The complexity of consecutive $\Delta-$coloring of bipartite graphs: 4 is easy, 5 is hard

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ABSTRACT. For a given graph G an edge-coloring of G with colors $1, 2, 3, \ldots$ is said to be a consecutive coloring if the colors of edges incident with each vertex are distinct and form an interval of integers. In the case of bipartite graphs this kind of coloring has a number of applications in scheduling theory. In this paper we investigate the question whether a bipartite graph has a consecutive coloring with Δ colors. We show that the above question can be answered in polynomial time for $\Delta \leq 4$ and becomes NP-complete if $\Delta > 4$.

1 Introduction

In this paper we consider a relatively new concept of graph coloring, namely the consecutive edge-coloring problem. Given a coloring of the edges of G with colors $1, 2, 3, \ldots$ the coloring is said to be a *consecutive coloring* if the colors received by the edges incident with each vertex are distinct and form an interval of integers. Not all graphs have such colorings. A simple counterexample is K_3.

The consecutive coloring problem has immediate applications in scheduling theory, in the case when an optimal schedule without waiting periods and idle times is desired. In this application the vertices of a graph correspond to processors or jobs, edges represent unit execution time tasks and colors correspond to assigned time units.

The consecutive edge-coloring problem apparently was first studied under the name of "interval coloring" by Asratian and Kamalian [1] and Sevastjanow [8]. Their papers were devoted mainly to bipartite graphs, and so is ours. In this article we consider a problem of deciding whether any given bipartite graph has a consecutive coloring with Δ colors. We show that this problem can be solved in polynomial time if G is a graph with $\Delta \leq 4$, but it already becomes NP-complete, if $\Delta = 5$. Though algorithms considered herein solve decision problems, but on the basis on this paper it is easy to convert our decision algorithms into optimization ones.

The fact that a given bipartite graph doesn't have a consecutive coloring with Δ colors does not mean that it is not consecutively colorable at all. Many known bipartite graphs have such a coloring. For example, trees, even cycles, complete bipartite graphs, even cacti and $(2, \Delta)$-regular bipartite graphs are consecutively colorable. As it was shown in [8], the problem of deciding whether a bipartite graph has consecutive coloring is NP-complete. The smallest, in the sense of the number of vertices, known bipartite graph which is not consecutively colorable has order 19, and the smallest with respect to the maximum degree has degree $\Delta = 14$. Bipartite graphs with small Δ seem to have consecutive colorings, but this was proved only for $\Delta \leq 3$ [4]. Even the question if all bipartite graphs $G(V_1, V_2)$, whose every vertex from V_1 has degree 3, and every vertex from V_2 has degree 4 have a consecutive coloring is an open problem [6].

2 Factors and matchings

In this article we use $G(V_1, V_2, E)$ to denote a bipartite graph with the partitions V_1 and V_2, and E as the set of its edges. Let $V = V_1 \cup V_2$ and let $V^{(i)} \subseteq V$ denote a set of vertices, with degree i. All graphs considered are simple, bipartite and have no vertices of degree 0, though they need not be connected.

In the following we will need some facts about factorization in graphs.

Definition 2.1. *Let $f : V \mapsto N$ be a function, where V is the vertex set of $G(V, E)$ an N is the set of positive integers. Then by an* f-factor *in G we mean a subset A of E such that for all vertices $v \in V$ the number of edges in A incident with v is equal to $f(v)$. If $k \in N$ then a subset $A \subseteq E$ is* k-factor *if A is f-factor for the constant function $f(v) = k$. A 1-factor in G is called also* a perfect matching.

We will show the polynomial solvability of decision problems concerning the existence of some factors in bipartite graphs. The following theorem is crucial for further considerations. As usual $n = |V|$ and $m = |E|$, and Δ (respectively δ) is the maximum (respectively minimum) vertex degree in graph.

