}| = 21$, (iv) $j = 6 \Rightarrow |Y_{i,j}| = 42$. How many of the Youden squares $y_1, y_2 \in Y_{i,j}$ are non-isomorphic?

4.1 Using $G(\delta_{i,j})$ to count non-isomorphic 6×7 Youden squares

We can use the automorphisms $\Pi_{\alpha 1}, \Pi_{\alpha 1}, \ldots, \in G(\delta_{i,j})$ to identify for equivalence(s) within each of the rows $r_1, r_2, \ldots, r_7 \in R$, columns $c_1, c_2, \ldots, c_7 \in C$ and symbols $a, b, \ldots, g \in L$ of $\delta_{i,j}$. In this way we can recognize equivalences within R, C, and L and consequently identify isomorphisms within the set of 6×7 Youden squares $Y_{i,j}$. In particular, if $Y^*_{i,j}$ represents the set of non-isomorphic 6×7 Youden squares obtainable from $\delta_{i,j}$ we investigate the order of $Y^*_{i,j}$ according to whether $G(\delta_{i,j})$ is trivial or non-trivial.

4.1.2 Non-isomorphic 6×7 Youden squares obtainable from 7×7 Latin squares with $|G(\delta_{i,j})| > 1$

Consider $\delta_{15,3}$. As $|G(\delta_{15,3})| = 2$, there is only one non-trivial automorphism $\Pi_\alpha \in G(\delta_{15,3})$ where $\Pi_\alpha = (\pi_r = (15)(23)(47),\ \pi_c = (17)(25)(46),\ \pi_l = (ab)(cf)(eg))$ and consequently Π_α fixes 1 row, 1 column and 1 letter. Furthermore, row 1 is equivalent to row 5, ($r_1 \equiv r_5$) in the sense that the 6×7 Youden square obtained by omitting row 1 is isomorphic to the Youden square that is obtained by omitting row 5, and likewise for rows 2 and 3 and rows 4 and 7. For columns we have: $c_1 \equiv c_7$, $c_2 \equiv c_5$ and $c_4 \equiv c_6$ and for symbols: $a \equiv b$, $c \equiv f$ and $e \equiv g$. Consequently 3 rows, 3 columns and 3 symbols are equivalent and we can obtain 12 non-isomorphic 6×7 Youden squares by omitting any one of the 3 rows: 1, 2, 4 or 6 from each of δ^*_{RCL}, δ^*_{LRC}, δ^*_{LCR}. Thus $\delta_{15,3}$ gives 12 non-isomorphic 6×7 Youden squares, that is $|Y^*_{15,3}| = 12$.

Remark. Clearly there is a connection between the cycle structure of $\Pi_\alpha \in G(\delta_{i,j})$ and the number of non-isomorphic 6×7 Youden squares obtainable from $\delta_{i,j}$. The number of non-isomorphic Youden squares will depend on the order of π_r, π_c and π_l. For example, a permutation of rows that consists of a single cycle of length m has order m and implies that $m - 1$ rows are equivalent. On the other hand permutation consisting of z cycles each of length m implies that $z\ (m-1)$ rows are equivalent. Moreover a Latin square with a large automorphism group will have more equivalences within each of its constraints and consequently give rise to fewer non-isomorphic Youden square. Indeed we have

$$|G(\delta_{i,j})| > 1 \Rightarrow |Y_{i,j}| > |Y^*_{i,j}| \quad (1)$$

$$\text{As } |G(\delta_{i,j})| \text{ Increases, } |Y^*_{i,j}| \text{ Decreases} \quad (2)$$

4.1.3 Non-isomorphic 6 × 7 Youden squares obtainable from 7 × 7 Latin squares with $|G(\delta_{i,j})| = 1$

If $G(\delta_{i,j})$ is trivial then there are no equivalences within the set of rows, columns and symbols and we have

$$|G(\delta_{i,j})| = 1 \Rightarrow |Y_{i,j}| = |Y^*_{i,j}| \tag{3}$$

and $|G(\delta_{i,j})| = 1 \Rightarrow |Y^*_{i,1}| = 7$, $|Y^*_{i,2}| = 14$, $|Y^*_{i,3}| = 21$ and $|Y^*_{i,6}| = 42$. The set $|Y^*|$ of all non-isomorphic Youden squares of order 6 × 7 is given by (4), where $j = \{1, 2, 3, 6\}$.

