A construction of holely perfect Mendelsohn designs*

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ABSTRACT. In this article we give a direct construction of HPMD. As an application, we discuss the existence of $(v, 6, 1)$-PMD and obtain an infinite class of $(v, 6, 1)$-PMD where $v \equiv 4 \pmod 6$.

1 Introduction

The concept of a perfect cyclic design was introduced by N.S. Mendelsohn [18]. This concept was further studied in a subsequent paper [6], where the notation of resolvability was discussed and associations made with certain classes of quasigroups and orthogonal array with interesting conjugacy properties. A development of the concept was made by D.F. Hsu and A.D. Keedwell [16], where the designs were called Mendelsohn designs. In what follows we shall adapt the terminology and notation in [16] and introduce the definitions involving the concept of Mendelsohn designs.

A set of k distinct elements $\{a_1, a_2, \ldots, a_k\}$ is said to be *cyclically ordered* by $a_1 < a_2 < \cdots < a_k < a_1$ and two elements a_i, a_{i+t} are said to be t-apart in a cyclic k-tuple $(a_1, a_2, \ldots, a_k)$ where $i + t$ is taken modulo k.

Let v, k and λ be positive integers. A (v, k, λ)-*Mendelsohn design* (briefly (v, k, λ)-MD) is a pair $(X, \mathcal{B})$ where X is a v-set (of *point*) and $\mathcal{B}$ is a collection of cyclically ordered k-subsets of X (called *blocks*) such that every ordered pair of points of X appears consecutively in exactly λ blocks of $\mathcal{B}$. The (v, k, λ)-MD is called *r-fold perfect* if each ordered pair of points of X appears t-apart in exactly λ blocks for all $t = 1, 2, \ldots, r$. A $(k-1)$-fold perfect (v, k, λ)-MD is called *perfect* and denoted it briefly by (v, k, λ)-PMD.

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In graph notation, a (v, k, λ)-MD is equivalent to the decomposition of the complete directed multigraph λK_v^* on v vertices into k-circuits. A (v, k, λ)-PMD is equivalent to the decomposition of λK_v^* into k-circuits such that for any r, $1 \leq r \leq k-1$, and for any two distinct vertices x and y there are exactly λ circuits along which the distance from x to y is r.

If we ignore the cyclic order of the elements in blocks, a (v, k, λ)-PMD becomes a $B(v, k, \lambda(k-1))$. Therefore, we can consider perfect Mendelsohn designs as a generalization of balanced incomplete block designs.

Since the complete directed multigraph λK_v^* contains $\lambda v(v-1)$ arcs and each block as a circuit contains k arcs, it is easy to see that the number of blocks in a (v, k, λ)-PMD is

$$\frac{\lambda v(v-1)}{k}.$$

This leads to an obvious necessary condition for the existence of a (v, k, λ)-PMD, that is,

$$\lambda v(v-1) \equiv 0 \pmod{k}. \tag{1}$$

This condition is known to be sufficient in many cases, but certainly not in all.

For $k = 3$, it has been shown in [3], [17] that the necessary condition for the existence of a $(v, 3, \lambda)$-PMD is sufficient, except for the non-existing $(6, 3, 1)$-PMD. An alternative proof can be found in [24].

For $k = 4$, Mendelsohn started in [18] the investigation of the existence of $(v, 4, 1)$-PMD, and noticed that a $(v, 4, 1)$-PMD is equivalent to the existence of a quasigroup of order v satisfying certain identities. A partial solution for $v \equiv 1 \pmod 4$ was obtained in Bennett [2]. Zhang [22] discussed the remaining case $v \equiv 0 \pmod 4$. An almost complete solution for the existence of a $(v, 4, \lambda)$-PMD was presented in [11], where $v = 12$ and $\lambda = 1$ is the only unsolved case. F.E. Bennett recently reported finding a $(12, 4, 1)$-PMD. So the necessary condition (1) for the existence of $(v, 4, \lambda)$-PMD is also sufficient, except for $v = 4$ and λ odd, $v = 8$ and $\lambda = 1$.

For $k = 5$, some new constructions by weighting and by k-difference sequence were introduced and an almost complete solution for the existence of a $(v, 5, \lambda)$-PMD was presented in [7], [8]. A $(90, 5, 1)$-PMD, a $(110, 5, 1)$-PMD and a $(130, 5, 1)$-PMD were found in [1] and [25]. A $(86, 5, 1)$-PMD, $(146, 5, 1)$-PMD and $(18, 5, 5)$-PMD was obtained in [13]. Chang [14] obtained a $(v, 5, 1)$-PMD for v = 26, 36, 46, 66, 126, 186, 206, 246. We summarize the results as follows: The necessary condition (1) for the existence of a $(v, 5, \lambda)$-PMD is sufficient, except for $v = 6$ and $\lambda = 1$, and the possible exceptions of (v, λ) where $\lambda = 1$ and $v \in \{10, 15, 20, 30, 50, 56\}$.