Theorem 2.2. [7] *A maximum cardinality matching in a bipartite graph can be found in $O(m\sqrt{n})$ time.* □

Corollary 2.3. *The problem of existence a perfect matching in a bipartite graph can be solved in polynomial time.* □

Let $G(V_1, V_2, E)$ be a bipartite graph and $f : V \mapsto \{1,2\}$. Suppose that vertices v_1 and v_2 are connected by an edge e and $f(v_1) = f(v_2) = 2$. We construct a bipartite graph G' by deleting edge e and replacing it by a path $v_1 \to v_1' \to v_2' \to v_2$, where v_1' and v_2' are new vertices. We extend function f to f' by setting $f'(v_1') = f'(v_2') = 1$. For these graphs we have

Lemma 2.4. *G has an f–factor if and only if G' has an f'–factor.*

Proof: $\Rightarrow$ Let A be an f–factor in G. If $e \in A$, then $(A - \{e\}) \cup \{\{v_1, v_1'\}, \{v_2, v_2'\}\}$ is an f'–factor in G'; otherwise it is the set $A \cup \{\{v_1', v_2'\}\}$.

$\Leftarrow$ Let A' be an f'–factor in G'. If $\{v_1', v_2'\} \in A'$, then $\{v_1, v_1'\} \notin A'$ and $\{v_2, v_2'\} \notin A'$, so $A' - \{\{v_1', v_2'\}\}$ is an f–factor in G. Otherwise $\{\{v_1, v_1'\}, \{v_2, v_2'\}\} \subseteq A'$, and $(A' \cap E) \cup \{e\}$ is an f–factor in G. □

Now, let G and f be as above. Suppose that there is a vertex u with $f(u) = 2$ and $f(v) = 1$ for every vertex v adjacent to u. We construct a new bipartite graph G' by adding a new vertex u' and connecting it by edges with all vertices which are adjacent to u in G. Consider the function f' such that $f'(u) = f'(u') = 1$ and f' is equal to f for all vertices of G distinct from u. For these graphs we have

Lemma 2.5. *G has an f–factor if and only if G' has an f'–factor.*

Proof: $\Rightarrow$ Let A be an f–factor in G. Then exactly two edges $e_1 \neq e_2$ belong to A and are incident with u. Let $e_2 = \{u, v_2\}$. The set $(A - \{e_2\}) \cup \{\{u', v_2\}\}$ is an f'–factor in G'.

$\Leftarrow$ Let A' be an f'–factor in G', and let $e = \{u, v\} \in A'$, $e' = \{u', v'\} \in A'$. Then the set $(A - \{e'\}) \cup \{\{u, v'\}\}$ is an f–factor in G. □

Theorem 2.6. *For any bipartite graph $G(V_1, V_2, E)$ and a function $f : V \mapsto \{1,2\}$ the problem of existence of f–factor in G is polynomially solvable.*

Proof: Let $e_1, \dots, e_k$ be all the edges of G such that the function f has value 2 on both endpoints of e_i, for $i = 1, \dots, k$. Using the construction of Lemma 2.4 k times we obtain a graph G_1 and a function f_1, such that the existence of an f–factor in G is equivalent to the existence of an f_1–factor in G_1. Every vertex v in G_1 for which $f_1(v) = 2$ is incident only with vertices u with $f_1(u) = 1$. Let $v_1, \dots, v_l$ be all vertices in G_1 with $f_1(v_i) = 2$. Using the construction from Lemma 2.5 l times we reach a graph G_2 and function f_2 such that the existence of f_1–factor in G_1 is equivalent to the existence of f_2–factor in G_2. But f_2 is a constant function equal to 1, so the last problem is simply that of finding a perfect matching, which is polynomial by Corollary 2.3. Since G_2 has at most twice as many vertices as G_1 and G_1 has at most $3m$ edges, so the proof is complete. □

Corollary 2.7. *The problem of the existence of 2–factor in a bipartite graph can be solved in polynomial time.* □

3 Polynomial cases

A finite subset A of Z is called an *interval* if it includes all integers between $\min A$ and $\max A$.

