A generalization of Favaron's theorem

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Abstract

A graph is well covered if every maximal independent set has the same size and very well covered if every maximal independent set contains exactly half the number of vertices. In this paper, we present an alternative characterization of a certain sub-class of well-covered graphs and show that this generalizes a characterization of very well covered graphs given by Favaron [3].

1 Introduction

In what follows, G denotes a simple, undirected, finite graph $G = (V, E)$ with $|V|$ vertices and $|E|$ edges. A vertex is *isolated* if it has no neighbours. For a graph G, the size of a maximum independent set is denoted by $\alpha(G)$, and the number of cliques in a minimum clique cover by $\kappa(G)$.

The concept of a *well-covered* graph was introduced by Plummer [4]. He defined a graph to be well covered if every maximal independent set in it has the same size. These graphs are of interest because the independence number problem, which is NP-complete for general graphs, can be solved efficiently for this family. Chvátal and Slater [2], and Stewart and the author [8], independently showed that the problem of recognizing a graph as being not well covered is NP-complete. Hence, it is unlikely that there exists a good characterization of this family. See [5] for an excellent survey on well-covered graphs.

A graph is said to be *very well covered* if every maximal independent set in it contains exactly half the vertices in the graph. Berge [1] showed that the size of a maximal independent set of a well-covered graph G without isolated vertices is bounded by $|V|/2$. Hence, any such graph can be transformed into a very well covered one by adding an appropriate number of isolated vertices. Therefore, we, like all previous researchers in this area, restrict our attention to very well covered graphs without isolated vertices.

Hence, in this paper, the term *very well covered* stands for *very well covered without isolated vertices.*

Very well covered graphs were studied independently by Staples [9] and Favaron [3]. Favaron gave a characterization of these graphs that showed that all such graphs have perfect matchings that obey a certain property P. In this paper, we give an alternative characterization of the family W_{AR}, a sub-class of well-covered graphs that properly contains the family of very well covered graphs. We show that all graphs G belonging to this family have clique partitions of size $\alpha(G)$ that obey a certain property Q. We also show that this characterization generalizes Favaron's characterization of very well covered graphs.

This paper is organized in the following manner. Section 2 introduces some definitions and states some preliminary results. Sections 3 presents an alternative characterization of the family W_{AR} and shows that this is a generalization of Favaron's characterization of very well covered graphs. Conclusions and future work make up Section 4.

2 Preliminaries

We first introduce some definitions and notation. We then define and characterize the family W_{AR} and state some results on this family. For the proofs of these results, and for more information on this and other related families of well-covered graphs see ([6],[7]). We conclude this section by stating Favaron's characterization of very well covered graphs.

$u \sim v$ denotes that the vertices u and v are adjacent. Given a vertex set $A \subseteq V$, $< A >$ denotes the subgraph induced by A. $N(v)$ and $N[v]$ denote the *open* and *closed* neighbourhoods, respectively, of a vertex $v \in V$, where $N(v) = \{x \mid x \in V \text{ and } x \sim v\}$ and $N[v] = N(v) \cup \{v\}$. $N(S)$ and $N[S]$ denote the *open* and *closed* neighbourhoods, respectively, of a set $S \subseteq V$, where $N(S) = \cup N(v)$, for all $v \in S$, and $N[S] = N(S) \cup S$. A vertex is *simplicial* if $< N(v) >$ is a clique.

A graph G is said to be *complete k-partite* if its vertex set can be partitioned into one or more disjoint independent sets, or *parts*, such that each vertex is adjacent to every other vertex that is not in the same part. It is said to be *complete k_n-partite* if it is complete k-partite with all parts having the same number of vertices. In the instances where values are assigned to k and n, k corresponds to the number of parts, and n to the number of vertices in each part, respectively.

Let the vertex set V of a graph G be partitioned into disjoint sets, or *layers*, $L_1, L_2, \ldots, L_t$, $1 \leq t \leq\mid V \mid$, such that the induced subgraphs, or *lgraphs*, $H_i =< L_i >$, $1 \leq i \leq t$, are complete k_n-partite. Then G is said to be partitioned into complete k_n-partite subgraphs. E_i denotes the edge set,

k_i the number of parts, and n_i the number of vertices in each part, of H_i, $1 \leq k_i \leq | L_i |$, $n_i = | L_i | / k_i$. H_i is written as $H_i = (P_{i1}, P_{i2}, \ldots, P_{ik_i}, E_i)$, where $P_{i1}, P_{i2}, \ldots, P_{ik_i}$ denote the parts in H_i. A part P_a is *adjacent* to a vertex v if v has a neighbour in P_a. Two parts P_a and P_b are *adjacent*, or *connected*, or are *neighbours*, if there exist $u \in P_a$ and $v \in P_b$ such that $u \sim v$. P_a is *completely connected* to P_b if $< P_a \cup P_b >$ is complete bipartite. Two layers are *adjacent* if there is a part in one that is adjacent to a part in the other.

