Several Resolvable BIB or PBIB Doubles Round Robin Tournament Designs

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ABSTRACT. Several unbiased tournament schedules for round robin doubles tennis are presented, in a form which can be useful to the urban league tournament director. The unbiased tournament affords less restriction than does the usual spouse-avoiding tournament (see [7]). As gender considerations are not necessary, it is most often the tournament of choice.

1 Introduction

Complex mathematical problems arise from a wide variety of sources in everyday life. A case in point is a recent study of scheduling for a bridge club by application of discrete optimization algorithms [1]. In this paper concern is directed to unbiased scheduling for the gender-unspecific round robin doubles tennis tournament. Such tournaments are less restrictive, hence occur more often in urban league tennis than does the spouse-avoiding, mixed-doubles round robin tournament, as defined in [7]. Unfortunately, one criticism of [7] as to style is that designs shown to exist are not then explicitly spelled out in a manner accessible to the non-specialist urban league tournament director. Thus, one objective of the present paper is to afford such utility, albeit for a different class of tournament.

The author's concern with the optimal scheduling problem arose from the frustrations of repeated pairings with weak opponents in urban league doubles tennis, where assurances by the tournament director that such occurrences are a law of nature became suspect. However, what first appeared as lackadaisical pairings in a tournament of twenty players competing on five courts, on closer scrutiny reveals a challenging problem in combinatorial design, whose solution by the non-specialist can be difficult.

The Scheduling Dilemma In mixed doubles parlance, a match is a simultaneous interaction (customarily four games) between a quadruple of four players. The two teams in a quadruple switch sides of the court after two games, retaining the same partner. Draws are avoided by certain win or loss on the point which succeeds 40, and for the match each person receives the score (0-4) earned by his team. Assuming $\nu = 4C$ players and a C-court facility, a round consists of C simultaneous matches, with a tournament consisting of R rounds.

The tournament directors dilemma is the following: What is the maximum number of rounds, and what choice of match pairings, will assure an unbiased tournament, in the sense that (i) each player completes one match in every round, and (ii) over the course of the tournament no two players interact together in a matched quadruple more than once? Thus, the tournament is unbiased if and only if no pair of players have a simultaneous court appearance more than once, outside of competing together as either partners or opponents (but not both) in some particular match.

It would be tempting to call the tournament unbiased if after several rounds of unbiased play as previously defined, pair repeats are allowed so long as over the course of the tournament each player experiences an equal number of such occurrences. However, this is a mathematical complication which will not be fully investigated here.

P-Optimal Versus K-Optimal Tournaments Given $\nu = 4C$ players, a tournament design which possesses the maximum possible number, P, of unbiased rounds is defined as a P-optimal tournament. On the other hand, it is K-optimal if the designer has only discovered a K-round unbiased design, for which he may be unaware it can be continued one or more rounds without introducing pair repeats. It is DK-optimal (a dead-end design) if it is impossible by any means to continue the current design for another round without introducing pair repeats. In such a case, one must either prove his design is P-optimal, or else attempt a completely new design.

It may be remarked that a P-optimal tournament can become the building block for a tournament design which is unbiased in the sense that each player has the same number of pair repeats: A tournament of NP rounds may be achieved simply by multiple repetition of the P-schedule. For example, by also alternating partners in a fixed quadruple, a 3P round tournament results, in which no person has had the same match partner twice, yet has played against each of his opponents three times, and no better design of 3P rounds is possible.

Consider now the deeper question of unbiased tournament design: How is it possible to determine P, the exponent of optimality? When is a DK-optimal design also P-optimal? For a fixed number of players, the work of Brouwer [4] makes it possible to find an upper bound on P, and sometimes such a bound can be proven sharp, by the simple expedient of counting pairs used. For $C = 4$, it is shown here that $P = 5$ is sharp. It is also shown that for $C = 5, P \geq 5$, whereas from Brouwer [4], $P \leq 6$.

Classification (Resolvable, Sometimes Balanced, Incomplete Block Design) The tournament scheduling problem concerns **block design** of a peculiar sort: depending upon the number of courts, C, a balanced incomplete block design (BIB) which is resolvable [5] may emerge, where every possible pair of players has been matched exactly once in a P-optimal tournament. For another C value, a balanced design may not be possible, due to competition for match opponents becoming stalled by a shortage of such. In this case the tournament possibly may be P-optimal, yet it must end with each participant having matched $N < 4C$ persons.

As some pairings have not been made, it is now difficult to establish P-optimality on the basis of pairs used. However, by defining two associate classes as (i) - two persons have played each other, or (ii) they have not played each other, perhaps this design class could be investigated as a **partially balanced, incomplete block design (PBIB)** having two associate classes [6].