Definition 3.1. *Let G be a graph with edge set E, and let Z be a set of integers. A function $c : E \mapsto Z$ is called a proper* edge-coloring *or simply a* coloring *if for every vertex v of graph G all edges incident with v have different colors.*

Definition 3.2. *A coloring c of graph G is* consecutive at vertex *$v \in V$ if the colors of edges which are incident with v form an interval. The coloring c is called* consecutive *if it is consecutive at every vertex of G.*

Every connected component of a bipartite graph with $\Delta \leq 2$ is a path or an even cycle, so the edges of any component can be consecutively colored with colors 1 and 2, alternately. Therefore we have

Theorem 3.3. *A bipartite graph with $\Delta \leq 2$ can be consecutively colored with Δ colors in linear time.* □

Before we begin considering more difficult cases, we need one more definition.

Definition 3.4. *A* doubling *of graph G* with respect to *$U \subseteq V(G)$ is a supergraph of G obtained by building an isomorphic copy of G, say G', and connecting by single edges all vertices from U with their copies in G'.*

For example a doubling of a path of 2 edges with respect to its two endpoints is a cycle C_6. A doubling of a connected graph with respect to the empty set is a graph which has two connected components, both isomorphic to G. It is easy to see that doubling of a bipartite graph is bipartite.

Theorem 3.5. *A bipartite graph with $\Delta \geq 2$ can be consecutively colored with Δ colors if and only if its doubling with respect to $V^{(1)}$ can also be consecutively colored.*

Proof: First notice that the doubling of G with respect to $V^{(1)}$ has the same maximum degree Δ.

⇒ Given a consecutive Δ-coloring of G with colors $\{1, \ldots, \Delta\}$ we can carry it over to the graph G'. Now we must assign approporiate colors from $\{1, \ldots, \Delta\}$ to the edges connecting vertices of degree 1 in G with their copies in G' in such a way the remaining coloring is consecutive.

⇐ Restriction of a consecutive coloring with Δ colors of the doubling of graph G to $E(G)$ is a consecutive Δ–coloring of G. □

Let us consider the case of bipartite graphs with $\Delta = 3$. Not all of these graphs have consecutive coloring with 3 colors, for example $K_{2,3}$ needs 4 colors in order to color it consecutively.

Lemma 3.6. *A bipartite graph G with $\Delta = 3$ and $\delta \geq 2$ has a consecutive Δ–coloring if and only if it has a perfect matching.*

Proof: $\Rightarrow$ Let c be a consecutive coloring of such G with colors $1, 2, 3$. Since every vertex in G has degree 2 or 3, it must be incident with exactly one edge colored 2. All such edges form a perfect maching in G.

$\Leftarrow$ Let A be a perfect matching in G. We color every edge from A with color 2. The set of remaining edges forms a subgraph, whose every vertex has degree 1 or 2. So its components are paths or even cycles. The edges of any component can be colored alternately with 1 and 3. Together with previously colored edges of A, this makes a consecutive Δ-coloring of graph G. □

Theorem 3.7. *The problem of deciding the existence of consecutive Δ–coloring in a bipartite graph with $\Delta = 3$ can be solved in polynomial time.*

Proof: Let $G^\star$ be a doubling of G with respect to $V^{(1)}$. Then $G^\star$ does not contain a vertex of degree 1. By Theorem 3.5 the existence of consecutive 3–colorings for G and $G^\star$ are equivalent. But by Lemma 3.6 this is equivalent to the existence of a perfect matching in $G^\star$, and the latter is polynomial by Corollary 2.3. □

Now we consider the case of bipartite graphs with $\Delta = 4$. Again not all of these graphs can be consecutively colored with 4 colors, for example it was shown in [5], that every consecutive coloring of $K_{3,4}$ requires at least 6 colors.

Lemma 3.8. *Let G be a bipartite graph with $\Delta \leq 4$. If G has a coloring with colors from set $\{1, \ldots, 4\}$ fulfilling the following properties:*

- *it is consecutive at every vertex of degree 3 or 4,*
- *at every vertex of degree 2 the difference between colors of its edges is 1 or 2,*

then G has a consecutive coloring with colors from set $\{1, \ldots, 4\}$.