The intersection R of a pair of maximal independent sets of G is said to be *maximal* if for every pair of maximal independent sets I_a and I_b that contain R, $I_a \cap I_b = R$.

Theorem 2.1 *The intersection R of a pair of maximal independent sets I_1 and I_2 of a well-covered graph G is maximal if and only if $< V - N[R] >$ is complete k_n-partite.*

We now define and characterize the family W_{AR}.

Definition 2.2 *A graph G is said to belong to the family W_{AR} if*
a) G is complete k_n-partite, or
b) G is well covered and for every maximal R, the intersection of a pair of maximal independent sets of G, $< N[R] >$ belongs to W_{AR}.

From Definition 2.2 and Theorem 2.1, it is easy to see that a graph G belonging to the family W_{AR} can be recursively decomposed into complete k_n-partite subgraphs such that the corresponding vertex sets partition the vertex set of G into layers. Clearly, there can be more than one such decomposition, since at every stage of such a decomposition, there can be more than one maximal intersection R for which $< N[R] >$ is in W_{AR}.

Theorem 2.3 *A graph G belongs to the family W_{AR} if and only if its vertices can be partitioned into layers $L_1, L_2, \ldots, L_t$, $1 \leq t \leq | V |$, that have the following properties:*
a) The layers induce complete k_n-partite subgraphs, with every layer except L_t having at least two parts. L_t has one or more parts.
b) For every two adjacent layers L_j and L_k, there exist parts $P_j \in L_j$ and $P_k \in L_k$ such that $| N(P_j) \cap L_k | = 0$ and $| N(P_k) \cap L_j | = 0$, and the parts of $L_j - P_j$ and $L_k - P_k$ are completely connected to each other.
c) The non-common neighbours of every pair of parts in every layer are completely connected to each other.

From [6] and [7], we recall some of the properties of a graph G belonging to this family: All recursive decompositions of G yield the same set of layers; that is, the layers obtained are unique. Isolated vertices, if any, form the layer L_t. Every maximal independent set of G contains exactly one part from each layer. For each layer L_i, $1 \le i \le t$, we can find a maximal intersection R such that $L_i = V - N[R]$.

In ([6],[7]), it is shown that W_{AR} properly contains the family of very well covered graphs. We now state Favaron's characterization of very well covered graphs.

Favaron's theorem ([3]) *For a graph G, the following are equivalent:*

a) G is very well covered.

b) There exists a perfect matching in G that satisfies P.

c) There exists at least one perfect matching in G and every perfect matching of G satisfies P.

Property P in the above theorem is defined as follows.

Property P *A matching M in a graph G satisfies property P if for every edge $(u, v) \in M$, $N(u) \cap N(v) = \phi$, and $N(u) - \{v\}$ is adjacent to all of $N(v) - \{u\}$.*

Favaron also defined the following equivalence relation for very well covered graphs.

Definition 2.4 *Let M be a perfect matching of a very well covered graph G. Two vertices x and y are called equivalent if either $x = y$ or if $(x, v), (y, u) \in M$ and $x \in N(u)$ and $y \in N(v)$.*

She showed that the resulting equivalence classes form a partition of the vertex set of G into independent sets with certain properties.

3 The generalization

Let $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$, $1 \le k \le n$, be a clique partition of a graph G, with the corresponding vertex set being $\{V_1, V_2, \ldots, V_k\}$. We denote by $C(v)$ the clique ($\in \mathcal{C}$), and by $V(v)$ the corresponding vertex set, that $v \in V$ belongs to. We define property Q as follows.

Property Q We say that a clique partition $\mathcal{C}$ satisfies property Q if:

a) $| N(v) \cap V_i | = 0$ *or* $| N(v) \cap V_i | = | V_i | - 1, \forall v \in V, 1 \le i \le k$.

b) $(w \in V(v), u \in N(v) - N(w)) \Rightarrow (N(u) \supseteq N(w) - N(v)), \forall v \in V$.