For $k = 6$. Miao and Zhu in [19] proved that $(v, 6, 1)$-PMD exists whenever $v > 6$ and $v \equiv 0, 1 \pmod 6$ with at most 150 possible exceptions of which 2604 is the largest. $(6, 6, 1)$-PMD does not exist. When $v \equiv 3, 4 \pmod 6$, although the Wilson's theory on PBD-closure [20] can be used to show that a $(v, 6, 1)$-PMD exists whenever v is in these classes and v is sufficiently large, neither a specific bound on v nor a specific value of v for $v \equiv 3, 4 \pmod 6$ is known. In Section 4 we will give an infinite class of $(v, 6, 1)$-PMD where $v \equiv 4 \pmod 6$.

For $k = 7$, a partial solution has been given in [5], [10]. For recent results on PMDs with some additional properties such as resolvability, incomplete PMDs, PMDs with holes, and perfect Mendelsohn covering designs, the reader is referred to [4], [9], [12], [23].

2 Construction by filling in holes

We denote by $K_{n_1,n_2,\ldots,n_h}$ the complete multipartite directed graph with vertex set $X = \bigcup_{1\le i\le h} X_i$, where X_i $(1 \le i \le h)$ are disjoint sets with $|X_i| = n_i$, $v = \sum_{1\le i\le h} n_i$, and where two vertices x and y from different sets X_i and X_j are joined by exactly λ arcs (x, y) and λ arcs (y, x).

If $K_{n_1,n_2,\ldots,n_h}$ can be decomposed into k-circuits such that for any r, $1 \le r \le k-1$, and for any two vertices x and y from different sets X_i and X_j, there are exactly λ circuits along which the (directed) distance from x to y is r, we call $(X, \mathcal{B})$ a *holely perfect Mendelsohn design*, where $\mathcal{B}$ is the collection of all circuits. We denote the design by (v, k, λ)-HPMD (or (k, λ)-HPMD). The set X_i $(1 \le i \le h)$ is called a *hole* and the vector $(n_1, n_2, \ldots, n_h)$ is said to be the *type* of the HPMD. We sometimes use an "exponential" notation to describe the type of the HPMD.

A (v, k, λ)-HPMD of type $(1, 1, \ldots, 1, n)$ is called an *incomplete perfect Mendelsohn design*, denoted by (v, n, k, λ)-IPMD. It is easy to see that a (v, k, λ)-PMD is indeed a (v, k, λ)-HPMD of type $(1, 1, \ldots, 1)$. We can construct PMD from IPMD by filling in holes.

Lemma 2.1. *If there exist both (v, n, k, λ)-IPMD and (n, k, λ)-PMD, then there exists a (v, k, λ)-PMD.*

Proof: Let $(X, Y, \mathcal{B})$ be the (v, n, k, λ)-IPMD where Y is the hole of size n. Let $(Y, \mathcal{B}_0)$ be the (n, k, λ)-PMD. Then, $(X, \mathcal{B} \cup \mathcal{B}_0)$ is the required (v, k, λ)-PMD. □

Lemma 2.2. *If there exist a (v, k, λ)-HPMD of type $(n_1, n_2, \ldots, n_h)$ and an $(n_i + m, m, k, \lambda)$-IPMD for $2 \le i \le h$, then there exists a $(v + m, n_1 + m, k, \lambda)$-IPMD. Moreover, if there exists an $(n_1 + m, m, k, \lambda)$-IPMD, then there exists a $(v + m, m, k, \lambda)$-IPMD.*

Proof: Let $(X, \mathcal{B})$ be the given (v, k, λ)-HPMD of type $(n_1, n_2, \ldots, n_h)$. X is partitioned into $X_1, \ldots, X_h$, $|Y| = m$, $Y \cap X = \emptyset$. Let $(X_i \cup Y, Y, \mathcal{B}_i)$ be the given (n_i+m, m, k, λ)-IPMD for $2 \le i \le h$. Then

$$(X \cup Y, X_1 \cup Y, (\cup_{2\le i\le h}\mathcal{B}_i) \cup \mathcal{B})$$

is the required $(v+m, n_1+m, k, \lambda)$-IPMD. If we further have the (n_1+m, m, k, λ)-IPMD $(X_1 \cup Y, Y, \mathcal{B}_1)$, then

$$(X \cup Y, Y, (\cup_{1\le i\le h}\mathcal{B}_i) \cup \mathcal{B})$$

is a $(v+m, m, k, \lambda)$-IPMD. □

Lemma 2.3. *If there exist a (v, k, λ)-HPMD of type $(n_1, n_2, \ldots, n_h)$ and an (n_i, k, λ)-PMD for $1 \le i \le h$, then there exists a (v, k, λ)-PMD.*