2 A Tournament With Sixteen Players

The case of a $C = 4$ court facility leads to solving a C^2-Kirkman problem, whose solutions classically have been obtained by heuristic methods which rely upon geometric intuition [2,3]. It is instructive to solve the problem analytically, by an approach which is applicable to both PBIB and BIB designs. The approach is similar to Raghavarao's [8] method of symmetric differences, but for PBIB designs the differences clearly can not be symmetric.

Consider the cyclic group $Z = \{1, 2, \ldots, M\}$, whose group operation is addition modulo M. For a specific n-tuple $X_0 = (x_1, x_2, \ldots, x_n)$ whose components $x_i \in Z$ are ordered by increasing magnitude, a permutation mapping, T, is introduced, which is pair dispersive over a cycle of length M. To be a cycle of length M , the M-th iterate is at most a permutation of X_0. $\{T, X_0\}$ is defined as pair dispersive of order Q if and only if the group of successive n-tuples

$$\{T, X_0\} = \{T^k(X_0) : k = 0, 1, \ldots, Q-1\} \tag{1}$$

have no repeated pairs. In the language of dynamical systems, X_0 is a vector of initial state. It is desired to determine the maximum value for Q such that the set of state vectors $\{T, X_0\}$ which are T-reachable from X_0 are free of pair repeats.

For present purposes, for $3 \leq n < M$ define the mapping, T(y), where y is an n-tuple from Z, as follows:

$$T(y_i) = y_i + 1, \ Mod \ M \tag{2}$$

The pair dispersive character of $\{T, X\}$ is determined by a characteristic difference table formed from the components $\{x_i : i = 1, 2, \dots, n\}$ of the initial vector:

$$\Delta_{ij} = x_j - x_i, \; Mod\ M : i, j = 1, 2, \dots, n \tag{3}$$

Sample difference tables for $\{M = 15, n = 4\}$ are given in Table 1:

(a) n =4, M = 15

	1	2	4	8
1	15	1	3	7
2	14	15	2	6
4	12	13	15	4
8	8	9	11	15

(b) n = 3, M = 15

	5	10	15
5	15	5	10
10	10	15	5
15	5	10	15

Table 1: Characteristic Difference Tables

Theorem 1: Pair Dispersiveness of $\{T, X_0\}^Q$.

(i) *The full cycle of $(Q = M)$ reachable states will have no pair repeats if and only if there are no off-diagonal pair repeats in the characteristic difference table for X_0;*

(ii) *If τ is the smallest table difference which is repeated, the set of reachable states will have no pair repeats before τ states are generated.*

Proof: Proposition (i) follows from the fact that the characteristic difference table associated with a state vector is invariant under the mapping T. Clarifying further, a pair repeat can arise only in the following ways: The initial state satisfies

(a) $x_i + \Delta = x_j,\ and\ x_j + \Delta = x_i,\ with\ \Delta = M/2,\ M\ even$, or

(b) $x_i - x_j = x_k - x_i = \Delta$, or

(c) $x_i - x_j = x_k - x_l = \Delta$, or

(d) The only repeat is on the diagonal: $\Delta_{ii} = M$.

In each case there is a pair repeat after Δ iterations, and only after such an occurrence; proposition (i) follows. Addressing proposition (ii), even if there are several difference repeats of classes (a-c), the presence of a specific repeated difference Δ causes no pair repeat until Δ iterations have occurred. Proposition (ii) follows;

Corollary. *If $\{T, X_0\}^M$ and $\{T, Y_0\}^K$, with $K < M$, possess characteristic difference tables whose entries have no common pair repeats smaller than M, with each set of reachable states being pair dispersive, then the union of the two sets of reachable states has no pair repeats.*

For an example where the theorem is useful, see Table 1, with $M = 15, K = 5$. In this case, Table 1-a permits a block design consisting of a set of 15 quadruples having no repeated pairs, while Table 1-b permits a set of 5 triplets having no repeated pairs. The union has no repeated pairs, and uses up all 105 distinct pairs generated by the integers $\{1, 2, \ldots, 15\}$. By adjoining to each triplet the number 16, the joint collection of quadruples has no pair repeats, and uses all 120 pairs which can be generated from the integers $\{1, 2, \ldots, 16\}$. This twenty-block design is resolvable: By appropriate rearrangment there is obtained the $P = 5$ round balanced, incomplete 5-optimal block-cyclic design whose first $4x4$ matrix block is given in Table 2,

5	10	15	16
1	2	4	8
6	7	9	13
11	12	14	3
6	11	1	16
2	3	5	9
7	8	10	14
12	13	15	4
7	12	2	16
3	4	6	10
8	9	11	15
13	14	1	5
8	13	3	16
4	5	7	11
9	10	12	1
14	15	2	6
9	14	4	16
5	6	8	12
10	11	13	2
15	1	3	7

Table 2: A Block-Cyclic, P=5 Optimal, BIB Design For Sixteen Players

and whose succeeding matrix blocks are T^*-iterates of the first block, where the *-mapping operates as does T, with the exception of having a fixed point at sixteen.