Proof: Let c be a coloring of G as required. Denote by $G[i, j]$ the subgraph of G generated by the edges of colors i and j. Obviously such a subgraph contains only vertices of degree not exceeding 2, so its components are paths and even cycles. Denote by $G[i, j](v)$ a component of $G[i, j]$ containing vertex v. Vertex u is a *final vertex* of such a component iff $G[i, j](v)$ is a path and one of its endpoints is u. The coloring c is not consecutive only at vertices of degree 2. The edges of such vertices have colors 1 and 3 or 2 and 4. These vertices will be called *vertices with a gap.* We will give a method to successively eliminate all gaps. First, note that by interchanging the colors of edges in any component of $G[2,3]$ or $G[1,4]$ we obtain again a coloring satisfying assumptions of the lemma. Observe that components of $G[2,3]$ can have final vertices only of degree 1 or 2 in G, and components of $G[1,4]$ only of degree $1,2$ or 3 in G. Now let v_1 be an arbitrary vertex with a gap (if there is no such vertex then the coloring is already consecutive). Then v_1 is a final vertex of a component $G[2,3](v_1)$. Let v_2 be its other final vertex. If v_2 is neither a vertex with a gap nor pendant then it is a final vertex of a component of $G[1,4](v_2)$. Let v_3 be its second final vertex, etc. In this way we form a sequence of vertices $v_1, v_2, \ldots, v_k$, such that v_i, v_{i+1} are both final vertices of the same component of subgraph $G[1,4]$ if i is even, and $G[2,3]$ if i is odd. We finish the construction at vertex v_k, $k > 1$ if one of the following situations holds:

1. v_k is the first vertex after v_1, at which there is a gap,
2. $deg_G(v_k) \in \{1,3\}$,
3. $v_1 = v_k$.

In Cases 1 and 2 we interchange colors successively in components $G[2,3]$ (v_1), $G[1,4](v_2)$, $G[2,3](v_3), \ldots, G[i,j](v_{k-1})$, where i,j depend on the parity of k. In Case 1 this eliminates gaps at v_1 and v_k, and in Case 2 this removes a gap at v_1. In both cases we do not obtain gaps at new vertices, so the number of vertices with gaps decreases. We will show that Case 3 is not possible, hence by repeating the above procedure we eliminate all gaps. Suppose that Case 3 holds. Then the successive edges of paths $G[2,3](v_1)$, $G[1,4](v_2)$, $G[2,3](v_3), \ldots, G[i,j](v_{k-1})$ form a closed chain which begins at v_1 and comes back to v_1. Every two of its successive edges (except the first and the last) have colors with difference 1 or 3, so their parities are different. Since G is bipartite, this chain has even length, and the first of its edges and the last (both incident to v_1) have colors of different parity. On the other hand, we know that the difference of these colors is 2 and we obtain a contradiction. This completes the proof. □

Next fact was first proved in [4] (see also [6]).

Corollary 3.9. *A bipartite graph with $\Delta = 3$ is consecutively colorable with at most 4 colors.*

Proof: These graphs being bipartite can be edge-colored with $1,2,3$, and this coloring meets the assumptions of Lemma 3.8. □

Lemma 3.10. *Let G be a biparite graph with $\Delta = 4$. Then G has a consecutive 4−coloring if and only if there is a consecutive 4-coloring of its doubling with respect to $V^{(2)}$.*

Proof: Note that the doubling of G has the same maximum degree 4.

$\Rightarrow$ The proof is similar to that of Theorem 3.5. Given a consecutive coloring of G with colors $1, \ldots, 4$, we can carry it over to its isomorphic copy G'. Now we assign appropriate colors to the edges connecting vertices of degree 2 in G with their copies in G'. Let v be such a vertex and let c and $c+1$ be colors of edges incident with v in G. Then at least one of numbers $c-1$ or $c+2$ belongs to $\{1, \ldots, 4\}$ and this color can be assigned to the edge joining v with its copy. We repeat this for all vertices from $V^{(2)}$.

$\Leftarrow$ If we have a consecutive coloring of a doubling of graph G with colors $\{1, \ldots, 4\}$, than the restriction of this coloring to $E(G)$ is a coloring of G which meets the assumptions of Lemma 3.8. □

Lemma 3.11. *Let G be a bipartite graph with $\Delta = 4$ and $\delta \geq 3$. Then G is consecutively 4-colorable if and only if G has a 2–factor.*

Proof: $\Rightarrow$ Suppose we have a consecutive coloring of G with colors $1, \ldots, 4$. Every vertex has degree 3 or 4, so it is incident with exactly one edge colored with 2 and exactly one colored with 3. This set of edges which have colors 2 or 3 is a 2–factor.

