A Sharp Lower Bound for the Number of Connectivity-Redundant Nodes

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ABSTRACT. We call a node of a simple graph *connectivity-redundant* if its removal does not diminish the connectivity. Studying the distribution of such nodes in a CKL-graph, i.e., a connected graph G of order ≥ 3 whose connectivity κ and minimum degree δ satisfy the inequality $\delta \geq (3\kappa - 1)/2$, we obtain a best lower bound, sharp for any $\kappa \geq 1$, for the number of connectivity-redundant nodes in G, which is $\kappa + 1$ or $\kappa + 2$ according to whether κ is odd or even, respectively. As a by-product we obtain a new proof of an old theorem of Watkins concerning node-transitive graphs.

1 Introduction

In this note we consider only finite simple graphs. As usual, by $V = V(G)$, $\delta = \delta(G)$, and $\kappa = \kappa(G)$ we denote the node set of a graph G, the minimum degree of a node in G, and the connectivity of G, respectively. For a subset $U \subset V$, by $G - U$ we denote the induced subgraph $\langle V - U \rangle$ of G. Generally, the terminology and notation in this note follow [4].

In order to measure the contribution of a single node v of a nontrivial graph G to $\kappa(G)$, Akiyama et al. introduced [1] the concept of the *cohesiveness* $c_G(v)$ of v in G, defined by $c_G(v) = \kappa(G) - \kappa(G - v)$. In this note we will call a node v of G *connectivity-essential*, or an *e-node*, if $c_G(v) > 0$; we will call v *connectivity-redundant*, or an *r-node*, if $c_G(v) \leq 0$. It can be easily seen that G may have at most one node v with $c_G(v) < 0$, which is the case if and only if G is obtained from some n-connected, $n \geq 1$, graph H by adding a node, v, and then joining it to some $n - 1$ nodes in H.

The following is an obvious lower bound for the number of e-nodes in G, which is certainly positive if G is a connected nontrivial graph:

$$\#ess(G) \geq \kappa. \tag{1}$$

In this note we find a positive lower bound for the number $\#red(G)$ of r-nodes of G, with placing some constraints on G. (Of course, $\#ess(G) + \#red(G) = |V|$.) We now mention two known related results. One is the content of Exercise 5.25 of [4], attributed to Chartrand, Kaugars, and Lick; it says essentially that if G has $\kappa \geq 2$ and satisfies the inequality:

$$3\kappa - 2\delta \leq 1, \tag{2}$$

then $\#red(G) \geq 1$. The other is Exercise 10 of [9, §2.2], whose result is due to Lozovanu and Syrbu; it says essentially that every graph G of connectivity 2 satisfying (2) has $\#red(G) \geq 4$.

In this note we are particularly concerned with connected graphs G of order $|V| \geq 3$, and of connectivity $\kappa \geq 1$ satisfying (2). We will refer to such graphs as CKL-graphs; "CKL" is for "Chartrand, Kaugars, Lick". We now state the principal result of this note:

Theorem. *For any CKL-graph G of connectivity κ, we have*

$$\#red(G) \geq 2\lceil(\kappa + 1)/2\rceil. \tag{3}$$

Furthermore, this bound is sharp for each value of κ ($\kappa \geq 1$).

Let us see immediately what this theorem gives if applied to trees. Clearly, a node v of a tree $G = T$ of order ≥ 3 is connectivity-redundant if and only if v is an *endpoint*, that is, a node having degree equal 1. Next observe that every T is a CKL-graph of connectivity 1, and thus (3), when applied to trees, is essentially equivalent to a well-known fact that every tree of order ≥ 2 has at least two endpoints (see [4]).

We include here an example of a class of graphs that attain all the bounds (1), (2), and (3). Let k be any positive integer congruent to 1 modulo 3. With K_n denoting the complete graph on n nodes, consider the graph $G = K_{(2k+1)/3} + 2K_{(k+2)/3}$, where $2K_{(k+2)/3}$ is the union of two disjoint copies of $K_{(k+2)/3}$, and "+" denotes the *join* of the summand graphs (see [4]). G has $\delta = (k+2)/3 - 1 + (2k+1)/3 = k$; $\kappa = (2k+1)/3 = (1+2\delta)/3$; $\#red(G) = 2(k+2)/3 = \kappa + 1$; and $\#ess(G) = \kappa$.

