A Note Following A Remark Of Harary Regarding A Certain Family Of Sum Graphs

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Sum graphs were introduced by Harary [3] in 1990. To use another of Harary's papers [4], the sum graph $G^+(X)$ of a finite set X of positive integers is the graph (V, E), where $V = X$ and E is the set of edges (u, v) for which $u + v \in V$. A graph isomorphic to such a graph is called a *sum graph*. In [4] the concept is generalized to cases where X contains nonpositive integers as well. A further generalization is considered by Alon and Scheinerman [1] where *sum* is replaced by any symmetric polynomial. In [4] Harary also considers the set $X = \mathbf{Z}^+(n)$, the set of all positive integers $\leq n$. This particular graph is denoted by G_n. The sum graphs G_1, G_2, G_3, G_4 and G_5 are presented as follows. G_1 is K_1, G_2 is $2K_1$, G_3 is $K_1 \cup K_2$, G_4 is $K_1 \cup P_3$ where P_t is a path on t vertices; the sum graph G_5 is given explicitly. The presentation of the first five G_n is followed by a line which is here reproduced verbatim.

We have tried without success to find a simple elegant description of the structure of the graph G_n.

We shall give a characterization of G_n by means of the sequence of the degrees of its vertices. This characterization enables us to decide, by considering the degrees of the vertices of a given, unlabeled graph of order n, whether it is a G_n or not. This will in turn suggest simple constructions of G_n based on recurrences with respect to either G_{n-1} or G_{n-2}. First some definitions.

Let $\mathbf{Z}(n)$ denote the set of all nonnegative integers $\leq n$. Let v be a vertex in a given graph G. Then $d(v)$ is the degree of v, $D(G)$ is the set of degrees of all the vertices of G. The term S_k^v denotes the k-star with v as its root. The vertex v itself may be termed S_0^v. Let G be a graph of order n. Put $\lfloor (n-1)/2 \rfloor = s$.

Theorem 1. *The set $D(G) = \mathbf{Z}(n-2)$ with s appearing twice defines the graph uniquely (up to isomorphism). Moreover, the graph G thus defined*

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turns out to be G_n.

Proof: The statement is clearly true for $n \leq 2$. Using induction on n we observe that G has one isolated vertex and another vertex of degree $n-2$. Deleting these two vertices from G, it is easy to see that we are left with a graph G' of order $n-2$, and such that $D(G') = \mathbf{Z}(n-4)$. There is only one nontrivial automorphism of G' which transposes the two vertices of equal degree. Using induction we may also assume that $G' = G_{n-2}$ thus proving the theorem.

We have in fact shown that G is the sum graph G_n if and only if $D(G) = \mathbf{Z}(n-2)$, the degree s appearing twice.

Let $\mathcal{S}_k$ denote the symmetric group of order k and let $\mathcal{E}_t$ denote the identity group on t elements. We then have

Corollary 1. *The graph described in Theorem 1 has the group*

$$\mathcal{S}_2[\mathcal{E}_{n-2}] \cong \mathcal{S}_2.$$

By using arguments similar to those already used, it is not difficult to show the following.

Proposition 1. *By deleting from the sum graph G_n a vertex of degree s, we arrive at the sum graph G_{n-1}.*

All the aforesaid suggests simple algorithms for constructing G_n from G_{n-2} or from G_{n-1}. The simplest is: Given G_n, add a vertex x, join it to every vertex in G_n and add a vertex y. The resulting graph is the sum graph G_{n+2}.

It might be of interest to note that G_n may be represented as an edge-disjoint union of stars in the following way.

$$G_n = \bigcup_{i=0}^{s} S_1^{n-i} \cup \bigcup_{j=1}^{n-s-1} S_{n-s-j-1}^{n-s-j}.$$

Consider G_n. By simultaneously deleting the two vertices of maximal and minimal degrees (the latter being the isolated vertex) as in the proof of theorem 1, and using induction, we may arrive at the following.

Proposition 2. *Let $n = 2m+1$ or $n = 2m$ according as n is odd or even respectively. We then have $G_n = \bigcup_{i=1}^{m+1} \Gamma_i$ with $\Gamma_i = K_i$ for $i = 1, 2, \ldots, m+1$ and with the exception that $\Gamma_{m+1} = K_m$ for even n; also $\Gamma_{j+1} \cap \Gamma_j = K_{j-1}$ for $i = 1, 2, \ldots, m$; where K_0 is the empty set.*

The odd and the even cases are treated separately.

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References

[1] N. Alon and E.R. Scheinerman, Generalized Sum Graphs, *Graphs and Combinatorics* 8 (1992), 23–29.

[2] F. Harary, "Graph Theory", Addison Wesley, Reading (1972).

[3] F. Harary, Sum Graphs and Difference Graphs, 19th Southeastern Conference on Combinatorics, *Graph Theory and Computing*, Baton Rouge (1988).

[4] F. Harary, Sum Graphs over all the integers, *Discrete Math.* **124** (1994), 99–105.