# $P_3$-Connected Graphs of Minimum Size

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ABSTRACT. A graph without 4-cycles is called *$C_4$-free*. A $C_4$-free graph is *$C_4$-saturated* if adding any edge creates a 4-cycle. Ollmann showed that any $n$-node $C_4$-saturated graph has at least $\frac{3}{2}n - 3$ edges. He also described the set of all $n$-node $C_4$-saturated graphs with $\lceil \frac{3}{2}n \rceil - 3$ edges. A graph is *$P_3$-connected* if each pair of nonadjacent nodes is connected by a path with exactly 3 edges. A $C_4$-saturated graph is $P_3$-connected, but not vice versa. We generalize Ollmann's results by proving that any $n$-node $P_3$-connected graph has at least $\frac{3}{2}n - 3$ edges. We also describe the set of all $n$-node $P_3$-connected graphs with $\lceil \frac{3}{2}n \rceil - 3$ edges. This is a superset of Ollmann's set as some $n$-node $P_3$-connected graphs with $\lceil \frac{3}{2}n \rceil - 3$ edges do have 4-cycles.

Let $P_k$ be a path on $k$ edges. Two nodes are *$P_k$-connected* if they are connected by a $P_k$. A graph is *$P_k$-connected* if all pairs of nonadjacent nodes are $P_k$-connected. The only $P_1$-connected graphs are complete graphs. A graph is $P_2$-connected if and only if its diameter is at most 2. In general, a $P_k$-connected graph has diameter at most $k$. However, a diameter $k$ graph need not be $P_k$-connected. For example, $P_k$ for $k \geq 3$ has diameter $k$ and yet is not $P_k$-connected.

Let $C_k$ be a cycle with $k$ edges. A graph without $C_k$ as a (not necessarily induced) subgraph is called *$C_k$-free*. A *$C_k$-saturated* graph is a maximal $C_k$-free graph in the sense that adding any edge creates a $C_k$. If adding an edge between two nodes creates a $C_k$, then a $P_{k-1}$ must connect the nodes. So a $C_k$-saturated graph is $P_{k-1}$-connected. However, a $P_{k-1}$-connected graph need not be $C_k$-saturated (e.g., Figure 3 (d) and (e)).

Ollmann [1] answered the question: *What is the minimum number of edges in an $n$-node $C_4$-saturated graph?* He showed that all $C_4$-saturated graph have at least $\frac{3}{2}n-3$ edges. He also described all $C_4$-saturated graphs with $\lceil\frac{3}{2}n\rceil-3$ edges showing the answer is $\lceil\frac{3}{2}n\rceil-3$ for all $n>4$. Tuza [2] gave simpler proofs of Ollmann's results.

*What is the minimum number of edges in an n-node $P_3$-connected graph?* Theorem 1 shows the answer is again $\lceil\frac{3}{2}n\rceil-3$ for all $n>4$. Since a $C_4$-saturated graph is $P_3$-connected, this is a generalization of Ollmann's bound. Theorem 2 describes all $P_3$-connected graphs with $\lceil\frac{3}{2}n\rceil-3$ edges. In addition to the three families given by Ollmann (Figure 3 (a), (b), and (c)), we find two new families (Figure 3 (d) and (e)) of $P_3$-connected graphs that are not $C_4$-saturated.

**Lemma 1.** *Let $G$ be an $n$-node connected graph where every edge is in a $C_3$. Then $G$ has at least $\frac{3}{2}(n-1)$ edges.*

**Proof:** Any connected graph has a spanning tree $T$ with $n-1$ edges. An edge of $G-T$ can form a $C_3$ with at most two edges of $T$. So $G-T$ has at least $(n-1)/2$ edges. □

**Theorem 1.** *Let $G$ be an $n$-node $P_3$-connected graph. Then $G$ has at least $\lceil\frac{3}{2}n\rceil-3$ edges.*

**Proof:** This is trivial for $n=1$. Assume $n\geq 2$. Let $\delta$ be the minimum degree of the nodes of $G$. Since $G$ is connected, $\delta>0$. If $\delta\geq 3$, then $G$ has at least $\frac{3}{2}n$ edges. So assume $\delta=1$ or 2.