The Theorem will be completely proven in Section 3, in which we study the structure of the set of r-nodes in a given CKL-graph; the sharpness of the bound (3) will be established in the next section. In Section 4 we develop the structural characterization of the sets of r-nodes and of e-nodes in an arbitrary graph. Finally, in Section 5 we address connectivity properties of node-transitive graphs.

2 Examples

Here we will construct three series of examples. For providing these we will employ a binary operation due to Sabidussi [7]–the *composition* $H_1[H_2]$ (also called the *lexicographic*, or *wreath*, *product*) of two graphs H_1 and H_2 with $V(H_1) \cap V(H_2) = \emptyset$, defined to be the graph with the node set $V(H_1[H_2]) = V(H_1) \times V(H_2)$, and the edge set $X(H_1[H_2]) = \{(u_1, u_2)(v_1, v_2) :$ either $[u_1v_1 \in X(H_1)]$ or $[u_1 = v_1$ and $u_2v_2 \in X(H_2)]\}$. (This operation is also useful for the study of the groups of graphs; one application will be addressed in Section 5.)

With C_4 denoting the cycle of length 4, our initial series consists of the graphs $E_0(n) = C_4[K_n]$, $n \geq 1$. We next "disbalance" $E_0(n)$, firstly by removing one arbitrary node u to obtain the series $E_1(n) = E_0(n) - u$ $(n \geq 1)$, and secondly by removing a pair of nonadjacent nodes $\{u, v\}$ to obtain $E_2(n) = E_0(n) - \{u, v\}$ $(n \geq 2)$. Clearly, regardless of the choice of the nodes to be removed, the graphs $E_1(n)$ and $E_2(n)$ are well-defined, up to isomorphisms. It is readily seen that $E_0(n)$ has connectivity $2n$, with all nodes essential; therefore $\kappa(E_1(n)) = 2n - 1$, and $\kappa(E_2(n)) = 2n - 2$. Furthermore, it is straightforwardly verified that $E_1(n)$ and $E_2(n)$ are CKL-graphs, while $E_0(n)$ is not (for any n). Finally, it is also directly verified that, given an odd [resp., even] value of κ, the graph $E_1(\frac{\kappa+1}{2})$ [resp., $E_2(\frac{\kappa+2}{2})$] attains both bounds (1) and (3).

3 A Structural Characterization of the Set of r-Nodes

Let G be a graph of connectivity κ, $\kappa \geq 1$. Its r-nodes break into equivalence classes, called *r-classes*; two r-nodes are in the same r-class provided they are joined by an *r-path*, that is, a path through only r-nodes. The *attachment* of a given r-class R, denoted by $att(R)$, is defined to be the set of e-nodes which are adjacent to at least one node in R; note that $|att(R)| \geq \kappa$. A subset $A \subset V$ is called a *separating set* provided $G - A$ is not connected or, in other words, has at least two components, in which case we say that A *separates* one node from another if those are in different components. Let A be a separating κ-set, that is, a separating set of cardinality κ, and let I be some component of $G - A$. The induced subgraph $S = \langle A \cup V(I) \rangle$ of G is called a *suspension*; A is called the *articulation set* of S, $art(S)$, and $V(I)$ the *interior set* of S, $int(S)$. When S is the union of some suspensions with a common articulation set, $art(S)$, we preserve the notation $int(S)$ for the union of the interior sets of the members of S. A suspension S is called *minimal* if, given a separating κ-set A', $A' \subset V(S)$ implies $A' = art(S)$. Note that any CKL-graph is certainly not complete, and hence has at least one separating κ-set.