The first condition states that if a vertex in G has a neighbour in some clique in the clique partition, then it is adjacent to all but one vertex in that clique. The second one states that for every two vertices in a clique, their non-common neighbours are completely connected to each other.

We now give an alternative characterization of the class W_{AR}. We say that a clique partition of a graph G is an α-clique partition if the number of cliques in the partition is $\alpha(G)$, the size of a maximum independent set in G.

Theorem 3.1 *The following are equivalent for a graph G.*

a) G belongs to W_{AR}.

b) There exists an α-clique partition of G that satisfies Q.

c) There exists an α-clique partition of G, and every α-clique partition of G satisfies Q.

Hence, if G is in W_{AR}, every α-clique partition of G satisfies Q. In order to prove this theorem, we need to state some definitions and establish some results.

Let C be a clique partition of a graph G, and let C satisfy Q. We define the following equivalence relation.

Definition 3.2 *We say that u and v are equivalent if either $u = v$ or $\mid V(u) \mid = \mid V(v) \mid$ and $x \sim v, y \sim u$, $\forall x \in V(u) - \{u\}, y \in V(v) - \{v\}$.*

That is, two vertices u and v are said to be equivalent if either they are the same vertex, or if their clique sizes are the same, and every vertex of $V(u) - \{u\}$ is adjacent to v, and every vertex of $V(v) - \{v\}$ is adjacent to u. Note that two distinct vertices u and v in the same clique cannot be equivalent as this would require each one to be adjacent to itself, which is not permitted.

We need to show that the above is indeed an equivalence relation. We first prove the following lemma.

Lemma 3.3 *Let C be a clique partition of a graph G and let C satisfy Q. Then, if u is equivalent to v, $u \neq v$, $< V(u) \cup V(v) >$ is complete k_2-partite, with $\{u, v\}$ forming one of the parts.*

Proof:
Since u and v are equivalent, we know that $\mid V(u) \mid = \mid V(v) \mid$. Also, $C(u)$ and $C(v)$ are cliques. Since u is adjacent to all of $V(v) - \{v\}$, from Property Q a), u is not adjacent to v. Similarly, v is adjacent to all but u in $V(u)$. Consider some vertex $x \in V(u) - \{u\}$. Since $x \sim v$, from Property Q a), x is adjacent to all but some $y \in V(v) - \{v\}$ in $V(v)$. Likewise, y is adjacent to all but x in $V(u)$. Therefore, the vertices of $V(u)$ and $V(v)$ can be paired into disjoint sets of two vertices each such that the neighbour set of a vertex in a pair consists of all but the other vertex in the pair. From the above, $\{u, v\}$ forms one such pair. This proves the lemma.

□

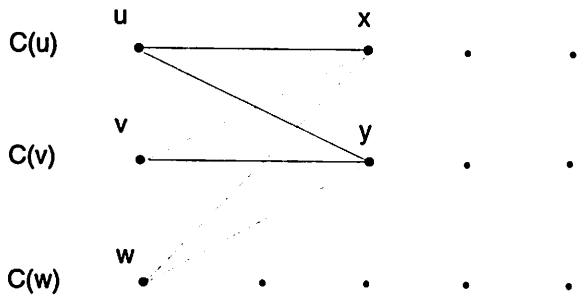


Figure 1: Equivalence of u and w

Now, let u be equivalent to v, $u \neq v$, and v be equivalent to w, $u, v \neq w$. Since $| V(u) |=| V(v) |$ and $| V(v) |=| V(w) |$, it follows that $| V(u) |=| V(w) |$. From Lemma 3.3, $< V(u) \cup V(v) >$ is complete k_n- partite, as is $< V(v) \cup V(w) >$. Also, $u \not\sim v$ and $v \not\sim w$; with Q $a)$, this implies that $C(u) \neq C(w)$. Consider a part $\{x, y\}$ in $< V(u) \cup V(v) >$, $x \in V(u), y \in V(v)$, $x \neq u, y \neq v$. Now, $v \sim x$ and $y \sim w$. Also, $v \not\sim w$ and $y \not\sim x$. Since v and y are in the same clique $C(v)$ and have non-common neighbours x and w respectively, from Property Q $b)$, $w \sim x$ (see Figure 1). Therefore, w is adjacent to all the vertices in $V(u) - \{u\}$. In a similar fashion, we can show that u is adjacent to all of $V(w) - \{w\}$. That is, u is equivalent to w. Therefore, the relation of Definition 3.2 is an equivalence relation.