$$|Y^*| = \sum_{i=1}^{147} |Y^*_{i,j}| \tag{4}$$

Remark. We have shown that the number of distinct transformation sets obtainable from a Latin square $\delta_{i,j}$ depends on the number of distinct adjugate sets within $\delta_{i,j}$ and $|G(\delta_{i,j})|$. If $G(\delta_{i,j})$ is trivial, then the number of non-isomorphic 6 × 7 Youden squares contained in $Y^*_{i,j}$ is unaffected. However, if $|G(\delta_{i,j})|$ is non-trivial, the number of non-isomorphic Youden squares will depend on the order $G(\delta_{i,j})$. What can we say about the number of species of 6 × 7 Youden squares?

5 Enumeration and classification of 6 × 7 Youden squares into species

Let S_i represent the set of species of 6 × 7 Youden squares obtained from species i of 7 × 7 Latin square. Then we have (5), and the set of all species of 6 × 7 Youden squares S is given by (6)

$$|Y_{i,j}| \geq |S_i| \tag{5}$$

$$|S| = \sum_{i=1}^{147} |S_i| \tag{6}$$

To classify the non-isomorphic Youden squares into species, we consider the number of species of 6×7 Youden square obtainable from a single species of 7 × 7 Latin square $\delta_{i,j}$. As in sections (4.1) and (4.2) we investigate the number of species of Youden squares obtainable from $\delta_{i,j}$ when $|G(\delta_{i,j})|$ is trivial and non-trivial respectively.

5.1 The number of species of 6 × 7 Youden squares obtainable from species of 7 × 7 Latin squares with $|G(\delta_{i,j})| = 1$

We have four possibilities to consider:

(i) If $\delta_{i,j}$ contains 1 adjugate set, then each of the 7 Youden squares represents a single species.

(ii) If $\delta_{i,j}$ contains 2 distinct adjugate sets, see (7):

$$\Delta^*_{\mathrm{RCL}} \equiv \Delta^*_{\mathrm{CLR}} \equiv \Delta^*_{\mathrm{LRC}} \not\equiv \Delta^*_{\mathrm{RLC}} \equiv \Delta^*_{\mathrm{LCR}} \equiv \Delta^*_{\mathrm{CRL}} \qquad (7)$$

Then interchanging the roles of constraints C and L of a Youden square y belonging to an adjugate set from the LHS of (7) will produce the dual of $y - y'$ that belongs to an adjugate set from the RHS of (7) and consequently any Latin square containing 2 adjugate sets and having the trivial automorphism group will give only 7 species of 6×7 Youden squares.

(iii) If $\delta_{i,j}$ contains 3 distinct adjugate sets, then interchanging the roles of constraints C and L of a Youden square from each adjugate set produces a self-dual Youden square, and a pair of Youden squares that are duals of one another. We therefore have 7 species produced from the self-dual Youden square and 7 species from the pair of Youden squares that are duals of one another. Consequently, a Latin square belonging to a species that contains 3 distinct adjugate sets and having non-trivial automorphism group produces 14 species of 6×7 Youden squares.

(iv) If $\delta_{i,j}$ contains 6 distinct adjugate sets, then interchanging the roles of constraints C and L of a Youden square from each adjugate set produces 3 pairs of duals giving 21 species of 6×7 Youden square. See Table 1:

Number of distinct adjugate sets in δ	Number of species of 6×7 Youden squares obtainable from δ
1	7
2	7
3	14
6	21

Table 1

5.2 The number of species of 6×7 Youden squares obtainable from species of 7×7 Latin squares with $|G(\delta_{i,j})| > 1$

Norton's species 15 of 7×7 Latin square has constraints columns and symbols equivalent [8]; consequently its 3 distinct adjugate sets are either Δ^*_{RCL}, Δ^*_{LRC}, Δ^*_{LCR}, or Δ^*_{RLC}, Δ^*_{CRL}, Δ^*_{CLR} and as $|G(\delta_{i,j})| = 2$, $\delta_{15,3}$

produces 12 non-isomorphic 6×7 Youden squares. See section 4.1.1 and Table 2.