Lemma 1. *Let G be a CKL-graph and let A be a separating κ-set. Assume that at least one of the components of $G - A$ corresponds to a minimal* *suspension, S_1, with $art(S_1) = A$, and denote by S_2 the union of the remaining suspensions having A as their common articulation set. If there is another separating κ-set B such that $B \cap int(S_1) \neq \emptyset$, then $A - B$ ($\neq \emptyset$) entirely belongs to the node set of only one component of $G - B$.*

Proof: Observe first that, since S_1 is minimal, we have $B \cap int(S_2) \neq \emptyset$. Assume for a contradiction that $A - B$ has members in, at least two, different components of $G - B$. Those components correspond to a number (≥ 2) of suspensions having B as their common articulation set. Denote one of those suspensions by T_1, and the union of the others by T_2. For $i = 1$ and 2, denote the sets $A \cap int(T_i)$ and $B \cap int(S_i)$ by A_i and B_i, the *interior parts* of A and B, and by a_i ($\neq 0$) and b_i ($\neq 0$) their cardinalities, respectively. Of course, $a_1 + a_2 = b_1 + b_2 = \kappa - |A \cap B|$. We have four principal cases to consider:

Case 1: $a_1 < \min\{b_1, a_2, b_2\}$.

Case 2: $a_1 = b_1 < a_2 = b_2$.

Case 3: $a_1 = b_1 > a_2 = b_2$.

Case 4: $a_1 = b_1 = a_2 = b_2 = (\kappa - |A \cap B|)/2$.

We shall establish the impossibility of each case shortly, but first we explain why these cases are indeed exhaustive. Case 1 corresponds to the existence of a strict minimum among a_i and b_j, and we will see shortly that it is enough to consider one, say a_1, of the numbers chosen as such minimum; the other choices are handled analogously. In Cases 2 and 3, the consideration is up to reversing "a_1" and "a_2", which is a matter of notation.

We will say that two subsets of V are adjacent if some node in one is adjacent to some in the other. Note that a component of $G-(A \cup B)$ cannot be adjacent to both A_1 and A_2 [or, resp., both B_1 and B_2], for otherwise $int(T_1)$ and $int(T_2)$ [resp., $int(S_1)$ and $int(S_2)$] would be joined by a path in $G-B$ [resp., $G-A$]. Hence each component I of $G-(A \cup B)$ is adjacent to at most two interior parts; if one is of A, then the other must be of B. Furthermore, I must be adjacent to *exactly* two interior parts, and maybe also to $A \cap B$, for otherwise G would be separated by some m-set with $m < \kappa$. For each pair $\{i, j\}$, where $i, j \in \{1, 2\}$, among the components of $G - (A \cup B)$ adjacent to both A_i and B_j, we select one of maximum order, which we designate by I_{ij}; it may happen that $V(I_{ij}) = \emptyset$, though.

We next proceed to complete the reductio ad absurdum by proving the impossibility of each of the four above-named cases. Regardless of the case in question, at least one of the following three alternatives must arise: (i) $V(I_{11}) = V(I_{12}) = \emptyset$, (ii) $V(I_{11}) \neq \emptyset$, (iii) $V(I_{12}) \neq \emptyset$.

Impossibility of Case 1. If (i) arises, then any node occurring in A_1 may be only adjacent to nodes in B_1 and B_2, and maybe also to ones in $A \cap B$, as well as other nodes in A_1 itself. Then it can be easily verified, regardless of whether κ is odd or even (using the fact that δ is an integer), that a node in A_1 has degree $< \kappa + \kappa/2$. Thus we come to a contradiction with the inequality (2). If (ii) or (iii) arises, then the nodes in $U = V(I_{11})$ or $V(I_{12})$, respectively, are adjacent to a total of at most $a_1 + \max\{b_1, b_2\} + |A \cap B| < \kappa$ nodes not in U, which is a contradiction.