Let $EC(u)$ denote the equivalence class of u, and let $CC(u)$ denote the corresponding clique class; that is, $CC(u)$ is made up of the cliques $C(v)$ corresponding to each vertex $v \in EC(u)$, together with the edges between the cliques. Let $VC(u)$ represent the vertex set of $CC(u)$ (see Figure 2). We now prove the following lemma.

Lemma 3.4 *Let C be a clique partition of a graph G, and let C satisfy Q. Then the following are true:*

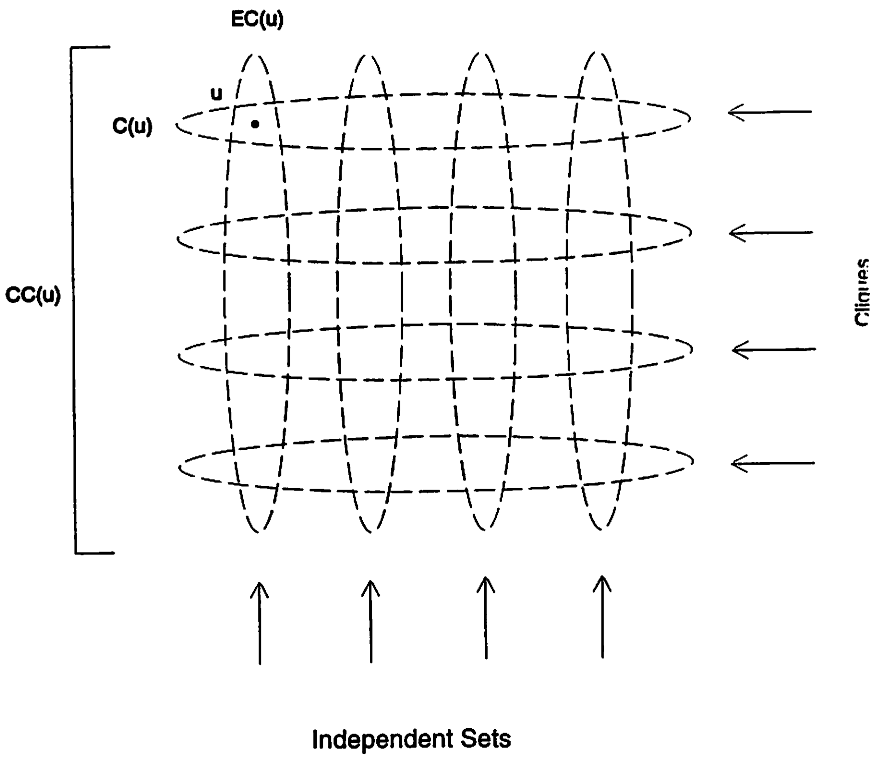
a) The equivalence classes partition V into independent sets.

b) The clique classes are complete k_n-partite with each part forming an equivalence class, and form a partition of G.

Proof:

a)

Consider the equivalence class $EC(u)$ of a vertex $u \in V$. From Lemma 3.3, the vertices in $EC(u)$ are pairwise disjoint, that is, $EC(u)$ is an independent set. As it is an equivalence relation, no vertex can appear in more than one equivalence class.



VC(u) = { x | x in C(y), for some y in EC(u) }

CC(u) = < VC(u) >

Note: edges between cliques are not shown

Figure 2: Equivalence and clique classes of the vertex u

b)