Number of adjugate sets distinct sets	$\lvert Y^*_{15,3}\rvert$	$\lvert S_{15}\rvert$
$\Delta^*_{\mathrm{RCL}} \equiv \Delta^*_{\mathrm{RLC}}$	4	4
$\Delta^*_{\mathrm{LRC}} \equiv \Delta^*_{\mathrm{CRL}}$	4	4
$\Delta^*_{\mathrm{LCR}} \equiv \Delta^*_{\mathrm{CLR}}$	4	/

Table 2

As constraints C and L are equivalent ($C \equiv L$), then interchanging the constraints C and L of a Youden square y obtained from Δ^*_{RCL} produces a Youden square y' and the 4 non-isomorphic Youden squares obtained from Δ^*_{RCL} are self-dual; each set represents a species. Further, interchanging the columns and symbols of a Youden square obtained from Δ^*_{LRC} produces a Youden square obtainable from adjugate set Δ^*_{LCR} and Youden squares from Δ^*_{LRC} and Δ^*_{LCR} are duals of one another, consequently, the 12 transformation sets of 6×7 Youden squares obtained from $Y_{15,3}$ fall into 8 species, that is, $|S_{15}| = 8$.

5.3 The total number of transformation sets and species of 6×7 Youden squares

Using the approach described in section 5 we enumerated the transformation sets and species of 6×7 Youden square. Tables 3 shows the total number of transformation sets and species according to the number of Youden squares obtained from Latin squares with trivial automorphism groups. Table 4 (i) and (ii) show the number of transformation sets and species according to the number of adjugate sets and order of automorphism group of the Latin squares from which each Youden square was obtained.

Number of adjugate sets	Frequency	$\lvert Y^*_{i,j}\rvert$	$\lvert S_i\rvert$
1	14	$14 \times 7 = 98$	$14 \times 7 = 98$
2	4	$4 \times 14 = 45$	$4 \times 7 = 28$
3	43	$43 \times 21 = 903$	$43 \times 14 = 602$
6	44	$44 \times 42 = 1848$	$44 \times 21 = 924$
Total	105	2905	1652

Table 3

Species (i)	$\|G(\delta_i)\|$	$\|Y^*_{i,1}\|$	$\|S_i\|$
1	294	1	1
13	24	2	2
70	24	2	2
129	12	2	2
135	4	4	4
136	4	4	4
138	3	3	3
145	4	4	4
146	168	1	1
	Total	25	25

Species (i)	$\|G(\delta_i)\|$	$\|Y^*_{i,3}\|$	$\|S_i\|$
2	6	6	4
7	2	12	8
14	3	9	6
15	2	12	8
50	2	12	8
69	4	12	8
71	2	12	8
91	2	12	8
105	2	15	1
106	2	12	8
107	2	12	8
127	5	9	6
130	2	15	10
131	3	9	6
134	2	15	10
141	8	9	6
143	2	15	10
144	8	9	6
	Total	207	138

Species (i)	$\|G(\delta_i)\|$	$\|Y^*_{i,2}\|$	$\|S_i\|$
147	5	6	3
	Total	6	3

Species (i)	$\|G(\delta_i)\|$	$\|Y^*_{i,6}\|$	$\|S_i\|$
3	2	24	12
18	2	24	12
20	2	24	12
32	2	24	12
38	2	24	12
39	2	24	12
40	2	24	12
52	2	24	12
56	2	24	12
59	2	24	12
88	2	24	12
98	2	24	12
114	2	24	12
116	2	24	12
	Total	336	168

Table 4

From Tables 3–4 we have $|Y^*| = 3479$ and $|S^*| = 1986$. Of these 1986 species, 493 are self-dual and 1493 are duals of each other.

6 Enumeration and classification of 6 × 7 DYRs

To enumerate all non-isomorphic 6×7 DYRs an exhaustive computerized procedure was used. We used the notion that Youden squares isomorphic to one another cannot come from different Latin squares. For each $y \in Y^*$ we searched exhaustively for all possible ways of placing a second set of symbols on y to form a DYR. It then remained to sift for isomorphisms amongst all DYRs obtained. The essence of our approach was to enumerate non-isomorphic 6×7 DYRs that are obtainable from a single species of 7×7 Latin square, and then to sift for isomorphisms amongst DYRS obtained from each species. Using such an approach avoids sifting for isomorphisms amongst all DYRs obtained and consequently saves on cpu time. There are two basic steps:

(i) $\forall\, y_i \in Y^*$ find the set of all solutions g_i^*;

(ii) within g^* search for isomorphisms.