Impossibility of Case 2. Suppose first that (i) arises. Then, in the same way as in Case 1 (in fact, we only used $a_1 \leq a_2$), we obtain that the degree of a node occurring in A_1 does not satisfy (2). If (ii) arises, then we obtain a contradiction like we did in Case 1. So we can suppose that (iii) arises, but (ii) does not. Then, taking B_1 as A_1 just above, we conclude that $V(I_{21}) \neq \emptyset$, for otherwise the members of B_1 would not satisfy (2). Therefore $V(I_{21})$ is the interior set of some suspension, which we designate by S_{21}, with $art(S_{21}) \subset A_2 \cup B_1 \cup (A \cap B)$. Thus

$$|art(S_{21})| \leq a_2 + b_1 + |A \cap B| = \kappa. \tag{4}$$

Clearly, strict inequality may not hold in (4), but if κ is attained, we come to a contradiction with the minimality of S_1.

Impossibility of Case 3. This is established as that of Case 2; merely reverse "1" and "2" in indices, beginning from the list of alternatives (i)–(iii) through to the discussion as to what happens if (iii) [which alternative now has the form $V(I_{21}) \neq \emptyset$] arises, but (ii) [now in the form $V(I_{22}) \neq \emptyset$] does not, in which place we now immediately have $V(I_{21}) \neq \emptyset$, and so that we can continue like we did in Case 2 to come to a contradiction.

Impossibility of Case 4. Again, similarly to Case 2, we conclude that $V(I_{11})$ or $V(I_{21})$ is nonempty, which is impossible by the minimality of S_1. The details are left to the reader.

The proof is complete. □

Observe that, unless G is complete, a node of G is connectivity-essential if it occurs in some separating κ-set, and connectivity-redundant if not.

Lemma 2. *Let G be a CKL-graph. If S is a minimal suspension in G, then $int(S)$ is an r-class, with its attachment having cardinality κ.*

Proof: Actually, our job is to prove that $int(S)$ is an r-class, in which event its attachment, coinciding with $art(S)$, certainly has cardinality κ. Suppose for a contradiction that $int(S)$ is not an r-class. Then it must contain some e-node which we designate by u. When $\kappa = 1$, this already contradicts the minimality of S. Thus we may suppose $\kappa \geq 2$, in which case u must occur in some separating κ-set B, $B \neq art(S)$, satisfying Lemma 1 with $art(S)$ as A, and S as S_1. Hence $art(S) - B$ belongs to one component of $G - B$. On the other hand, every node v (if any) in $V(G) - (V(S) \cup B)$ must be joined to at least one node in $art(S) - B$ by a path avoiding B, for otherwise v would be separated from the nodes of $int(S)$ by the set $B - int(S)$ having cardinality $< \kappa$. By a similar argument, thanks to the minimality of S, one can establish that $B \cap V(S)$ cannot separate any node of $int(S) - B$ (if such exists) from the nodes of $art(S) - B$. But now it follows that $G - B$ is connected, the final contradiction. □

Proof of the Theorem: By (2), given an r-class R in G, $|att(R)| = \kappa$ implies $|R| \geq \lceil \frac{\kappa+1}{2} \rceil$. Thus it suffices to prove that G has at least two such r-classes. Let A be a separating κ-set. Pick first some two suspensions S_1 and S_2 with $art(S_1) = art(S_2) = A$, and next pick two minimal ones, one in S_1 and the other in S_2. Now applying Lemma 2 finishes the proof of (3). Thus, with the sharpness of the bound established in Section 2, the proof is complete. □

4 The Redundancy-Essence Graph

Like the set of r-nodes breaks into r-classes, the set of e-nodes breaks into *e-classes* (merely replace "r-" by "e-" in the definition of the preceding section). Denote by R_i the r-classes of a given connected graph G, and by E_j its e-classes. We will call an r-class together with its attachment, $R_i \cup att(R_i)$, an *extended r-class*, and denote it by R_i^*. In order to display the distribution of r-nodes and e-nodes globally, we associate with G the *redundancy-essence graph*, $re(G)$, defined as follows: $V(re(G)) = \{R_i^*\} \cup \{E_j\}$, with two nodes adjacent provided that the three conditions are satisfied: (i) one node corresponds to an extended r-class R_i^*, (ii) the other corresponds to an e-class E_j, and (iii) $E_j \cap R_i^* \neq \emptyset$. Note that $re(G)$ is a bigraph unless a trivial graph.