$\Leftarrow$ Let A be a 2–factor in G. The edges of A form a 2– regular bipartite subgraph, so its components are even cycles. Their edges can be colored alternately by 2 and 3. The subgraph generated by the edges not belonging to A is bipartite and all its vertices have degree 1 or 2, hence its components are paths or even cycles. We color them alternately by 1 and 4. The union of these two partial colorings forms a consecutive coloring of graph G. □

Corollary 3.12. *A bipartite graph G with $\Delta = 4$ and no vertex of degree 3 is consecutively colorable with 4 colors.*

Proof: Assume that $G(V, E)$ has no pendant vertices. Then G is Eulerian and has an even number of edges. So we can mark the edges in any Eulerian cycle of G with black and white, alternately. Then the set $A \subseteq E$ of all black edges is an $f-$ factor in G, where $f(v) = deg(v)/2$ for all $v \in V$. Let $G^*(V^*, E^*)$ be the doubling of G with respect to $V^{(2)}$ composed of G and its isomorphic copy G'. Set $A^* \subseteq E^*$ containing all the edges from A, the isomorphic copy of A in G' and the edges connecting vertices of degree 2 in G with their copies in G' is a 2–factor in G^*. Hence G^* meets the assumptions of Lemma 3.11 and by Lemma 3.10 the corollary follows.

If G has vertices of degree 1 then its doubling with respect to $V^{(1)}$ has no pendant vertices and therefore is consecutively colorable with 4 colors. Restriction of such a coloring to $E(G)$ is a consecutive coloring of G. □

Theorem 3.13. *The problem of deciding the existence of consecutive Δ–coloring of a bipartite graph with $\Delta = 4$ can be solved in polynomial time.*

Proof: Denote by $G^*(V_1^*, V_2^*)$ a doubling of G with respect to $V^{(1)}$. Then G^* contains no pendant vertex. By Theorem 3.5 the existence of consecutive 4−colorings for G and G^* are equivalent. Next let G^{**} be a doubling of G^* with respect to $V^{*(2)}$. Then G^{**} contains no vertex of degree 1 or 2 and by Lemma 3.10 the existence of consecutive 4−colorings for G^* and G^{**} are equivalent. Lemma 3.11 implies that the later problem is equivalent to the existence of a 2− factor in G^{**}, which is polynomial by Corollary 2.7. □

4 NP-completeness result

We will use the following fact shown in [2]:

Theorem 4.1. *Let G a bipartite graph with $\Delta = 3$ in which some of pendant edges are precolored with 0 or 1. The problem of deciding whether that coloring can be extended to a proper coloring (not necessarily consecutive) of graph G using colors $0, 1, 2$ is NP-complete.* □

Corollary 4.2. *The above problem remains NP-complete even if restricted to graphs having no vertices of degree 2.*

Proof: Let G be a graph meeting the assumptions of Theorem 4.1 with some pendant edges precolored with 0 and 1, and let G' be a graph obtained from G by attaching to each vertex of degree 2 a new pendant edge. Then the existence of a coloring using colors $0, 1, 2$ is equivalent to the existence of such a coloring for graph G'. □