**Case 1:** $\delta=1$. Let $v$ be a degree 1 node with neighbor $w$. Let $T$ be the breadth-first tree rooted at $v$. Let $X$ and $Y$ be the nodes that are distance 2 and 3, respectively, from $v$. Since all nodes are distance at most three from $v$, all nodes save $v$ and $w$ are in $X$ or $Y$. Thus $|X|+|Y|=n-2$. Let $x\in X$. Since $x$ is not adjacent to $v$, there is a path $v,w,q,x$. Since $q$ is distance 2 from $v$, we have $q\in X$ and hence the nodes in $X$ have degree 1 or more within $X$.

Let $Y_1$ be the degree 1 nodes of $Y$ and $k=|Y_1|$. Let $Y_2=Y-Y_1$. Let $X_1$ be the nodes of $X$ adjacent to $Y_1$ and let $X_2=X-X_1$. Then for all $y_1,y_2\in Y_1$, there is a path $y_1,x_1,x_2,y_2$ where $x_1,x_2\in X_1$. Since $x_1\neq x_2$, each node in $X_1$ is adjacent to exactly one node in $Y_1$. Further, every pair of nodes in $X_1$ is adjacent. Thus $|X_1|=k$ and $X_1$ is a clique (see Figure 1). Since each node in $X_2$ and $Y_2$ is incident to at least one edge not in $T$, the number of edges not in $T$ is

$$e(G-T)\geq\binom{|X_1|}{2}+\frac{|X_2|+|Y_2|}{2}=\binom{k}{2}+\frac{n-2-2k}{2}\geq\frac{n-4}{2}.$$

Thus the number of edges in $G$ is $e(G)\geq(n-4)/2+n-1=\frac{3}{2}n-3$.

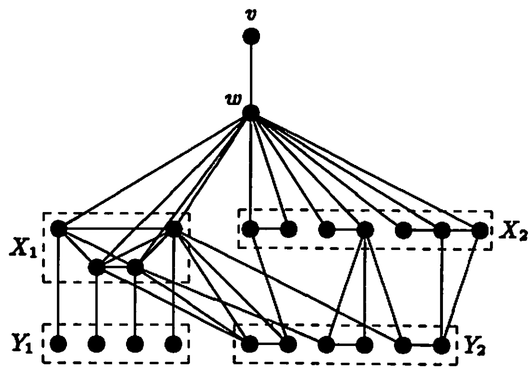


**Figure 1.** The construction in Case 1 of Theorem 1.

**Case 2: $\delta = 2$ and a degree 2 node is not in a $C_3$.** Let $v$ have degree 2 with nonadjacent neighbors $w_1$ and $w_2$. Let $T$ be the breadth-first tree rooted at $v$. Let $X$ and $Y$ be the nodes which are distance 2 and 3, respectively, from $v$ (see Figure 2). Since a $P_3$ connects $v$ to $x \in X$, there is an edge incident to $x$ that is not in $T$. Further since $\delta = 2$, each $y \in Y$ is also incident to an edge not in $T$. So $e(G-T) \geq (|X|+|Y|)/2 = (n-3)/2$, and hence the number of edges in $G$ is $e(G) \geq n-1+(n-3)/2 = \frac{3}{2}n - \frac{5}{2}$.

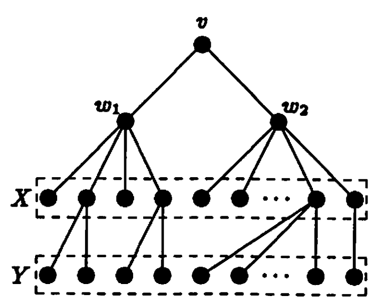


**Figure 2.** The tree formed in Case 2 of Theorem 1.

**Case 3: $\delta = 2$ and each degree 2 node is in a $C_3$.** Let $S$ be the subgraph of all $C_3$'s with one or more degree 2 nodes. Let $S_1, S_2, \ldots, S_m$ be the components of $S$. Lemma 1 shows that $e(S_i) \geq \frac{3}{2}(|S_i| - 1)$. Since the degree 2 nodes of two components are $P_3$-connected, there is an edge between every pair of components of $S$. So there are at least $\binom{m}{2}$ edges between components of $S$. Then the number of edges in the subgraph induced on the nodes of $S$ is

$$e(\langle S\rangle) \geq \sum_{i=1}^{m} e(S_i) + \binom{m}{2} \geq \sum_{i=1}^{m} \frac{3}{2}(|S_i| - 1) + \binom{m}{2}$$
$$= \frac{3}{2}|S| + \frac{m^2}{2} - 2m \geq \frac{3}{2}|S| - 2.$$