Let v be in $EC(u)$, $v \neq u$. Consider $x \in V(u)$, $x \neq u$. Since v is in $EC(u)$, using Lemma 3.3, we have that $< V(u) \cup V(v) >$ is complete k_n-partite, with $\{u, v\}$ forming one of the parts. Therefore, there has to be a $y \in V(v)$, $y \neq v$, that forms a part with x in $< V(u) \cup V(v) >$. Thus, y is adjacent to all of $V(u) - \{x\}$, and x is adjacent to all of $V(v) - \{y\}$. Also, $| V(u) |=| V(v) |$, since v is equivalent to u. Hence, y is equivalent to x. So for each vertex $v \in EC(u)$, we can find a $y \in V(v)$ that is equivalent to x. Now, u is in $C(x)$, by our choice of x. Using a similar argument, we can show that for each $z \in EC(x)$, we can find a $w \in V(z)$ that is equivalent to u. Therefore, $| EC(u) |=| EC(x) |$. From $a)$, $EC(u)$ and $EC(x)$ are mutually disjoint independent sets. Thus, each vertex x in $V(u)$ yields an equivalence class $EC(x)$, $| EC(x) |=| EC(u) |$, whose vertices are from $VC(u)$. Therefore, the $EC(x)$'s partition $VC(u)$ into mutually disjoint independent sets, all of which have the same size, with each $EC(x)$ having exactly one vertex from each clique ($\in \mathcal{C}$) in $CC(u)$. Since every two vertices in $EC(u)$ are equivalent to each other, from Lemma 3.3, the cliques ($\in \mathcal{C}$) in $CC(u)$ form pairwise a complete k_2-partite graph. From the above, $CC(u)$ is complete k_n-partite with $n =| EC(u) |$. It is easy to verify that for every vertex $v \in VC(u)$, $v \neq u$, the clique class $CC(v)$ is the same as $CC(u)$. Therefore, when we refer to the clique classes, we are referring to the distinct clique classes obtained from the equivalence classes. Clearly, every clique in the clique partition $\mathcal{C}$ belongs to some clique class, and no clique can belong to more than one clique class. Hence, the clique classes form a partition of G (see Figure 3 for an example).

□

We are now ready to prove Theorem 3.1.

Proof*(of Theorem 3.1)*:

a) $\rightarrow$ *c)*

G belongs to W_{AR}. Therefore, the vertices of G can be partitioned into layers L_1 to L_t that satisfy properties $a)$ to $c)$ of Theorem 2.3. From Theorem 2.3 $a)$, the corresponding lgraphs are complete k_n-partite. Hence, each lgraph can be decomposed into cliques giving an α-clique partition of G since, by [7], every maximal independent set of G contains exactly one part from each layer. Let one such α-clique partition be given by $\mathcal{C} = \{C_1, C_2, \ldots, C_{\alpha(G)}\}$. We observe that each clique in $\mathcal{C}$ appears in exactly one lgraph and has exactly one vertex in each part of that lgraph. That the vertices obey property Property Q $a)$ follows from Theorem 2.3 $a)$ and $b)$; that they obey Q $b)$ follows from Theorem 2.3 $c)$. Hence, there exists an α-clique partition in G that satisfies Q. Since, for any graph, the size of a minimum clique partition is greater than or equal to the size of a maximum