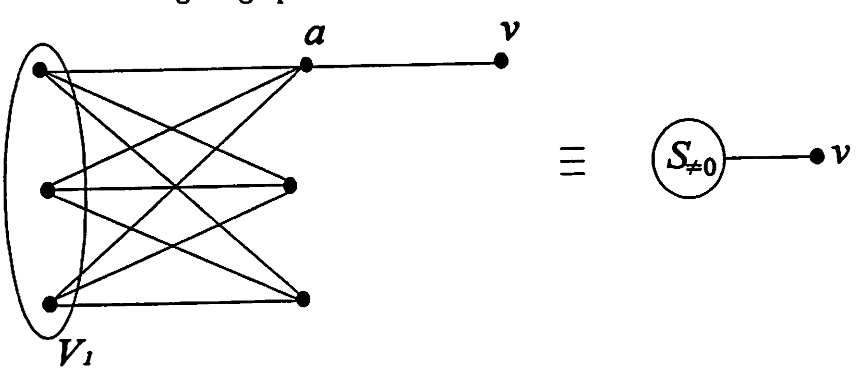

Figure 4.1. Graph $S_{\neq 0}$

Now let us consider some bipartite graphs with $\Delta = 4$ which can be colored consecutively using 5 colors from set $L = \{-2, -1, 0, 1, 2\}$. Note that every interval containing at least 3 elements from this set must include number 0. The first graph under consideration is shown in Figure 4.1. This graph can be colored by taking any edge-coloring of $K_{3,3}$ with $-2, -1, 0$ and assigning to edge $\{a, v\}$ color 1. By adding 1 to all colors we obtain a coloring with color 2 on edge $\{a,v\}$ and by multiplying them by -1 we obtain colorings with colors respectivelly -1 or -2 on this edge. Of course, all these four colorings are consecutive and only use colors from L. Note that graph cannot be colored consecutively with colors from L by assigning to the pendant edge color 0. In fact, since such a coloring is consecutive, exactly one edge outgoing from each vertex of V_1 has color 0. Therefore a is connected by edge of color 0 to one vertex of V_1 and therefore this color cannot be used for edge $\{a,v\}$. This graph will be denoted by $S_{\neq 0}$.

The graph S_0 is obtained by joining two graphs $S_{\neq 0}$ by their pendant vertices and adjoining to this common vertex a new pendant edge as shown in Figure 4.2. This graph can be consecutively colored with colors from L only if the pendant edge has color 0.

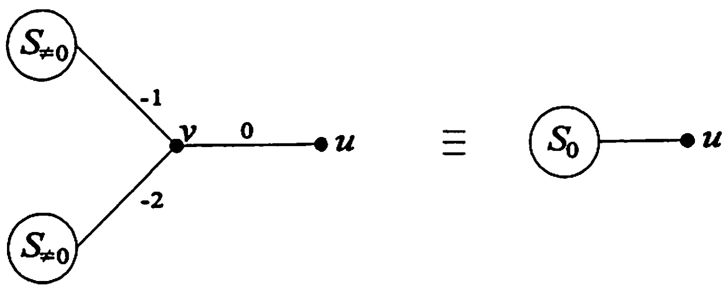

Figure 4.2. Graph S_0

Figure 4.3. Graph $S_{\pm 1}$

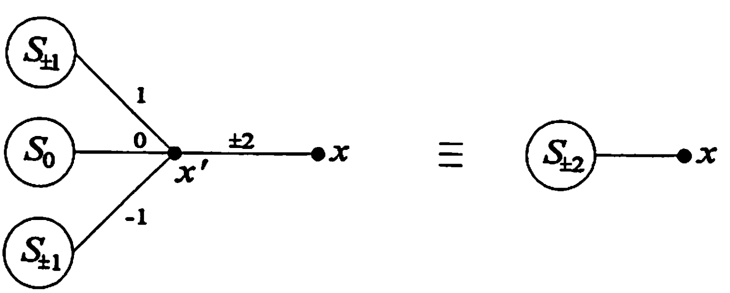

Figure 4.4. Graph $S_{\pm 2}$

The graph $S_{\pm 1}$ is constructed by attaching another pendant edge to vertex u in S_0. This graph can be consecutively colored with colors from L and the new pendant edge may have only colors 1 or -1.

Finally, we define the graph $S_{\pm 2}$ as shown in Figure 4.4. The pendant edge must have color 2 or -2. This is so because vertex x' is of degree 4, and one of the edges incident to it must have color 2 or -2, but it cannot be any of the edges of graphs $S_{\pm 1}$ and S_0.