Since all degree 2 nodes are in $S$, nodes not in $S$ have degree 3 or more. Thus the number of edges in $G$ is $e(G) \geq \frac{3}{2}|S| - 2 + \frac{3}{2}(n - |S|) = \frac{3}{2}n - 2$. □

**Theorem 2.** *Let $G$ be a $P_3$-connected graph with $n$ nodes and $\lceil \frac{3}{2}n \rceil - 3$ edges. Then $G$ is in one of the families of graphs in Figure 3.*

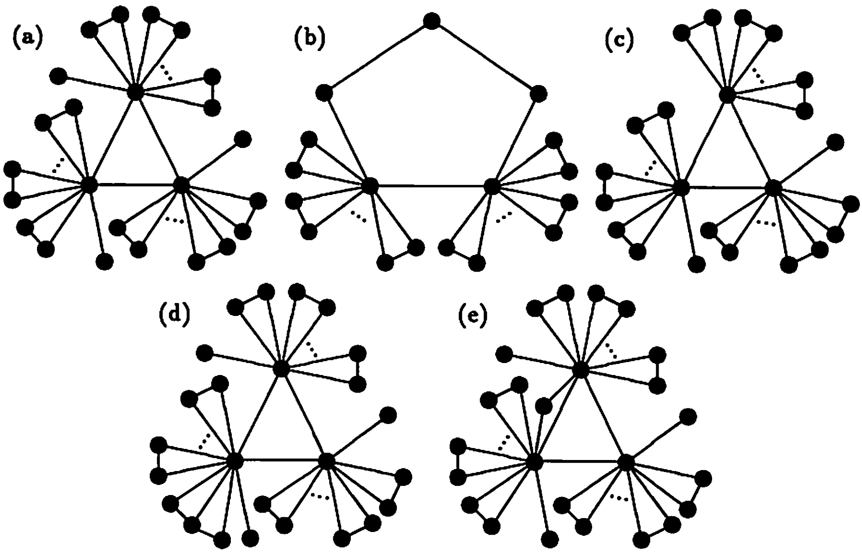


**Figure 3. Families of $n$-node $P_3$-connected graphs with $\lceil \frac{3}{2}n \rceil - 3$ edges.** The dots indicate an arbitrary number (including zero) of pendant $C_3$'s.

**Proof:** In Theorem 1, Case 3 had at least $\frac{3}{2}n - 2$ edges which is greater than $\lceil \frac{3}{2}n \rceil - 3$ for all $n$. So we only need consider Cases 1 and 2.

**Case 1:** $\delta = 1$. Let $v$, $w$, $X$, $Y$, $X_1$, $Y_1$, $X_2$, $Y_2$ and $T$ be as they were in Case 1 of Theorem 1. For $e(G) = \lceil \frac{3}{2}n \rceil - 3$, either $|Y_1| = 1$ or 2, and either $e(G-T) = (n-4)/2$ if $n$ is even or $e(G-T) = (n-3)/2$ if $n$ is odd.

**Case 1a:** $|Y_1| = 1$. Let $X_1 = \{x_1\}$ and $Y_1 = \{y_1\}$. Since every node in $X$ and $Y_2$ has degree 1 or more in $G - T$, we have $e(G-T) \geq (n-3)/2$. So $n$ is odd and $e(G-T) = (n-3)/2$. Since $e(G-T)$ is half the number of nodes in $X$ and $Y_2$, every node in $X$ and $Y_2$ has degree one in $G-T$. Further, since each node of $X$ is adjacent to a node of $X$ and these edges are not in $T$, no edge of $G-T$ connects a node of $X$ to a node of $Y_2$. Thus every node in $Y_2$ is adjacent to exactly one node in $X$ (this edge is in $T$) and exactly one node in $Y_2$ (this edge is not in $T$). Further, every node in $X$ is adjacent to exactly one node in $X$.