3 A Tournament With Twenty Players

How many unbiased rounds can be played, assuming twenty players compete on a five-court facility? According to the design below, at least five unbiased rounds can be played ($P \geq 5$). Brouwer's [4] results indicate $P \leq 6$. On round six, four courts can be assigned without bias, whereas all players on court five previously have been assigned together.

5	10	15	20
1	2	4	8
6	7	9	13
11	12	14	18
16	17	19	3
6	11	16	1
2	3	5	9
7	8	10	14
12	13	15	19
17	18	20	4
7	12	17	2
3	4	6	10
8	9	11	15
13	14	16	20
18	19	1	5
8	13	18	3
4	5	7	11
9	10	12	16
14	15	17	1
19	20	2	6
9	14	19	4
5	6	8	12
10	11	13	17
15	16	18	2
20	1	3	7

Table 3: A Block-Cyclic PBIB Design For Twenty Players

The method of Section 2 readily produces an unbiased, five round tournament schedule classed as a resolvable, unbalanced, incomplete block design. It is not known how to prove $P = 5$ is optimal. This design consists of the

block-cyclic group of matrices $\{T(A_k) : k = 0, 1, 2, 3, 4\}$, where the first matrix appears in the first block of Table 3.
Here, the mapping T

$$T({a_{ij}}^{k+1}) = {a_{ij}}^k + 1, \; Mod\, 20 \tag{4}$$

which now has no fixed points, maps each block of Table 3 into its successor, Mod 5, and the group operation is the composition induced by combining powers of T in the usual way.

The author has discovered a further four design methods which yield distinct DK-optimal, five-round, unbiased tournament schedules. In each case an additional unbiased round is impossible, as it can be proven that any sixth round, at best, must exhibit one pair repetition for every member of one match quadruple, with no pair repetition in the other four matches.

Of the four, that method whose requirements are most elementary will be presented. Let

$$A = [a_i, b_i, c_i, d_i \; ; \; i = 1, 2, 3, 4, 5] \tag{5}$$

be the matrix whose rows specify the first round match assignments, where in future the elements a_i, b_i, c_i, d_i of any fixed row must never be assigned together.

Solution By A Cylinder Method The idea of the solution is to "bend" the matrix array so as to form a cylinder, with the elements $a_i; i = 1, 2, 3, 4, 5$ appearing on a ring which is a generator of the cylinder (so likewise, for the bi, ci, di). The persons assigned to a fixed initial match now appear parallel to each other, but on distinct rings of the cylinder. To generate a next round with distinct pairs, holding the a-ring fixed, one simply rotates, (either clockwise or) counter-clockwise, the b-ring by one, the c-ring by two, and the d-ring by three positions. It is fairly easy to convince oneself (see **Conclusions**) that this can be repeated four times, each time producing a new round such that no pair from any round has been previously assigned together. Mathematically, one way to describe such a process is

$$A^p = [a_i, b_{i+p}, c_{i+2p}, d_{i+3p}; \; i = 1, 2, 3, 4, 5] \tag{6}$$

where $p = 0, 1, 2, 3, 4$. Here, $p = 0$ represents the initial round, and index values such as $i + jp$ are evaluated with modulo 5 arithmetic. Again, a cyclic group of matrices has been produced.

A Dead-End, DK-optimal Design The tournament of the present design has the property that each person from the a-group has after 5 cycles ($p = 0, 1, 2, 3, 4$) been assigned to match exactly once with each member of the b-group, c-group, and d-group. Likewise, each member of any particular letter-designated group has been assigned to match exactly once with each member of every other letter-designated group, yet has not been assigned with anyone having the same letter designation. Consequently, on cycle six of the assignment process any four members of a letter-designated group can be assigned together without pair repetition: one person necessarily must be assigned with other persons he has matched during the first five cycles.

Thus, on cycle six only four matches can be scheduled without pair repetition; all members of match group five of necessity have interacted within the first five rounds. This is the proof that a DK-optimal design with $K = 5$ has been obtained. This design is conjectured to be $P = 5$-optimal. The conjecture is supported by a result for the 5 x 5 Kirkman problem which appears in the sequel.

It is remarked that the 5-round design is a group divisible design; and it is readily established that a 6-round group divisible design having no repeated pairs is impossible. Moreover, it is impossible for a BIB design having no repeated pairs to exist, for $C = 5$. Thus, over the design class of BIB and/or group divisible designs, this design is p-optimal with $p = 5$.