Let Y_i be a Youden square obtained from species i of 7×7 Latin square and g be a solution of y, such that if g is superimposed on y_i, the resulting design is a DYR of order 6×7. Further, let $R = \{r_1, r_2, \ldots, r_6\}$ and let $L = \{1, 2, \ldots, 6\}$ be the sets of rows and symbols of y. Then our sequential process for obtaining all the solutions g_i^* begins by placing the integer k twice in r_k and once in each of the remaining $|R| - 1$ rows, thus forming a 'frame' of type k, f_k. For frames of type k that do not violate the conditions of a DYR, we place the integer $k+1$ twice in r_{k+1} and once in the remaining $|R| - 1$ rows. For frames of type $k+1$ that do not violate the conditions of a DYR we place the integer $k+2$ twice in r_{k+2} and once in the remaining $|R| - 1$. We continue in this way until we obtain, for a given y_i, all frames of type 6 ($t_6 = g_i$). We then use an isomorphism testing program of McKay [7] to check for isomorphism amongst the members of the set g of solutions for y_i. The basic algorithm is given in section (6.1).

6.1 A exhaustive search algorithm for 6×7 DYRs

The algorithm consists of 5 basic steps:

(1) Set $k = 1$.

(2) Place k in two positions of r_k and once in each of the remaining $|R| - 1$ rows.

(3) Check if f_k is a possible solution.

(4) Put $k = k + 1$.

(5) If $k < r$ go to (2).

If $k = r$ fill in the spaces with the integer 6; this is a solution.

6.2 The total number of non-isomorphic 6 × 7 DYRs

If Ω represents the set of non-isomorphic 6×7 DYRs, then the total number of non-isomorphic DYRs is given by (8):

$$|\Omega| = |g_1^*| + |g_2^*| + \cdots |g_{147}^*| \qquad (8)$$

and $|\Omega| = 2971$. See Table 5, where δ_{147} is the "missing square" found by Sade [11].

δ_i	$\|g_i^*\|$	δ_i	$\|g_i^*\|$	δ_i	$\|g_i^*\|$	δ_i	$\|g_i^*\|$	δ_i	$\|g_i^*\|$
1	-	2	2	3	12	4	2	5	14
6	2	7	6	8	24	9	30	10	48
11	42	12	50	13	2	14	4	15	4
16	22	17	16	18	10	19	50	20	18
21	28	22	52	23	26	24	26	25	46
26	14	27	32	28	12	29	28	30	38
31	54	32	20	33	28	34	62	35	32
36	68	37	10	38	28	39	6	40	14
41	19	42	12	43	4	44	6	45	14
46	42	47	32	48	36	49	12	50	8
51	7	52	16	53	40	54	46	55	9
56	20	57	18	58	36	59	18	60	14
61	12	62	58	63	26	64	16	65	30
66	11	67	9	68	24	69	6	70	-
71	16	72	34	73	52	74	38	75	22
76	14	77	2	78	30	79	14	80	12
81	30	82	20	83	8	84	11	85	22
86	38	87	32	88	18	89	52	90	34
91	5	92	36	93	32	94	42	95	26
96	30	97	16	98	20	99	54	100	-
101	6	102	6	103	20	104	11	105	10
106	2	107	6	108	54	109	13	110	13
111	8	112	8	113	14	114	6	115	62
116	20	117	22	118	4	119	32	120	12
121	20	122	34	123	48	124	14	125	22
126	22	127	15	128	12	129	-	130	7
131	-	132	8	133	16	134	10	135	-
136	2	137	4	138	1	138	14	140	17
141	3	142	14	143	6	144	-	145	-
146	-	147	20						

Table 5

6.3 Classification of 6×7 DYRs

The basic approach used by researchers in combinatorial design theory to show that two designs with the same parameters are non-isomorphic involves the identification of some special property that remains invariant under permutation of the symbols of the design and distinguishes between the two designs. Ideally our special property would remain invariant under the permutation of the constraints and that it would be easy to compute but effective in discriminating between non-isomorphic DYRs.

