On incidence matrices of finite affine geometries

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ABSTRACT. It is known that each incidence matrix between any two levels of the boolean lattice and the lattice of flats of a finite projective geometry has full rank. We show that this also holds for the lattice of flats of a finite affine geometry.

Let P be a partially ordered set with minimal element $\hat{0}$, maximal elements $\hat{1}$ and rank function r. We denote the set resp. the number of all rank-k elements in P by P_k resp. W_k. ($W_k = |P_k|$ are the Whitney numbers of P). For subsets H and K of P let $M_{H,K}$ be the incidence matrix with columns indexed by H and rows indexed by K, i.e. the x, y-entry of $M_{H,K}$ equals 1 if $x \geq y$ and 0 otherwise. Let in particular $M_{k,l} := M_{P_k,P_l}$ be the incidence matrix between the k-th and the l-th level of P. As usually $\mu : P \times P \to \mathbb{C}$ denotes the Möbius function of P.

Kantor [3], Kung [5] and others proved that for the Boolean lattice $B(n)$, the affine lattice $AG(n-1, q)$ and the projective lattice $PG(n-1, q)$, the rank of the incidence matrix $M_{k,l}$ equals W_k whenever $0 \leq k < l \leq n-k$ (n is the rank of all three lattices). Since $B(n)$ and $PG(n-1, q)$ are isomorphic to their order duals, the incidence matrices $M_{k,l}$ have full rank for $k+l > n$, too. We simplify the proof of Kung [5], and show, that also for the affine lattice $AG(n-1, q)$ the incidence matrices between any levels have full rank.

Let us first repeat two Lemmas from [5], which are fundamental in both Kung's and our proof.

We define the Radon transformation of a partially ordered set as the linear map

$$T : \{f : P \to \mathbb{C}\} \to \{f : P \to \mathbb{C}\}$$

defined by

$$Tf(x) = \sum_{y : y \leq x} f(y).$$

A function $f : P \to \mathbb{C}$ is called reconstructible from its Radon transform Tf resricted to the subset $K \subseteq P$ if f is uniquely determined by the function $Tf\,|_K : K \to \mathbb{C}$.

Then we have the following lemma (see [4, 5]).

Lemma 1. *Let H and K be subsets of a partially ordered set P. The incidence matrix $M_{H,K}$ has full rank $|H|$ iff every function $f : P \to \mathbb{C}$ with $f\,|_{P\setminus H} \equiv 0$ is reconstructible from $Tf\,|_K$.* □

If Tf is known everywhere in P we can reconstruct f by Möbius inversion:

$$f(x) = \sum_{y:y\leq x} \mu(y,x)Tf(y).$$

If Tf is known in an upper part of P we will use the following Möbius function identity (see [1, 4, 5]):

Lemma 2. *Let $f : P \to \mathbb{C}$ be defined on a finite lattice P. Then*

$$\sum_{y:x\leq y\leq z} \mu(y,z)Tf(y) = \sum_{u:u\vee x=z} f(u).$$

□

Now let P be a finite lattice satisfying the following regularity property:

(R) For elements $x, z \in P$ with $r(x) = k < l = r(z)$ and $x < z$, the number of elements y with $r(y) = m$ and $x \leq y \leq z$ depends only on k, l and m.

Let $f : P \to \mathbb{C}$ be an arbitrary function with $f(x) = 0$ unless $r(x) = k$ and let Tf be given for all $x \in P$ with $r(x) = l > k$. The regularity property (R) enables us to determine Tf for all $x \in P$ with $r(x) > l$: Let such an x be given. Then

$$\sum_{y:r(y)=l,y\leq x} Tf(y) = \sum_{y,z:r(y)=l,r(z)=k,z\leq y\leq x} f(x) = c(k,r(x),l)Tf(x),$$

where $c(k,l,m)$ denotes the constant occuring in the above definition of (R). This constant is always $\neq 0$ and therefore Tf is known for all x with $r(x) \geq l$.

Let P have two additional properties:

(S) $r(x) + r(y) \geq r(x \vee y)$ for all $x, y \in P$

(T) $\mu(x,y) \neq 0$ for all $x \leq y$

Furthermore let $k + l \leq n := \text{rank}(P)$.