Let $x_2 \in X_2$ so that $x_1x_2$ is an edge. Let $y_2y_3$ be an edge in $Y_2$. Since $x_1$ is not adjacent to any node in $X$ other than $x_2$, for $y_2$ to be $P_3$-connected to $y_1$, we have that either $y_2x_2$ or $y_3x_1$ is an edge.

First assume $y_2x_2$ is an edge. Since each node in $Y_2$ is incident to only one node of $X$ and only one node of $Y_2$, the only nodes adjacent to $y_2$ are $y_3$ and $x_2$. Then for $y_3$ to be $P_3$-connected to $y_1$, we have that $y_3x_2$ is an edge. Further since $y_3$ is adjacent to only one node of $X$, we have that $y_3$ is not adjacent to $x_1$. Otherwise assume $y_3x_1$ is an edge. Then we can similarly show that $y_2x_1$ is an edge, and $y_2$ is not adjacent to $x_2$. Either way, $y_2$ and $y_3$ are either both adjacent to $x_1$ or both adjacent to $x_2$. So the ends of any edge in $Y_2$ are either both adjacent to $x_1$ or both adjacent to $x_2$. Thus $G$ is in Family (c) where $w, x_1, x_2$ is the central $C_3$, and $vw$ and $x_1y_1$ are pendant edges.

**Case 1b:** $|Y_1| = 2$. Let $X_1 = \{x_1, x_2\}$ and $Y_1 = \{y_1, y_2\}$ where $x_1y_1$ and $x_2y_2$ are edges. Since $X_1$ is a clique, $x_1x_2$ must also be an edge.

First assume $n$ is even, so $e(G-T) = (n-4)/2$. For the inequalities to be equalities, each of the $n-4$ nodes in $X$ and $Y_2$ has degree 1 in $G-T$, and there are no edges between $X_1$ and $X_2$. Then arguments similar to those in Case 1a show that every node in $Y_2$ is adjacent to exactly one node in $X$ and exactly one node in $Y_2$, every node in $X$ is adjacent to exactly one other node in $X$, and the ends of any edge in $Y_2$ are either both adjacent to $x_1$ or both adjacent to $x_2$. So $G$ is in Family (a) where $w, x_1, x_2$ is the central $C_3$ and $vw$, $x_1y_1$ and $x_2y_2$ are pendant edges.

Otherwise $n$ is odd and $e(G-T) = (n-3)/2$. Since $|X| + |Y_2| = n-4$ and each node in $X$ and $Y_2$ has degree 1 or more in $G-T$, every node except one in $X$ and $Y_2$ has degree 1 in $G-T$. The exceptional node has degree 2.

First assume the exceptional node is in $X_1$ and without loss of generality, assume it is $x_2$. Since $x_1x_2$ is an edge, $x_2$ is adjacent in $G-T$ to some other node in $X_2$ or $Y_2$. If $x_2$ is adjacent to a node of $X_2$, which we can call $x_3$, then by arguments similar to Case 1a, we have that $G$ is in Family (e) where $w, x_1, x_2$ is the central $C_3$, $vw$, $x_1y_1$ and $x_2y_2$ are pendant edges, and $x_3$ is adjacent to both $w$ and $x_2$. If $x_2$ is adjacent to a node of $Y_2$, which we can call $y_3$, then $x_1y_3$ must be an edge in order that $y_3$ is $P_3$-connected to $y_2$. So by arguments similar to Case 1a, $G$ is in Family (e) where $w, x_1, x_2$ is the central $C_3$, $vw$, $x_1y_1$ and $x_2y_2$ are pendant edges, and $y_3$ is adjacent to both $x_1$ and $x_2$.