Christofi [2] used the order of the automorphism group of a DYR and an isomorphism testing program of McKay [7] as invariants to sift for isomorphisms among DYRs of order 4×5 and 5×6. However, the classification of the 2971 into species involves obtaining the adjugates of all 2971 DYRs and then testing for isomorphism amongst all 11884 DYRs. Due to the large number of DYRs, a similar approach to that used in [2] would be impracticable and we therefore require a more efficient method of extracting isomorphisms.

6.3.1 An invariant for DYRs

For the classification of DYRs we redefine an intercalate to mean a 2×2 Latin square embedded in either set of symbols of a DYR. We propose a characteristic of DYRs that involves the number of intercalates in each set of symbols and is both quick and easy to compute. More importantly, our special property remains invariant under:

(i) The permutation of the rows, columns, first and second sets of symbols (or any combination of these).

(ii) The interchange of constraints Q and S.

(iii) The interchange of constraints P and T.

We begin by showing that the number of intercalates in each set of symbols is unaffected by any permutation of the rows, columns or two sets of symbols. Let $P = \{r_1, r_2, \ldots, r_m\}$, $Q = \{c_1, c_2, \ldots, c_n\}$, $S = \{A, B \ldots, n\}$, and $T = \{\alpha, \beta, \ldots, m\}$ represent the sets of rows, columns, first and second set of symbols respectively of a DYR ω; then the adjugates of ω $^*\omega$, ω^* and $^*\omega^*$ can be obtained as described in section (2.3). Let $N(\omega_i, I_n)$ and $N(\omega_i, \eta)$ represent the number of intercalates in S and T respectively of a DYR ω_i. As Π_γ is one-one, the number of intercalates in each set of symbols of ω is unaffected by Π_γ and we have (9):

$$\omega_1 \equiv \omega_2 \Rightarrow \text{(i) } N(\omega_1, I_n) = N(\omega_2, I_n) \text{ and (ii) } N(\omega_1, \eta) = N(\omega_2, \eta) \quad (9)$$

Remark. The converse of (9) is not necessarily true. Two DYRs satisfying (9) are possibly isomorphic. On the other hand, two DYRs that do not satisfy (9) cannot be isomorphic to each other.

Let (r_1, c_1, A, γ), (r_1, c_2, B, δ), $(r_2, c_1, B, \varepsilon)$ and (r_2, c_2, A, ξ) represent the location of intercalate I_n in the set S of ω (see Table 6 (i)); then interchanging the constraints Q and S moves In to the new location (r_1, A, c_1, γ), (r_1, B, c_2, δ), $(r_2, B, c_1, \varepsilon)$ and (r_2, A, c_2, ξ) in $^*\omega$, see Table 6 (ii). An intercalate involving the symbols A, B and in columns c_1 and c_2 is moved to an intercalate involving "symbols" c_1 and c_2 and in "columns" A and B and the number of intercalates in the first set of symbols is unaffected by the interchange of the constraints Q and S. Similarly let (r_1, c_1, C, α), (r_1, c_2, D, β), (r_2, c_1, E, α) and (r_2, c_2, F, β) represent the intercalate η in set T of ω (see Table 7 (i)). Then interchanging the constraints P and T of ω moves η to the new location (α, c_1, C, r_1), (β, c_2, D, r_1), (α, c_1, E, r_2) and (β, c_2, F, r_2) in ω^* (see Table 7 (ii)). The number of intercalates in set T is unaffected by the interchange of the constraints P and T.

ω	c_1	c_2	$^*\omega$	A	B
r_1	$A\gamma$	$B\delta$	r_1	$c_1\gamma$	$c_2\delta$
r_2	$B\varepsilon$	$A\xi$	r_2	$c_2\varepsilon$	$c_1\xi$

(i) Before interchange (ii) After interchange

Table 6

ω	c_1	c_2	ω^*	C_1	C_2
r_1	$C\alpha$	$D\beta$	α	Cr_1	Dr_2
r_2	$E\beta$	$F\alpha$	β	Er_2	Fr_1

(i) Before interchange (ii) After interchange

Table 7

We can use our invariant to distinguish whether a DYR is isomorphic to any of its adjugates and classify all transformation sets into species. There are three possibilities:

(i) ω belongs to a species containing a single transformation set if (10) and (11) are satisfied;

$$N(\omega, I_n) = N(^*\omega, I_n) = N(\omega^*, I_n) = N(^*\omega^*, I_n) \qquad (10)$$

$$N(\omega, \eta) = N(^*\omega, \eta) = N(\omega^*, \eta) = N(^*\omega^*, \eta) \qquad (11)$$

(ii) ω belongs to a species containing two transformation sets if any pair of ω, $^*\omega$, ω^*, $^*\omega^*$ have the same number of intercalates in each set of symbols. For example,

$$N(\omega, I_n) = N(^*\omega^*, I_n) \text{ and } N(\omega, \eta) = N(^*\omega^*, \eta) \qquad (12)$$

(iii) ω belongs to a species containing four transformation sets if (13) and (14) are satisfied:

$$N(\omega, I_n) \neq N(^*\omega, I_n) \neq N(\omega^*, I_n) \neq N(^*\omega^*, I_n) \tag{13}$$
$$N(\omega, \eta) \neq N(^*\omega, \eta) \neq N(\omega^*, \eta) \neq N(^*\omega^*, \eta) \tag{14}$$

Remark. If any pair of ω, $^*\omega$, ω^* and $^*\omega^*$ have the same number of intercalates in sets S and T, then (10) or (11) is satisfied. In particular, if $N(\omega, I_n) = N(^*\omega^*, I_n)$ and $N(\omega, \eta) = N(^*\omega^*, \eta)$ are satisfied this implies that (10) and (11) are satisfied.

A 6×7 DYR and its adjugates are shown in Table 8. The number of intercalates in sets S and T for all 4 DYRs is given in Table 9 and we can immediately note that (13) and (14) are satisfied and that ω and its adjugates are all non-isomorphic and consequently each of the DYRs shown in Table 8 belongs to a different species.

C1	G2	E3	F4	A5	D6	B3	E5	G3	A1	F6	C3	D4	B2
E2	D3	G6	A1	C4	B5	F2	D1	F5	E4	B3	A2	G2	C6
F3	C6	A2	G5	B1	E4	D4	C2	E1	B6	G4	F4	A3	D5
G4	E5	D1	B2	F6	A3	C5	F3	D2	G5	C1	B5	E6	A4
D5	F1	B4	E6	G3	C2	A6	G6	C4	F2	A5	D6	B1	E3
B6	A4	F5	C3	D2	G1	E1	B4	A6	D3	E2	G1	C5	F1
			(ω)							($^*\omega$)			
C1	F5	D4	A2	B3	G6	E6	D2	E3	A1	C4	G6	B5	F6
E2	G1	A3	B4	D6	C5	F2	C3	D4	F5	E6	A2	G2	B1
F3	D2	E1	C6	G5	A4	B1	F4	G1	D6	B2	C1	A3	E5
G4	A6	B5	F1	C2	E3	D3	B6	C5	E2	G3	F3	D1	A4
D5	E4	F6	G3	A1	B2	C4	E1	F2	G4	A5	B4	C6	D3
B6	C3	G2	E5	F4	D1	A5	G5	A6	B3	F1	D5	E4	C2
			(ω^*)							($^*\omega^*$)			

Table 8

	$N(I_n)$	$N(\eta)$
ω	6	7
$^*\omega$	6	12
ω^*	7	7
$^*\omega^*$	7	12

Table 9

Using our invariant we classified the 2971 DYRs into 772 species. The results are shown in Table 10.

|Aut(ω)|	No. of Trans. sets in species 1	2	4	Total
1	7	40	711	758
2	1	2	1	4
3	0	1	0	1
5	0	2	6	8
6	1	0	0	1
Total	9	45	718	772