Now we are ready to prove that the question whether a given bipartite graph with $\Delta = 5$ can be consecutively 5–colored is NP-complete. We will show even more:

Theorem 4.3. *Let G be a bipartite graph with $\Delta = 5$. It is NP-complete to decide whether G has a consecutive 5-coloring even if G is consecutively colorable.*

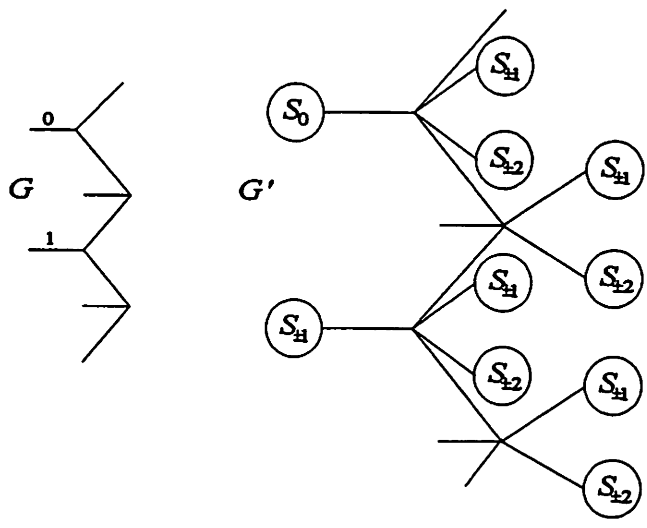

Figure 4.5. Examples of reduction

Proof: We will give a polynomially bounded reduction from the coloring problem of Corollary 4.2. Let G be a bipartite graph with $\Delta = 3$ having no vertex of degree 2 and such that some of its pendant edges are precolored with 0 or 1. We build a supergraph G' of graph G as follows: to every vertex of degree 3 we adjoin two graphs $S_{\pm 1}$ and $S_{\pm 2}$ (we call them *stabilizing graphs*) and every pendant edge precolored with 0 or 1 is replaced by a pendant edge outgoing from a copy of S_0 or $S_{\pm 1}$, respectively (we call them *coloring graphs*). An example of such a reduction is given in Figure 4.5. Of course, G' is bipartite with maximum degree 5. Because $S_0, S_{\pm 1}, S_{\pm 2}$ are consecutively colorable, so G' has a consecutive coloring (possibly with more then 5 colors). We now need to show that a precoloring of G can be extended to a 3-coloring with colors $0, 1, 2$ iff graph G' can be consecutively colored with colors from L.

$\Rightarrow$ Suppose we have a proper coloring of graph G with colors $0, 1, 2$. Then the colors of the edges at any vertex of degree 3 form interval $\{0, 1, 2\}$. All stabilizing graphs $S_{\pm 1}$ and $S_{\pm 2}$ can be colored by giving to their outgoing edges colors -1 and -2, respectively. Similary, all coloring graphs S_0 and $S_{\pm 1}$ are colored so that their outgoing edges are assigned 0 and 1, which meets precoloring conditions. This coloring is cleary consecutive.

$\Leftarrow$ Suppose we have a precoloring of some pendant edges of G and a consecutive coloring of G' with colors $-2, -1, 0, 1, 2$. Let us assign to each edge of G a number equal to the modulus of its color in G', i.e. $0, 1$ or 2. From the properties of the graphs S_0 and $S_{\pm 1}$ it follows that the pendant edges which were precolored in G get numbers equal to their colors in this precoloring. Now, we have just to examine if the assigned numbers give a proper coloring of the graph G, i.e. no vertex of degree 3 has two edges with the same number. Let v be such a vertex. Note that $deg_{G'}(v) = 5$ and there are 5 different colors on its edges: one with color 0, two with colors ± 1 and two with colors ± 2. Exactly two of the edges incident to v do not belong to G - they are outgoing from two stabilizing graphs: $S_{\pm 1}$ (it has color ± 1) and $S_{\pm 2}$ (it has color ± 2). The remaining 3 edges have colors as follows: the first 0, second ± 1 and third ± 2. Then after computing absolute values we obtain a proper coloring of G. □

Using similar methods we can generalize the above theorem to any $k > 4$.

Corollary 4.4. *Let k be any integer greater then 4 and G a bipartite graph with $\Delta = k$. It is NP-complete to decide whether G has a consecutive k-coloring even if G is consecutively colorable.* □

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