Proof: It follows by applying Lemma 2.2 with $m = 0$. □

3 A direct construction of HPMD

In this section we give a direct construction on HPMD.

Let $U = \{1, 2, \ldots, u\}$ and $(U, \mathcal{B})$ be a $(u, k, q\lambda_0)$-PMD, where q is a prime power and $q \ge u+2$. Let $d = \binom{u}{2}$. We will construct a (k, λ_0)-HPMD with u holes of size q^d.

Let V be a vector space of dimension d over $GF(q)$ (the field with q elements). As the point set of the (k, λ_0)-HPMD we take $X = V \times U$. The holes will be X_i, $i \in U$ where $X_i = V \times \{i\}$. It remains to describe the set $\mathcal{A}$ of k-circuits.

Since $d = \binom{u}{2}$, we can label the coordinates of each vector of V with the unordered pairs of U. We usually use bold letters to denote a vector in V. The vector $\mathbf{h} \in V$ has a representation

$$\mathbf{h} = \left(h_I : I \in \binom{U}{2}\right),$$

where $\binom{U}{2}$ denotes the set of all unordered pairs of distinct elements of U. Let us define a subset H of V by

$$H = \left\{\mathbf{h} \colon \sum_{I\in\binom{U}{2}} h_I = 0\right\},$$

then H has q^{d-1} vectors.

For any given unordered pair $\{a, b\} \in \binom{U}{2}$ $(a < b)$ and r $(1 \le r \le k-1)$. Since $(U, \mathcal{B})$ is a $(u, k, \lambda_0 q)$-PMD, (a, b) appears r-apart in exactly $\lambda_0 q$ of the blocks of $\mathcal{B}$. So, there is a λ_0-to-one correspondence from these blocks to $GF(q)$. Let

$$f^r_{(a,b)} \colon \{B \in \mathcal{B} \colon (a, b) \text{ appears } r\text{-apart in } B\} \mapsto GF(q)$$

be a λ_0-to-one mapping. As

$$\{B \in \mathcal{B}\colon (a,b) \text{ appears } r\text{-apart in } \mathcal{B}\} \\ = \{B \in \mathcal{B}\colon (b,a) \text{ appears } (k-r)\text{-apart in } \mathcal{B}\},$$

define $f^{k-r}_{(b,a)}(B) = -f^{r}_{(a,b)}(B)$, for any B in which (a,b) appears r-apart. Then $f^{k-r}_{(b,a)}$ is a λ_0-to-one mapping from these blocks to $GF(q)$.

Then for any given B, define a mapping $\phi_B\colon B \mapsto V$ as follows: $\phi_B(i) = (a_I\colon I \in \binom{U}{2})$ and

$$a_I = \begin{cases} f^{t}_{(i,i')}(B) & \text{if } (i,i') \text{ appears } t\text{-apart in } B, \text{ where } I = \{i,i'\} \text{ and } i < i', \\ 0 & \text{otherwise.} \end{cases}$$

For any $i \in U$, we define a map $T_i\colon V \mapsto V$. Let g be primitive element in $GF(q)$. Since $q \geq u+2$, the powers $g, g^2, \ldots, g^u$ are all distinct and different from 1. Now if $\mathbf{h} = (h_I\colon I \in \binom{U}{2})$ is any vector of V, defined each component of $T_i(\mathbf{h})$ by

$$(T_i(\mathbf{h}))_I = \begin{cases} h_I & \text{if } i \in I, \\ h_I g^i & \text{otherwise.} \end{cases}$$

For each triple $(z, \mathbf{h}, B) \in V \times H \times \mathcal{B}$, define a k-circuit

$$A(z, \mathbf{h}, B) = ((z + T_i(\mathbf{h}) + \phi_B(i), i)\colon i \in B)$$

based on X and define a set $\mathcal{A}$ of k-circuits as

$$\mathcal{A} = \{A(z, \mathbf{h}, B)\colon (z, \mathbf{h}, B) \in V \times H \times \mathcal{B}\}.$$