Next assume the exceptional node, which we will call $x_3$, is in $X_2$. Then $x_3$ cannot be adjacent to $x_2$ because then $x_2$ would also have degree 2 in $G-T$. Similarly $x_3$ is not adjacent to $x_1$. Suppose $x_3$ is adjacent to $y_3 \in Y_2$. Then $y_3$ cannot be adjacent to any node of $Y_2$, because otherwise $y_3$ would have degree 2 in $G-T$. Since $y_3$ is $P_3$-connected to $y_1$, we must have that $y_3x_2$ is an edge. Since $y_3$ is $P_3$-connected to $y_2$, we must have that $y_3x_1$ is an edge. However, at most one of edges $y_3x_2$ and $y_3x_2$ can be in $T$. Then $y_3$ would have degree 2 or more in $G-T$. So $x_3$ is not adjacent to any node in $Y_2$ in $G-T$. The only other possibility is that $x_3$ is adjacent to two other nodes in $X_2$, which we will call $x_4$ and $x_5$. Then by arguments similar to Case 1a, $G$ is in Family (d) where $w, x_1, x_2$ is the central $C_3$, $vw$, $x_1y_1$ and $x_2y_2$ are pendant edges, and $x_4, x_3, x_5$ is the $P_2$ adjacent to $w$.

Finally, assume the exceptional node which we will call $y_3$ is in $Y_2$. Suppose $y_3$ is adjacent to $x \in X$. Since $x$ is adjacent to a node in $X$, its degree within $G-T$ is two or more. So $y_3$ can only be adjacent to two nodes of $Y_2$, called them $y_4$ and $y_5$. Then by arguments similar to Case 1a, $G$ is in Family (d) where $w, x_1, x_2$ is the central $C_3$, $vw$, $x_1y_1$ and $x_2y_2$ are pendant edges, and $y_4, y_3, y_5$ form the $P_2$ which is adjacent to either $x_1$ or $x_2$.

**Case 2: $\delta = 2$ and a degree 2 node is not in a $C_3$.** Let $v$, $w_1$, $w_2$, $X$, $Y$ and $T$ be as defined in Case 2 of Theorem 1. Since $e(G) \geq \frac{3}{2}n - \frac{5}{2}$, we have that $n$ is odd, $e(G-T) = (n-3)/2$, and every node of $X$ and $Y$ has degree 1 in $G-T$. Since $v$ is $P_3$-connected to each $x \in X$, every $x \in X$ has a neighbor in $X$. Since all nodes of $X$ are adjacent to another node of $X$ and these edges are in $G-T$, there are no edges between $X$ and $Y$ in $G-T$. So each node in $Y$ is adjacent to exactly one node of $X$ (by an edge of $T$) and exactly one node of $Y$ (by an edge of $G-T$).

Let $X_i$ be the neighbors of $w_i$ in $X$. Then $X_1$ and $X_2$ are both nonempty as both $w_1$ and $w_2$ have degree 2 or more. Suppose $x \in X_1$ and $x \in X_2$. Then both $xw_1$ and $xw_2$ are edges. However, only one of these edges can be in $T$. Since $x$ has degree 1 in $G-T$ and $x$ is adjacent to a node of $X$, node $x$ cannot exist. So $X_1$ and $X_2$ are disjoint.

Suppose there exists a pair of nodes $x_1 \in X_1$ and $x_2 \in X_2$ where $x_1$ has no neighbors in $X_2$ and $x_2$ has no neighbors in $X_1$. Since $x_1$ cannot be adjacent to $x_2 \in X_2$, there is a path $x_1, a, b, x_2$. Our choice of $x_1$ and $x_2$ guarantees that $a \notin X_2$ and $b \notin X_1$. Further, $a$ and $b$ cannot both be in $X$, because otherwise $a$ and $b$ would both have degree 2 within $X$. Without loss of generality, assume $a \notin X$. If $a = w_1$, then $b \in X_1$, a contradiction. Thus $a \in Y$. Since no node in $Y$ is adjacent to two nodes in $X$, we have $b \in Y$. Then $b$ is only adjacent to $a$ and $x_2$, and $a$ is only adjacent to $b$ and $x_1$. Since $b$ is adjacent to $x_1$, there is a path $b, p, q, x_1$. If $p = a$, then $q = x_1$, a contradiction. If $p = x_2$, then $q$ is adjacent to both $x_1$ and $x_2$. If $q \in X$, then both edges $qx_1$ and $qx_2$ are in $G-T$. Since $q$ has degree one in $G-T$, this is impossible. If $q \in Y$, then one of the edges $qx_1$ and $qx_2$ is not in $T$. Since there are no edges of $G-T$ between $X$ and $Y$, this is also impossible. So $b$ is not $P_3$-connected to $x_1$, a contradiction. Thus no pair of nodes are of this type.