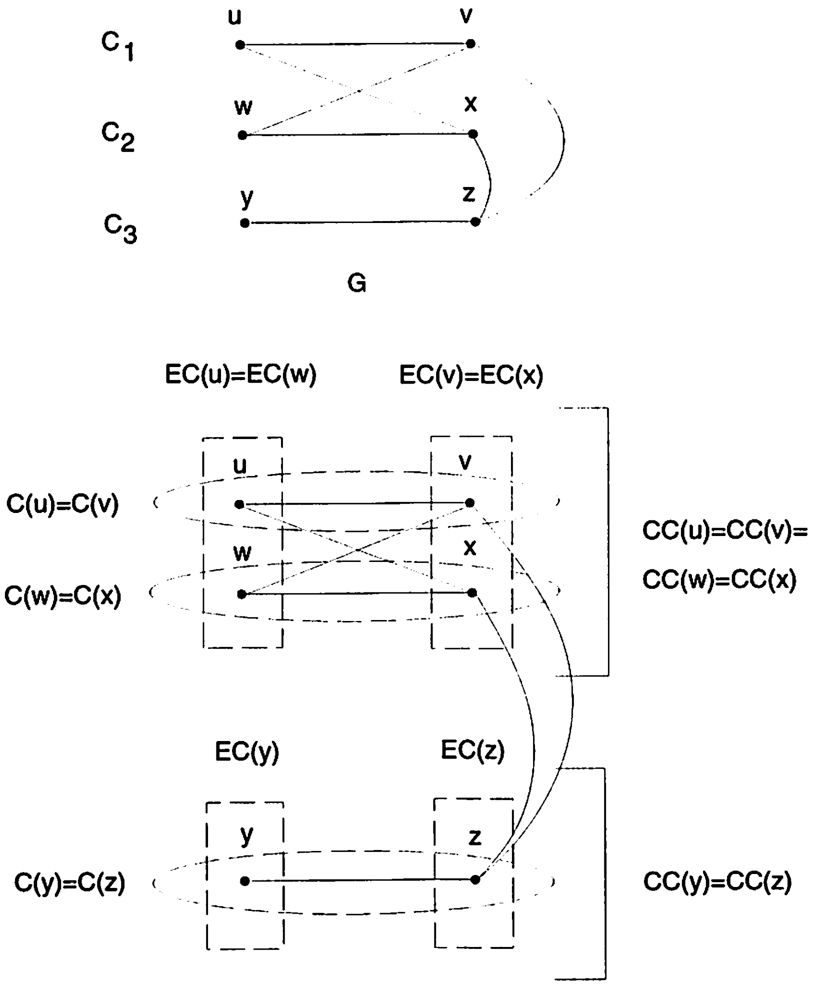


Figure 3: Equivalence and clique classes for the graph G

independent set, $\kappa(G) = \alpha(G)$.

Consider any α-clique partition $C' = \{C'_1, C'_2, \ldots, C'_{\alpha(G)}\}$ of G. Since $\kappa(G) = \alpha(G)$ and G is in W_{AR}, every maximal independent set of G contains exactly one vertex from each clique in C'. Consider any decomposition of G into layers L_1 to L_t. We show that the cliques in C' can be arranged to form the corresponding lgraphs H_1 to H_t. We ignore isolated vertices since each one forms a separate clique in the clique partition, and together form the layer L_t in the decomposition. Consider some layer L_i, $1 \leq i \leq t$. As stated in Section 2, we can find a maximal intersection R, the intersection of a pair of maximal independent sets of G, such that $L_i = V - N[R]$ (see [6] for a proof). We now show that $N[R]$ consists of vertices from only those cliques in C' that have a vertex in R. Assume not. Then there exists $C_j \in C'$, $1 \leq j \leq \alpha(G)$, that has no vertex in R, but has at least one vertex v in $N[R]$. Since G is in W_{AR}, so is $G_2 =< N[R] >$. Since R is a maximal independent set of G_2, every maximal independent set of G_2 has size $\mid R \mid$. Now, every maximal independent set of G contains exactly one vertex from each clique in C'. Therefore, starting with v, we can form an independent set that contains exactly one vertex from each clique ($\in C'$) that has a vertex in R. This set has size $>\mid R \mid$ implying that G_2 is not in W_{AR}, which is a contradiction. Hence, $N[R]$ consists of vertices from only those cliques in C' that have a vertex in R. That is, the lgraph H_i, induced by $L_i = V - N[R]$, contains exactly those cliques from C' that do not have a vertex in R. Since the number of vertices in R is given by $\alpha(G) - n_i$, there are $\alpha(G) - n_i$ cliques from C' in $< N[R] >$. As the number of cliques in C' is also $\alpha(G)$, there are exactly n_i cliques from C' in H_i.

Now, the lgraphs are complete k_n-partite. The size of a part in H_i is n_i, and the maximum possible size of a clique in it is given by k_i, the number of parts in it. Hence, the minimum number of vertex disjoint cliques required to cover the vertices of H_i is n_i, each being of size k_i. Since the cliques in C' are vertex disjoint, this means that the n_i cliques in H_i have exactly k_i vertices each, and form a partition of the vertices of H_i. Therefore, each lgraph in the decomposition contains *whole* cliques from C' such that the cliques form a clique partition of that lgraph. Since the sum of the n_i's is $\alpha(G)$, and there are $\alpha(G)$ cliques in C', and the lgraphs are vertex disjoint, each clique in C' appears in exactly one lgraph in the decomposition. That C' satisfies Property Q follows from the fact that the layers satisfy properties $a)$ to $c)$ of Theorem 2.3.

c) → b)

Follows.

b) → a)

There exists a clique partition C of G that satisfies Q. From Lemma 3.4, the equivalence classes form a partition of the vertex set of G into independent sets. From the same lemma, each clique class is complete k_n-partite, with each part forming an equivalence class. A clique class is constructed by taking an equivalence class and picking all the cliques in C that contain the vertices of the equivalence class. Since the clique classes are disjoint, every equivalence class is in exactly one clique class. From Lemma 3.4 *b*), the clique classes are complete k_n-partite and form a partition of the vertex set of G into layers. We show that G is in W_{AR} by showing that the subgraphs induced by these layers, that is, the clique classes, satisfy properties *a*) to *c*) of Theorem 2.3.

property a)

