Hamiltonicity of k-connected graphs with independent claws [1]

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Abstract

In this paper, we prove that if G is a k-connected ($k \geq 2$) graph of order n such that the sum of degrees of any $k+1$ independent vertices is at least $n+k$, and if the set of claw centers of G is independent, then G is hamiltonian.

Key words: hamiltonian graph, claw center

In this paper, we will consider only finite, undirected graphs, without loops or multiple edges. We use the notations and terminology in [3]. In addition, if G is a graph, we denote by $V(G)$ the vertex set of G, by $E(G)$ the edge set of G. For any $v \in V(G)$, $A \subseteq V(G)$ and $B \subseteq V(G)\backslash A$, we put

$$\begin{aligned} N(v) &= \{u \in V(G) : uv \in E(G)\}, \qquad d(v) = |N(v)|, \\ N(A) &= \{u \in V(G)\backslash A : uv \in E(G)\}, \\ \sigma_k(G) &= \min\Big\{\sum_{x \in I} d(x) : I \text{ is an independent set of order } k \text{ in } G\Big\}, \\ e(A,B) &= \Big|\{uw \in E(G) : u \in A \text{ and } w \in B\}\Big|. \end{aligned}$$

In a graph G, the subgraph induced by $A \subseteq V(G)$ will be denoted by $G[A]$. Let $C = c_1c_2\ldots c_pc_1$ be a cycle. We put $c_i^+ = c_{i+1}$ and $c_i^- = c_{i-1}$ ($i = 1, 2, \ldots, p$, $c_{p+1} = c_1$, $c_0 = c_p$). Set $A^+ = \{a^+ : a \in A\}$ and $A^- = \{a^- : a \in A\}$ for any $A \subseteq V(C)$. A graph is said to be claw-free (or $K_{1,3}$-free) if it does not contain $K_{1,3}$ as an induced subgraph. The vertex with degree 3 in a $K_{1,3}$ is called a claw center.

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In recent years, there have been a lot of results dealing with claw-free graphs, in particular in finding sufficient conditions in claw-free graphs for various cycle and path properties. Many of these are related to traditional conditions on degrees, neighborhood unions, connectivity, independence number, etc. for hamiltonian graphs. For the recent related topics, people can see [7], [9], etc. We are interested in the following result and conjecture.

Theorem 1. [10] If G is a k-connected claw-free graph of order n such that $\sigma_{k+1}(G) \geq n-k$, then G is hamiltonian.

Conjecture 2. [8] Every 4-connected claw-free graph is hamiltonian.

A weaker version for Conjecture 2 is the following.

Conjecture 3. [2] Every 4-connected line-graph is hamiltonian.

In this paper, we consider the class of graphs in which the set of claw centers is independent. A graph in this class is called a claw center independent graph. It is clear that a claw-free graph is also a claw center independent graph.

In [5], Li, Lu and Sun proved the following theorem.

Theorem 4. Let G be a 2-connected graph of order n and minimum degree δ such that $n \leq 4\delta - 3$. If the set of claw centers of G is independent, then either G is hamiltonian or G is in one of three families of exceptional graphs.

Li and Tian [6] proved the following theorem, which solves a conjecture proposed by Broersma, Ryjáček and Schiermeyer in [4] for $n \geq 79$.

Theorem 5. Let G be a 2-connected graph of order n ($n \geq 79$) such that $\sigma_3(G) \geq n-2$. If the set of claw centers of G is independent, then either G is hamiltonian or G is in one of two families of exceptional graphs.

Now, for $k \geq 2$, we construct a graph G_k as follows (see Fig. 1). Let H_0 be a complement of K_{k-1} with $V(H_0) = \{x_1, x_2, \ldots, x_{k-1}\}$. For any $1 \leq i \leq k+1$, let H_i be a complete graph of order $k-1$, M_i a matching between H_i and H_0. Set

$$\begin{aligned} V(G_k) &= V(H_0) \cup \Big(\bigcup_{i=1}^{k+1} V(H_i)\Big) \cup \{x_k\}, \\ E(G_k) &= \Big(\bigcup_{i=1}^{k+1} M_i\Big) \cup \Big(\bigcup_{i=1}^{k+1} E(H_i)\Big) \cup \Big\{vx_k : v \in \bigcup_{i=1}^{k+1} V(H_i)\Big\}. \end{aligned}$$

Obviously, G_k is a k-connected non-hamiltonian claw center independent graph. This means that Conjecture 2 can not be extended to claw center independent graphs. Therefore, it is meaningful to find sufficient conditions for the hamiltonicity of claw center independent graphs.