The so-introduced concept of the redundancy-essence graph $re(G)$ has some resemblance with the concept of the block-cutpoint graph $bc(G)$ originally introduced in Harary and Prins [5] and also in Gallai [2]. Our definition of $re(G)$ is converted to Harary's definition of $bc(G)$ (see [4, Chapter 4]) by replacing "extended r-class, R_i^*" by "block, B_i", and "e-class, E_j" by "cutpoint, c_j". However, even when G is a graph of connectivity equal 1 without adjacent cutpoints, $re(G)$ need not coincide with $bc(G)$. In fact, graphs which are the block-cutpoint graphs of some 1-connected graphs have been characterized [5] as trees in which the distance between any two endpoints is even, whereas $re(G)$ of a 1-connected graph G need not be a tree at all. For example, let F be a graph obtained from $K_{2,4}$ by the removal of a pair of independent edges; clearly, F is well-defined (up to isomorphisms); also observe that it contains a cycle of length 4 and has $re(F)$ isomorphic with F itself. (Note that F is a CKL-graph of connectivity 1, and also note that $re(G)$ is never isomorphic with G for any 2-connected CKL-graph G; the latter is derived easily from the above proof of the Theorem.)

Recently a characterization of the redundancy-essence graphs $re(G)$ has been obtained [6] in dependence on the connectivity of G: A nontrivial graph H is the redundancy-essence graph of some graph G having a prescribed positive value n of connectivity if and only if (i) for $n = 1$, H is a connected bigraph with one part of the bipartition entirely consisting of cut nodes, or (ii) for $n \geq 2$, H is a connected bigraph.

5 An Application to the Groups of Graphs

Denote by $\#orb(G)$ the number of transitivity classes ("orbits") into which $V(G)$ splits under the action of the automorphism group of a given graph G. G is said to be *node-transitive* if $\#orb(G) = 1$, in which event G is necessarily *regular*, that is, the degree of each node is the same, denoted by $\rho(G) = \rho$ $(= \delta)$. Let us now return to the examples constructed in Section 2; since both factors of $C_4[K_n]$ (with n fixed) are node-transitive graphs, then $E_0(n)$ is too. (With regards to transitivity properties of graph compositions, we refer the interested reader to [8].) As for the CKL-graphs $E_1(n)$ and $E_2(n)$, none of these is node-transitive; in fact, observe that $\#orb(E_1(n)) = 3$, and $\#orb(E_2(n)) = 2$ (for any n).

Since no automorphism of a graph sends an r-node onto an e-node (or vice versa), and since inequalities (1) and (3) both hold for any CKL-graph, it follows that if G is connected and node-transitive, then G is certainly not a CKL-graph. Hence, if such a graph G has order ≥ 3, then it does not satisfy inequality (2), and thus $\frac{\rho}{\kappa} < \frac{3}{2} - \frac{1}{2\kappa}$. Furthermore, for the graphs $E_0(n)$, all of which are node-transitive, we have $\rho(E_0(n))/\kappa(E_0(n)) = \frac{3}{2} - \frac{1}{2n} \to \frac{3}{2}$ (as $n \to \infty$). Thus we are led to the following result:

Corollary. *For every connected node-transitive graph G, other than K_1 and K_2, we have $\frac{\rho}{\kappa} < \frac{3}{2} - \frac{1}{2\kappa} < \frac{3}{2}$. Furthermore, although the bound of $\frac{3}{2}$ is never attained, it is still best possible.*

In essence, this result is originally due to Watkins [8] who employed another method. In a more general form the bound of $\frac{3}{2}$ is also presented in [3], in which book (pp. 170-171) it is established that every connected node-transitive graph G admits the so-called atomic partition (defined in [3]), which yields $\frac{\rho}{\kappa} < \frac{3}{2}$.

Finally, note that the family of CKL-graphs is the utmost family of graphs which is free from node-transitive members, in the strong sense of an infinity of such members possible (e.g., $E_0(n)$, $n = 1, 2, 3, \ldots$) with bound (2) minimally weakened (i.e., raised from 1 to 2).

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