4 More General Resolvable Design Problems

The cylinder approach of section 3 can be firmly established by the **Race-Track Proposition:**
Suppose four greyhounds are to race on (cylinder stacked) parallel circular tracks with unit markers at the angular intervals $j\Omega : j = 0, 1, \ldots, C-1$, where $\Omega = 2\pi/C$. Their successive speeds are $\{0, 1, 2, 3\}$ angular increments per unit time, where the angular unit is Ω radians. If all positive integers $k \leq 3$ are relatively prime to C, then there will not be a time T smaller than that time taken by hound #2 (the slowest **moving hound**) to traverse the track, for which a faster hound overtakes a slower one, exactly in coincidence with one of the integer angular unit markers.

Proof: The locations at which a faster hound overtakes a slower one are given by $\Theta_{jk} = jC/(k-1)\Omega$, where $j \geq 1$ is the cycle index, and $\{k = 2, 3, 4\}$. Thus, for consonance not to occur at an angular unit marker during the first cycle ($j = 1$), it is necessary that C and all positive $k \leq 3$ be relatively prime.

Referring to the cylinder method of the section 3, for $C > 4$ the cylinder argument can be extended to obtain C-round optimal schedules for the arbitrary 4C court facility, provided C and all positive integers $k \leq 3$ are relatively prime. The argument is as follows: By analogy with the racing hounds result, it is seen that an arbitrary pair which is together during the first cycle of design, by virtue of being members of a match quadruple, can not again be paired during the first cycle of parallel rotation of cylinder rings. Here, for any quadruple which is produced by rotation of cylinder rings, the racing hounds proposition shows no pair repeat can occur during a cycle measured relative to this quadruple, and certainly not in whatever fraction of the first cycle there remains to be generated.

The amusing thing is that all hounds are in coincidence with the slowest one, exactly as the slowest returns to the starting marker, after traversing the track any number of times.

5 A 7-Court, 28-Player BIB Design

The race track proposition of Section 4 guarantees existence of a seven-round unbiased design for this case. However, the Brouwer result [4] points to a possibility of a nine-round design. Indeed, such a design is possible [8], by employing symmetric differences on four symbols with the Galois field $GF(3^2)$ as base module, where also a fixed point is involved. By using a homeo-morphism to decode a design given in Raghavarao [8], one arrives at the following results:

1	2	12	15
7	5	17	13
10	11	21	24
16	14	26	22
19	20	3	6
25	23	8	4
0	9	18	27
2	0	13	16
8	3	15	14
11	9	22	25
17	12	24	23
20	18	4	7
26	21	6	5
1	10	19	27
0	1	14	17
6	4	16	12
9	10	23	26
15	13	25	21
18	19	5	8
24	22	7	3
2	11	20	27

Table 4: A Seven Court BIB Design, Rounds 1,2,3

3	4	17	11
0	7	10	15
12	13	26	20
9	16	19	24
21	22	8	2
18	25	1	6
5	14	23	27
4	5	15	9
1	8	11	16
13	14	24	18
10	17	20	25
22	23	6	0
19	26	2	7
3	12	21	27
5	3	16	10
2	6	9	17
14	12	25	19
11	15	18	26
23	21	7	1
20	24	0	8
4	13	22	27

Table 5: A Seven Court BIB Design, Rounds 4,5,6

8	6	10	13
5	0	12	11
17	15	19	22
14	9	21	20
26	24	1	4
23	18	3	2
7	16	25	27
6	7	11	14
3	1	13	9
15	16	20	23
12	10	22	18
24	25	2	5
21	19	4	0
8	17	26	27
7	8	9	12
4	2	14	10
16	17	18	21
13	11	23	19
25	26	0	3
22	20	5	1
6	15	24	27

Table 6: A Seven Court BIB Design, Rounds 7,8,9

6 Conclusions

There has been obtained optimal schedules for unbiased round robin doubles tennis tournaments on physical facilities of four, five and seven courts, respectively. However, the method of differences used in obtaining the four court schedule experiences difficulty when six courts (24 players) is considered. It is expected that there is some tie-in between this problem and that of finding six-by-six orthogonal latin squares, which are known not to exist. Fortunately, the cylinder method of design permits some extension.

As a note of historical interest, we see that the general C^2-Kirkman problem is solvable, for C a prime integer, using the cylinder method and the race-track argument with C hounds, whereby one obtains C days of walking. Moreover, the classically expected additional one day walk (one more unbiased covering) is now obtainable, through use of the matrix transpose operation. As the matrix transpose operation is unfruitful for quadruples and a 5×4 array, this explains why one less unbiased covering than Brouwer's results [4] seem to predict is actually realizable, in a resolvable 5-court design.

It is unknown to the author whether this Kirkman-type result, or this approach to its proof, has previously appeared.

Finally, it may be concluded by a similar argument that the generalized Kirkman problem, on any set of size $N \times K$ with $K < N$, permits and N-day resolvable design for unbiased walking about in K-tuplets, provided N and all positive integers $k \leq K - 1$ are relatively prime. In particular, this is always true if N is prime.

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