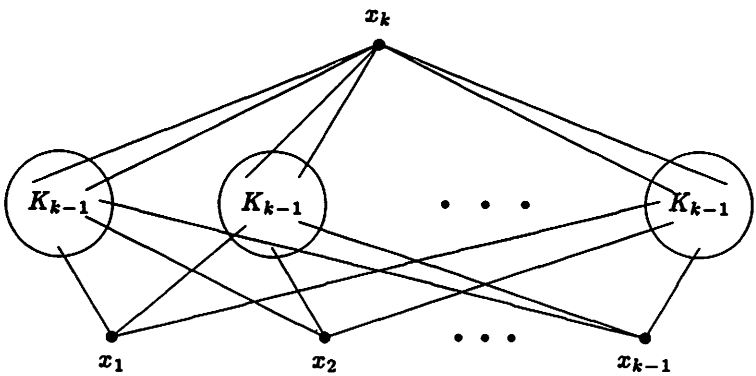

Fig.1: A k-connected claw center independent non-hamiltonian graph.

In this paper, we prove the following result.

Theorem 6. Let G be a k-connected ($k \geq 2$) graph of order n such that $\sigma_{k+1}(G) \geq n + k$. If the set of claw centers of G is independent, then G is hamiltonian.

For $G = G_k$, we have $n = k^2 + k - 1$, $d(v_i) = k$ for any $v_i \in V(H_i)$ ($i > 0$), $d(x_j) = k + 1$ for any $x_j \in V(H_0)$ and $d(x_k) = k^2 - 1$. Then $\sigma_{k+1}(G) = k^2 + k = n + 1$. This shows that Theorem 6 is best possible when $k = 2$.

Now, we assume that G satisfies the conditions of Theorem 6 and is not hamiltonian. In order to prove Theorem 6, we introduce some additional notations. Let $\Omega := \{v \in V(G) : v \text{ is a claw center}\}$, which by the condition of Theorem 6, is an independent set.

For a cycle C, we denote by $\overrightarrow{C}$ the cycle C with a given orientation. If $u, v \in V(C)$, then $u\overrightarrow{C}v$ denotes the consecutive vertices on C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We will consider $u\overrightarrow{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets.

Let C be a longest cycle in G, H a component of $G \backslash V(C)$. Since G is k-connected, we have $h = |N(H) \cap V(C)| \geq k$. Let $v_1, v_2, \ldots, v_h$ be the vertices of $N(H) \cap V(C)$, occurring on $\overrightarrow{C}$ in consecutive order. By the maximality of C, we know $v_{i+1} \neq v_i^+$, where $v_{h+1} = v_1$.

We call a vertex $v \in v_i^+\overrightarrow{C}v_{i+1}^-$ insertible if there exists $u \in v_{i+1}\overrightarrow{C}v_i^-$ such that $\{u, u^+\} \subseteq N(v)$.

In order to prove Theorem 6, we need the following two lemmas.

Lemma 1. [1] For any $1 \le i \le h$, there exists a non-insertible vertex in $v_i^+ \overrightarrow{C} v_{i+1}^-$.

Along $v_i^+ \overrightarrow{C} v_{i+1}^-$, let x_i be the first non-insertible vertex, $R = G \backslash V(C)$ and $Q = \bigcup_{i=1}^h (v_i^+ \overrightarrow{C} x_i)$. Choose $x_0 \in V(H)$. Let $I = \{x_0, x_1, x_2, \ldots, x_h\}$.

Lemma 2. [1]

(i) No two distinct vertices of I are joined by a path whose internal vertices (if any) are in $R \cup Q$. In particular, I is an independent set;

(ii) For any $1 \le i < j \le h$ and $v \in x_i \overrightarrow{C} x_j$, $v^+ \notin N(v_i^+ \overrightarrow{C} x_i)$ or $v \notin N(v_j^+ \overrightarrow{C} x_j)$.