From Lemma 3.4, we know that the clique classes are complete k_n-partite. Isolated vertices, if present, will all be in the same equivalence class, and hence will form a separate clique class.

property b)

Let $CC(x)$ and $CC(y)$ be two different clique classes. Let $x \in EC(x)$ from the class $CC(x)$ be adjacent to $y \in EC(y)$ from the class $CC(y)$. From Property Q *a*), there exists y_1 in $C(y)$ that x is not adjacent to. Since $CC(y)$ is complete k_n-partite, y_1 is adjacent to each $z \in EC(y)$. Since y and y_1 are in the same clique $C(y)$, using Property Q *b*), x is adjacent to all such z (see Figure 4). Therefore, x is adjacent to all of $EC(y)$. By a similar argument, y, and every other vertex in $EC(y)$, is adjacent to all of $EC(x)$. Thus, $< EC(x) \cup EC(y) >$ is complete bipartite.

Therefore, if parts from different clique classes are adjacent, they are complete bipartite. This enables us to do the following reduction on the clique classes: replace each part in a clique class by a single vertex, thus reducing each clique class to a single clique; replace the set of edges between two adjacent parts by a single edge. Clearly, this transformation preserves the relationship between the clique classes. Since the clique classes now consist of single cliques, the clique class $CC(u)$ of a vertex u is the same as $C(u)$, the corresponding vertex set $VC(u)$ is the same as $V(u)$, and the equivalence class $EC(u)$ consists of the single vertex u. Hence, we will use $C(u)$, $V(u)$ and $E(u)$ to refer to the clique class, the corresponding vertex set and the equivalence class, respectively, of a vertex $u \in V$.

We are now ready to prove Theorem 2.3 *b*). Two clique classes are said to be adjacent if there is a part in one that is adjacent to a part in the other. Consider a clique class $C(x)$ that is adjacent to another clique class $C(y)$. Let $i =| V(x) |$ and $j =| V(y) |$, with $i \leq j$. Let p be the number of vertices of $V(x)$ that have a neighbour in $V(y)$, and q be the number of vertices of $V(y)$ that have a neighbour in $V(x)$. Since $C(x)$ and $C(y)$ are adjacent, $0 < p \leq i$ and $0 < q \leq j$. Moreover, from Property Q *a*), $q \geq j - i$, $p \geq i - 1$, and the number of edges between $C(x)$ and $C(y)$ is $p(j-1) = q(i-1)$. This implies in particular that $p \leq q$.