Table 10

1.	A5	B4	C1	D1	E2	F3	G6	2.	A2	B1	C1	D3	E4	F5	G6
	C3	A2	B6	E4	D5	G1	F2		B4	A3	D6	E2	F2	G1	C5
	D2	F6	E3	G3	A1	B5	C4		C3	E5	G2	A6	D1	B3	F4
	E6	D3	G2	C5	F4	A4	B1		D5	G4	A4	F1	B6	C2	E3
	F1	G5	D4	A6	B3	C2	E5		E1	C6	F3	G5	A5	D4	B2
	G4	E1	F5	B2	C6	D6	A3		F6	D2	B5	C4	G3	E6	A1
3.	A2	B1	C1	D3	E4	F5	G6	4.	A6	B1	C5	D2	E3	F1	G4
	B3	A4	G5	C6	D2	E2	F1		B2	E2	D6	F4	G5	A3	C1
	D6	F3	A3	E1	B5	G4	C2		C3	A5	F2	G3	D1	E4	B6
	C4	E6	F2	A5	G3	D1	B4		D4	C4	B3	A1	F6	G2	E5
	E5	G2	B6	F4	A1	C3	D5		E1	G6	A4	B5	C2	D5	F3
	G1	C5	D4	B2	F6	A6	E3		F5	D3	G1	E6	B4	C6	A2
5.	B4	C2	F1	E5	G1	D3	A6	6.	A6	B2	C1	D5	E4	F1	G3
	C3	E6	G4	F2	B2	A5	D1		B5	C6	E2	G1	D3	A4	F2
	D2	F5	E3	G6	A4	C1	B3		C3	D1	G4	A2	F6	B3	E5
	E1	D4	B5	A3	C6	G2	F4		D4	E3	F5	B4	C2	G6	A1
	F6	G3	A2	B1	D5	E4	C5		E1	G5	B6	F3	A5	D2	C4
	G5	A1	D6	C4	F3	B6	E2		G2	F4	A3	E6	B1	C5	D6
7.	A3	B6	C1	D1	E2	F5	G4	8.	B1	G2	E4	F5	C3	A1	D6
	B2	C4	E5	G3	D6	A1	F2		C4	D5	F2	A6	G1	B2	E3
	C5	D2	G6	A4	F1	B3	E3		D3	C1	G6	B3	F4	E5	A2
	D4	E1	F4	B5	C3	G2	A6		E2	F3	D1	G4	A5	C6	B4
	F6	A5	D3	C2	G5	E4	B1		F6	A4	B5	E1	D2	G3	C5
	G1	F3	A2	E6	B4	C6	D5		G5	E6	A3	C2	B6	D4	F1

Table 11

9.	F1	A1	E3	G6	C4	D5	B2	10.	B1	C3	D4	E5	G2	A1	F6
	D6	F2	A4	B5	G3	C2	E1		E4	G1	F2	C6	D5	B2	A3
	G2	D3	B1	A3	F5	E4	C6		D3	F4	A6	B3	C1	G5	E2
	E5	G4	F6	C1	A2	B3	D4		C2	A5	E1	G4	F3	D6	B4
	B4	C5	G5	E2	D1	A6	F3		G6	D2	B5	F1	A4	E3	C5
	C3	E6	D2	F4	B6	G1	A5		F5	E6	G3	A2	B6	C4	D1
11.	F1	A1	B3	C4	D5	G6	E2	12.	F1	A1	E4	G2	C3	D6	B5
	G3	F2	D6	E5	A4	C2	B1		D5	F2	A6	B4	G1	C2	E3
	C6	D3	E1	A3	G2	B4	F5		G6	D3	B2	A3	F4	E5	C1
	B5	G4	A2	F6	C1	E3	D4		E2	G4	F3	C6	A5	B1	D4
	E4	C5	G5	B2	F3	D1	A6		B3	C5	G5	E1	D2	A4	F6
	D2	E6	F4	G1	B6	A5	C3		C4	E6	D1	F5	B6	G3	A2
13.	B1	G5	E3	F6	C4	A1	D2	14.	B3	C1	A4	G1	D6	E2	F5
	C6	D4	F2	A5	G3	B2	E1		C6	A5	B2	E4	F1	G3	D2
	D3	C2	G1	B3	F5	E4	A6		D1	E3	G5	F3	C2	A6	B4
	E5	F1	D6	G4	A2	C3	B4		E5	F2	D3	B6	G4	C4	A1
	F4	A3	B5	E2	D1	G6	C5		F4	G6	E1	A2	B5	D5	C3
	G2	E6	A4	C1	B6	D5	F3		G2	D4	F6	C5	A3	B1	E6