Lemma 3. For any $0 \le i < j \le h$, we have $N(x_i) \cap N(x_j) \subseteq \Omega$

Proof. Suppose z satisfies $|N(z) \cap I| \ge 2$. By Lemma 2(i), we have $z \in V(C) \backslash Q$.

If $z \in N(x_0) \cap N(x_j)$ for some $j > 0$, then $z^+, z^- \notin N(x_0)$ by the maximality of C, and $x_0 x_j \notin E(G)$ by Lemma 2(i). If $z = v_j$, then $x_j z^- \notin E(G)$ as x_j is non-insertible. We have that $G[\{v_j, x_0, x_j, v_j^-\}]$ is a claw. If $z \ne v_j$, then $x_j z^+ \notin E(G)$ by Lemma 1(i). We have that $G[\{z, x_0, x_j, z^+\}]$ is a claw. In either case, we have $z \in \Omega$.

If $z \in N(x_i) \cap N(x_j) \backslash N(x_0)$ for some $1 \le i < j \le h$, without loss of geniality we assume that $z \in x_i \overrightarrow{C} x_j$, then $z \in x_i \overrightarrow{C} v_j^-$ by Lemma 2(i). We have $x_i z^+ \notin E(G)$ by Lemma 2(ii), and $x_j z^+ \notin E(G)$ as x_j is non-insertible. Then the subgraph $G[\{z, z^+, x_i, x_j\}]$ is a claw. Hence $z \in \Omega$.

Let

$$I' = \{x_0, x_1, \ldots, x_k\}.$$

Define

$$S_i = \{v \in V(G) : |N(v) \cap I'| = i\} \qquad \text{and} \qquad s_i = |S_i|.$$

By Lemma 2(i), we have $I \subseteq S_0$, and hence

$$s_0 \ge |I| = h + 1 \ge k + 1.$$

It is easy to see that $\sum_{i=0}^{k+1} s_i = n$ and

$$\sum_{i=0}^{k+1} i s_i = \sum_{i=0}^{k} d(x_i) \ge \sigma_{k+1}(G) \ge n + k.$$

Hence

$$k \leq \sum_{i=0}^{k+1}(i-1)s_i = \sum_{i=2}^{k+1}(i-1)s_i - s_0. \tag{1}$$

Now, we turn to prove Theorem 6.

As the set of claw centers is independent, we have $|N(x_i) \cap \Omega| \leq 2$ for any $x_i \in I'$. By Lemma 3, we have $\bigcup_{i=2}^{k+1} S_i \subseteq \Omega$. Then

$$\begin{aligned} \sum_{i=2}^{k+1} is_i &= e(I', \bigcup_{i=2}^{k+1} S_i) \leq e(I', \Omega \backslash (S_0 \cup S_1)) \\ &\leq e(I', \Omega) - e(I', (S_0 \cup S_1) \cap \Omega) \\ &\leq 2(k+1) - \left|S_0 \cap \Omega\right|. \end{aligned}$$

If $\sum_{i=2}^{k+1} s_i < 2$, then at most one of s_i, say $s_j = 1$, is not equal to 0. We have $\sum_{i=2}^{k+1}(i-1)s_i = j-1 \leq k$; and if $\sum_{i=2}^{k+1} s_i \geq 2$, then

$$\sum_{i=2}^{k+1}(i-1)s_i = \sum_{i=2}^{k+1} is_i - \sum_{i=2}^{k+1} s_i \leq 2(k+1) - 2 = 2k. \tag{2}$$

In either case, we have (2). Combining (2) with (1), we have

$$k \leq \sum_{i=2}^{k+1}(i-1)s_i - (h+1) \leq 2k - h - 1 \leq k - 1.$$

a contradiction.

The proof of Theorem 6 is complete.

Remark We do not know whether Theorem 6 is true or not under the condition $\sigma_{k+1}(G) \geq n+2$. If true, the graph G_k (Fig. 1) shows that it is best possible.

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