Lemma 3.1. *Suppose that q is a prime power, u is a positive integer such that $q \geq u+2$, and $d = \binom{u}{2}$. Suppose that there exists a $(u, k, q\lambda_0)$-PMD. Then there is a (k, λ_0)-HPMD with type $(q^d)^u$.*

Proof: We verify that $(X, \mathcal{A})$ is a (k, λ_0)-HPMD with type $(q^d)^u$. Direct calculation shows that $\mathcal{A}$ contains $\lambda_0 q^{2d} u(u-1)$ ordered pairs which are r-apart in circuits of $\mathcal{A}$. We only need to show that each ordered pair not contained in a hole appears r-apart in at least λ_0 circuits of $\mathcal{A}$ for all $r = 1, 2, \ldots, k-1$.

For any ordered pair $\{(\mathbf{x}, i), (\mathbf{y}, j)\}$ not contained in a hole, and integer r such that $1 \leq r \leq k-1$, let $\mathbf{w} = \mathbf{x} - \mathbf{y} = (w_I\colon I \in \binom{U}{2})$. Since $f^{r}_{(i,j)}$ is a λ_0-to-one mapping, it takes all values λ_0 times in $GF(q)$ as B ranges through all blocks in which $\{i,j\}$ appears r-apart. So, there are exactly λ_0 blocks $B_1, B_2, \ldots, B_{\lambda_0}$ such that

$$f^{r}_{(i,j)}(B_s) = w_{\{i,j\}}, \quad s = 1, 2, \ldots, \lambda_0.$$

If $i < j$, by the definition of $\phi_{B_s}(i)$

$$(\phi_{B_s}(i) - \phi_{B_s}(j))_{\{i,j\}} = f^{r}_{(i,j)}(B_s) = w_{\{i,j\}}, \quad s = 1, 2, \ldots, \lambda_0.$$

If $i > j$, then

$$(\phi_{B_s}(i) - \phi_{B_s}(j))_{\{i,j\}} = -f^{k-r}_{(j,i)}(B_s) = f^{r}_{(i,j)}(B_s) = w_{\{i,j\}}, \quad s = 1, 2, \ldots, \lambda_0.$$

Define $\mathbf{d}_s$ by

$$\mathbf{d}_s = \mathbf{w} - \phi_{B_s}(i) + \phi_{B_s}(j), \quad s = 1, 2, \ldots, \lambda_0.$$

Then $\mathbf{d}_s$ is a vector of V with $\{i,j\}$ component 0. We define h_s by

$$(\mathbf{h}_s)_I = \begin{cases} (\mathbf{d}_s)_I(g^i - g^j)^{-1} & \text{if } i, j \notin I, \\ (\mathbf{d}_s)_I(1 - g^j)^{-1} & \text{if } i \in I \text{ but } j \notin I, \\ (\mathbf{d}_s)_I(g^i - 1)^{-1} & \text{if } j \in I \text{ but } i \notin I, \\ -\sum_{J \in \binom{U}{2} \setminus \{i,j\}} (\mathbf{h}_s)_J & \text{if } I = \{i,j\}. \end{cases}$$

Since $\sum_{I \in \binom{U}{2}} (h_s)_I = 0$, we obtain $\mathbf{h}_s \in H$, $s = 1, 2, \ldots, \lambda_0$.

By the definition of $T_i(\mathbf{h}_s)$, we have

$$(T_i(\mathbf{h}_s))_I = \begin{cases} -\sum_{J \in \binom{U}{2} \setminus \{i,j\}} (\mathbf{h}_s)_J & \text{if } I = \{i,j\}, \\ (\mathbf{d}_s)_I(1 - g^j)^{-1} & \text{if } i \in I \text{ but } j \notin I, \\ (\mathbf{d}_s)_I(g^i - 1)^{-1} g^i & \text{if } j \in I \text{ but } i \notin I, \\ (\mathbf{d}_s)_I(g^i - g^j)^{-1} g^i & \text{if } i, j \notin I. \end{cases}$$

and

$$(T_j(h_s))_I = \begin{cases} -\sum_{J \in \binom{U}{2} \setminus \{i,j\}} (\mathbf{h}_s)_J & \text{if } I = \{i,j\}, \\ (\mathbf{d}_s)_I(1 - g^j)^{-1} g^j & \text{if } i \in I \text{ but } j \notin I, \\ (\mathbf{d}_s)_I(g^i - 1)^{-1} & \text{if } j \in I \text{ but } i \notin I, \\ (\mathbf{d}_s)_I(g^i - g^j)^{-1} g^j & \text{if } i, j \notin I. \end{cases}$$