Suppose $|X_1| > 1$ and $|X_2| > 1$. Then at least two edges are between $X_1$ and $X_2$, for otherwise a pair of the type precluded above would occur. Let $x_1x_3$ and $x_2x_4$ be these edges where $x_1, x_2 \in X_1$ and $x_3, x_4 \in X_2$. Since $x_i$ is adjacent to only one node in $X$, we have $x_1$ is not adjacent to $x_2$ and $x_3$ is not adjacent to $x_4$. So there are paths $x_1, a, b, x_2$ and $x_3, c, d, x_4$. Outside of $Y$, the only nodes adjacent to $x_1$ are $w_1$ and $x_3$. If $a = w_1$, then $b = v$ or $b \in X_1$. Since $x_2$ is not adjacent to $v$ nor any node in $X_1$, we have $a \neq w_1$. If $a = x_3$, then $b = w_2$ or $b \in Y$. Since $x_2$ is not adjacent to $w_2$, and each node in $Y$ is adjacent to exactly one node in $X$, we have $a \neq x_3$. So $a \in Y$. Similarly, $b, c, d \in Y$. Since each node in $Y$ is adjacent to exactly one node in $Y$, we have $a, b, c, d$ are distinct. Since the neighbors of $b$ are $a$ and $x_2$, the neighbors of $c$ are $d$ and $x_3$, and there are no edges between $\{a, x_2\}$ and $\{d, x_3\}$, there is no $P_3$ between $b$ and $c$, a contradiction. So either $|X_1| = 1$ or $|X_2| = 1$.

Without loss of generality, assume $X_1 = \{x_1\}$ and $X_2 = \{x_2, x_3, \ldots, x_k\}$ where $x_1x_2$ is an edge. Let $Y_1$ be the nodes of $Y$ adjacent to $x_1$. Recall that each node of $Y$ is adjacent to exactly one node in $X$ and exactly one node in $Y$.

Assume $|X_2| = 1$. Let $Y_2 = Y - Y_1$. Since $w_1$ is $P_3$-connected to nodes of $Y_1$, every node in $Y_1$ is adjacent to another node in $Y_1$. Similarly, every node in $Y_2$ is adjacent to another node in $Y_2$. So the edges in $Y_1$ are independent, the edges in $Y_2$ are independent, and there are no edges between $Y_1$ and $Y_2$. So $G$ is in Family (b) with $v, w_1, x_1, x_2, w_2$ forming the central $C_5$ with only $x_1$ and $x_2$ having pendant $C_3$'s.

Otherwise $|X_2| > 1$. Let $y_1 \in Y_1$. Let $y_2$ be the neighbor of $y_1$ in $Y$. If $y_2x_1$ is an edge, then $y_2$ is not $P_3$-connected to $x_3$. If $y_2x_2$ is an edge, then $y_2$ is not $P_3$-connected to $w_2$. If $y_2x_i$ is an edge for $i > 2$, then $y_2$ is not $P_3$-connected to $x_1$ because $x_2$ is not adjacent to $x_i$. So $y_2$ is not adjacent to $X$ and hence $y_1$ cannot exist. Therefore, the only neighbors of $x$ are $w_1$ and $x_2$. Since $w_1$ is adjacent to only $v$ and $x_1$ which in turn are only adjacent to one other node, we can replace $v$ by $w_1$ in the argument from the previous paragraph. So $G$ is in Family (b) with $v, w_1, x_1, x_2, w_2$ forming the central $C_5$ with only $x_2$ and $w_2$ having pendant $C_3$'s. □

## References

[1] L.T. Ollmann, $K_{2,2}$-saturated graphs with a minimal number of edges, *Proceedings of the Third Southeast Conference on Combinatorics, Graph Theory, and Computing* (1972), 367–392.

[2] Z. Tuza, $C_4$-saturated graphs of minimum size, *Acta Universitatis Carolinae-Mathematica et Physica* (1989), 161–167.