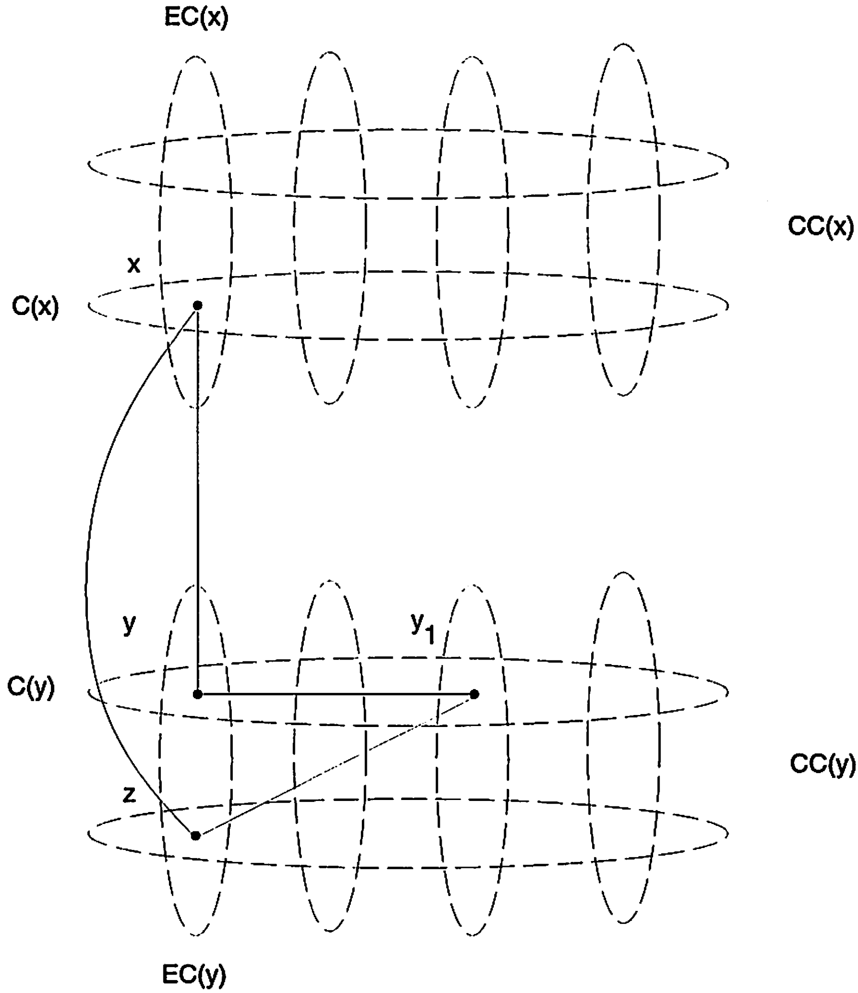


Figure 4: x is adjacent to each $z \in EC(y)$

If $p = i$, then p divides q and thus $p = q = i = j$. The edges between $C(x)$ and $C(y)$ are those of a complete bipartite graphs minus a perfect matching. Let (x, w) be the edge of this perfect matching incident to x. The vertex x is adjacent to all but w in $C(y)$ and w is adjacent to all but x in $C(x)$. Hence, x and w are equivalent, contradicting the fact that every equivalence class contains exactly one vertex.

Hence, $p = i - 1$, $q = j - 1$, and there exists exactly one vertex u in $V(x)$ that has no neighbours in $V(y)$ and one vertex v in $V(y)$ that has no neignbours in $V(x)$, and the edges between $V(x) - \{u\}$ and $V(y) - \{v\}$ are those of a complete bipartite graph. Therefore, the clique classes obey property $b)$ of Theorem 2.3.

property c)

Consider a clique class $CC(x)$. Every part in $CC(x)$ has exactly one vertex from each clique $C(v)$, $\forall v \in EC(x)$. From $b)$, which we have just proved, if parts from different clique classes are adjacent, they are completely connected. Property $c)$ follows from this and Property Q $b)$.

□

We now show that Theorem 3.1 is a generalization of Favaron's theorem. We define the family W_{AR2} as follows.

Definition 3.5 *A graph G belongs to W_{AR2} if G belongs to W_{AR} and each layer in any decomposition of G contains exactly 2 parts.*

Assume that the graph G in the above theorem belongs to W_{AR2}; that is, the graph is very well covered. An α-clique partition is now a perfect matching in G. That is, each clique in an α-clique partition C of G is a K_2. Using Property Q $a)$, we see that a vertex in a clique in C can be adjacent to at most one vertex in any other clique in C. Therefore, the vertices in a clique in C do not have a common neighbour. We can use this fact to rewrite Property Q for graphs that have a perfect matching. A perfect matching M is said to satisfy Property Q if the following condition is true.

$$(w \in V(v), u \in N(v), u \neq w) \Rightarrow (u \notin N(w)) \text{ and } (N(u) \supseteq N(w) - N(v)),\ \forall v \in V.$$

We see that this is the same as the Property P defined by Favaron. That is, the theorem reduces to Favaron's theorem for very well covered graphs.

Now let us see what happens to the equivalence relation of Definition 3.2 when G is very well covered graph without isolated vertices. From Favaron's theorem, there exists a perfect matching in G. Hence, any clique partition of G consists of K_2's. Therefore, the equivalence relation reduces to the following.

u and v are equivalent if either $u = v$, or $u \in N(V(v) - \{v\})$ and $v \in N(V(u) - \{u\})$.

This is the same as the equivalence relation defined by Favaron (Definition 2.4), and hence the equivalence classes obtained are the same.

4 Conclusions and future work

We have given an alternative characterization of the class W_{AR}, a sub-class of well-covered graphs, in terms of a clique partition of size α that obeys a certain Property Q. We have shown that when the cliques in the partition are K_2's, the clique partition reduces to a perfect matching, Property Q reduces to Property P, and the characterization reduces to Favaron's characterization of very well covered graphs. This is an interesting result since it generalizes the structure of very well covered graphs as characterized by Favaron. We note here that this generalization does not change the complexity of the recognition problem for the class W_{AR} as compared to that for the class of very well covered graphs; it remains tractable for both the classes (see [6]). An interesting problem is to see if the class W_{AR} can be generalized further, perhaps by relaxing properties *b*) and/or *c*) of Theorem 2.3, while still preserving the tractability of the recognition problem.

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