Thus, $(T_i(\mathbf{h}_s) - T_j(\mathbf{h}_s))_I = (\mathbf{d}_s)_I$ for any $I \in \binom{U}{2}$, i.e.,

$$T_i(\mathbf{h}_s) - T_j(\mathbf{h}_s) = \mathbf{d}_s = \mathbf{x} - \mathbf{y} - \phi_{B_s}(i) + \phi_{B_s}(j)$$

for $s = 1, 2, \ldots, \lambda_0$. Let

$$\mathbf{z}_s = \mathbf{x} - (T_i(\mathbf{h}_s) + \phi_{B_s}(i)), \quad s = 1, 2, \ldots, \lambda_0.$$

Then, for $s = 1, 2, \ldots, \lambda_0$

$$\begin{cases} \mathbf{x} = \mathbf{z}_s + T_i(\mathbf{h}_s) + \phi_{B_s}(i) \\ \mathbf{y} = \mathbf{z}_s + T_j(\mathbf{h}_s) + \phi_{B_s}(j). \end{cases}$$

By the definition of $A(\mathbf{z}_s, \mathbf{h}_s, B)$, it is easy to see that the ordered pair $((\mathbf{x}, i), (\mathbf{y}, j))$ appear r-apart in λ_0 k-circuits $A(\mathbf{z}_s, \mathbf{h}_s, B_s)$ $(s = 1, 2, \ldots, \lambda_0)$. We complete the proof. □

Corollary 3.2. *Suppose that q is a prime power, u is a positive integer such that $q \geq u+2$ and $d = \binom{u}{2}$. Suppose that there exist a $(u, k, q\lambda_0)$-PMD and (q^d, k, λ_0)-PMD. Then there is a (uq^d, k, λ_0)-PMD.*

Proof: By Lemma 3.1 there is a (k, λ_0)-HPMD with type $(q^d)^u$. Applying Lemma 2.3 we have a (uq^d, k, λ_0)-PMD. □

4 An application

In this section we apply Corollary 3.2 to $(v, 6, 1)$-PMD, and obtain an infinite class v such that $v \equiv 4 \pmod 6$ and a $(v, 6, 1)$-PMD exists.

Lemma 4.1. (Mendelsohn [18]). *Let v be any prime power and $k > 2$ be such that k is a divisor of $v - 1$, then there exists a $(v, k, 1)$-PMD.*

J. Yin [21] investigated the case $\lambda = 3$, we summarized the results as follows:

Lemma 4.2. *For $v \geq 6$, there exists a $(v, 6, 3)$-PMD except for $v \in \{6, 10, 12, 16, 18, 20, 22, 24, 26, 28, 30, 32, 33, 34, 38, 39, 40, 42, 44, 45, 48, 51, 52, 54, 55, 60, 62\}$.*

Lemma 4.3. **(Greig [15]).** *A $B(42t + 22, 7, 4)$ exists for any positive integer $t \neq 1, 2, 5, 11$.*

Lemma 4.4. *Let t and λ be positive integers and $t \neq 1, 2, 5, 11$, $\lambda \geq 6$. There exists a $(42t + 22, 6, \lambda)$-PMD.*

Proof: By Lemma 4.1 there exists a $(7, 6, 1)$-PMD. Any integer $\lambda \geq 6$ can be written as $\lambda = 3a + 4b$, where a, b are non-negative integers. By Lemma 4.2 and Lemma 4.3, there exists a $(42t + 22, 6, \lambda)$-PMD for $t \neq 1, 2, 5, 11$ and $\lambda \geq 6$. □

Theorem 4.5. *Let $v = 42t+22$, $t \neq 1, 2, 5, 11$ and l be an integer satisfying $v + 2 \leq 7^l < 7(v + 2)$. Then there exist a $(v7^{l\binom{v}{2}}, 6, 1)$-PMD.*

Proof: Let $v = 42t + 22$, $t \neq 1, 2, 5, 11$. Let $q = 7^l$. By Lemma 4.4, there exists a $(v, 6, q)$-PMD. Applying Lemma 3.1, we get a $(6, 1)$-HPMD with type $(q^{\binom{v}{2}})^v$. As a $(q^{\binom{v}{2}}, 6, 1)$-PMD exists by Lemma 4.1, a $(vq^{\binom{v}{2}}, 6, 1)$-PMD exists by Corollary 3.2, where

$$vq^{\binom{v}{2}} \equiv v \equiv 4 \pmod 6.$$

□

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