Table 11. cont

ω	$\lvert \text{Aut}(\omega) \rvert$	Generator(s) of Aut ω				No. of T. sets in ω
		P	Q	S	T	
1	2	(12)(35)(46)	(23)(47)(56)	(AC)(DF)(EG)	(12)(35)(46)	2
2	2	(13)(25)(46)	(12)(36)(45)	(AE)(BC)(FG)	(13)(25)(46)	1
3	2	(13)(25)(46)	(15)(23)(67)	(AB)(CF)(DE)	(13)(25)(46)	2
4	2	(13)(24)(56)	(12)(37)(46)	(BC)(DE)(FG)	(13)(24)(56)	4
5	3	(135)(264)	(246)(375)	(BDF)(CGE)	(135)(264)	2
6	5	(12643)	(16374)	(BCFED)	(12643)	2
7	5	(12643)	(16374)	(BCFED)	(12643)	2
8	5	(12543)	(16374)	(AFCGD)	(12543)	4
9	5	(12543)	(16374)	(AFCGD)	(12543)	4
10	5	(12543)	(16374)	(AFCGD)	(12543)	4
11	5	(12543)	(16374)	(AFCGD)	(12543)	4
12	5	(12543)	(16374)	(AFCGD)	(12543)	4
13	5	(12543)	(16374)	(AFCGD)	(12543)	4
14	6	(12)(36)(45)	(23)(47)(56)	(BC)(DG)(EF)	(12)(36)(45)	1
		(13)(25)(46)	(24)(36)(57)	(BD)(CF)(EG)	(13)(25)(46)	

Table 12

Table 11 shows a representative ω for each of the 14 species of 6×7 DYRs with non-trivial automorphism group. For each ω, Table 12 gives the order of the automorphism group and the number of transformation sets contained within ω.

Remark. Bailey [1, problem 14] asks for more DYRs to be found, "preferably in infinite series". Tables 11 and 12 provide a starting point for an investigation into the construction of an infinite series of $(n-1) \times n$ DYRs. We could begin by scrutinizing the structure of those DYRs that are isomorphic to their adjugates.

7 Conclusion

At present there is little in the literature concerning DYRs of order 6×7. Hedayat et. al. [4] described a method to construct DYRs of order $(n-1) \times n$. Their technique, known as *prolongation*, utilizes orthogonal Latin squares (OLS) of order $(n-1)$ to construct DYRs of order $(n-1) \times n$ and is consequently dependent on the existence of (OLS). However as there does not exist a pair of (OLS) of order 6, their method cannot be used to construct DYRs of order 6×7. Although one reason for the scarceness of 6×7 DYRs may be the non-existence of (OLS) of order 6, unawareness of the number of non-isomorphic 6×7 Youden squares has undoubtedly hindered the availability of DYRs of this size.

In this paper we have enumerated and classified all non-isomorphic Youden squares and DYRs of order 6×7. We used all 147 Latin squares of order 7×7 to obtain all non-isomorphic 6×7 Youden squares. We showed that, given two Latin squares each with the same number of adjugate sets, the square with trivial automorphism group will yield more non-isomorphic 6×7 Youden squares than the square with non-trivial automorphism group. For each of the 3479 non-isomorphic Youden squares, an exhaustive search was carried out to enumerate the number of non-isomorphic ways of placing a second set of symbols to form a DYR. We concluded that there are 2971 non-isomorphic DYRs of order 6×7. To classify all non-isomorphic DYRs, we proposed an invariant that involves the number of intercalates in each set of symbols. Using this invariant, we classified the 2971 DYRs into 772 species.

MacMahon [6] showed that the number of Latin squares of order n is

$$U_n = n!(n-1)T_n \tag{15}$$

where T_n is the number of reduced Latin squares of order n. Preece [9] pointed out that there are as many standard $(n-1) \times n$ as there are reduced Latin squares. Thus (15) also represents the number of standard $(n-1) \times n$ Youden squares. More recently, Kolesova et. al. [5] used (15) together with Burnside's theorem to verify the results of Wells [12] concerning the number of 8×8 Latin squares. If a symmetry group acts on the set of Latin squares, then the set is divided into orbits and and the size of each orbit may be determined if one knows the size of the automorphism group of any Latin square in the orbit. As the automorphism group of each non-isomorphic

6×7 Youden square and DYR is now known, the approach of Kolesova et. al. can be used to verify the results obtained in this paper.

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