Then Tf can also be evaluated for all x with $r(x) < l$ (this is clear for $r(x) < k$, since we have there $Tf(x) = 0$). We proceed inductively as follows:
Consider $x \in P$ with $r(x) < l$. Let $Tf(y)$ for $y > x$ be given or already evaluated. Using Lemma 2 we obtain

$$\sum_{y:x \leq y} \mu(y, \hat{1}) Tf(y) = \sum_{u:u \vee x = \hat{1}} f(u).$$

For all u with $r(u) = k$ we have $r(u \vee x) \leq k + l - 1 \leq n - 1$, i.e. $u \vee x \neq \hat{1}$. Therefore the right hand side of the above identity vanishes and we can evaluate $Tf(x)$:

$$Tf(x) = -\frac{1}{\mu(x, \hat{1})} \sum_{y:x<y\leq z} \mu(y, \hat{1}) Tf(y).$$

We have constructed f from $\dot{T}f\,|_{W_l}$. Lemma 1 yields the following theorem, which is due to Kung [5]:

Theorem 1. *Let P be a finite lattice with the properties (R), (S), (T). Then for $0 \leq k < l \leq n - k$, the rank of the incidence matrix $M_{k,l}$ equals W_k.* □

$B(n)$, $AG(n-1, q)$ and $PG(n-1, q)$ are geometric lattices, so they satisfy (S) and (T). Furthermore it is easy to see that they satisfy the regularity property (R), too. Since B_n and $PG(n-1, q)$ are isomorphic to their order duals, we obtain a result of Kantor [3]:

Corollary 1. *Every incidence matrix $M_{k,l}$ of the Boolean lattice $B(n)$ or the the projective lattice $PG(n-1, q)$ has full rank $(0 \leq k, l \leq n)$.* □

It remains to consider the "top half" of $AG(n-1, q)$, i.e. $M_{k,l}$ for $k+l > n$. Note that the dual lattice of $AG(n-1, q)$ does not satisfy (S), but (R) and (T). We profit from the fact that $AG(n-1, q)$ is weakly modular ([6]), that is: $x \wedge y = \hat{0}$ or x, y form a modular pair, i.e. $r(x) + r(y) = r(x \vee y) + r(x \wedge y)$. Consequently $AG(n-1, q)$ satisfies the following property (S'):

(S') $x \wedge y = \hat{0}$ or $r(x) + r(y) \leq r(x \wedge y) + n$.

Theorem 2. *Let P be a finite lattice satisfying (R), (S') and (T). Then for $0 \leq k < l \leq n$ and $k + l > n$, the rank of every incidence matrix $M_{k,l}$ equals W_l.*

Proof: We have to show that every incidence matrix $M_{k,l}$ of the dual lattice P^* has full rank W_k whenever $0 \leq k < l < n - k$. We proceed analogously to the above proof.

Consider $f : P^* \to \mathbb{C}$ with $f(x) = 0$ for $r(x) \neq k$ (r is the rank function of P^*). Let $Tf(x)$ be known for $r(x) = l$. Then we can evaluate Tf on the entire lattice. This is done for all x with $r(x) > l$ using (R) and for all x with $k \leq r(x) < l$ inductively:

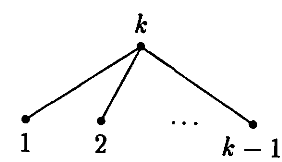
Choose $z \in P^*$ with $r(z) = n - 1$ and $x \leq z$. Since (S') holds, we have for $x, u \in P^*$ with $r(x) < l$ and $r(u) = k$: $x \vee u = \hat{1} \neq z$ or $r(x \vee u) \leq r(x) + r(u) \leq l + k - 1 \leq n - 2$, i.e. $x \vee u \neq z$. The right hand side in Lemma 2 vanishes and $Tf(x)$ can be evaluated:

$$Tf(x) = -\frac{1}{\mu(x,z)} \sum_{y:x<y\leq z} \mu(y,z)Tf(y).$$

Thereby we have constructed f from $Tf\ |_{W_l}$, and Lemma 1 finishes the proof. □

Corollary 2. *Every incidence matrix $M_{k,l}$ of the affine lattice $AG(n-1,q)$ has full rank.* □

There is another lattice satisfying (R), (S') and (T). It is the direct product of n factors given in the figure below with a zero element added.



In the case $k = 3$ we obtain the lattice of all faces of an n-dimensional cube ordered by inclusion. Note that these lattices do not satisfy the condition (S). In fact, the incidence matrices $M_{k,l}$ of the above lattices have full rank iff $k + l \geq n$. This was shown by Engel [2], who determined all ranks explicitly.

Kung conjectures that in every geometric lattice the incidence matrices $M_{k,k+1}$ have full rank whenever $0 \leq k < n/2$. He proved in [5] that this is true for the partition lattice.

Acknowledgement

The author wishes to thank Konrad Engel for the help and advice he offered in the preparation of this paper.

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