Fractal Patterns in Gaussian and Stirling Number Tables

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ABSTRACT. Geometrical representations of certain classical number tables modulo a given prime power (binomials, Gaussian q-binomials and Stirling numbers of 1st and 2nd kind) generate patterns with self-similarity features. Moreover, these patterns appear to be strongly related for all number tables under consideration, when a prime power is fixed.

These experimental observations are made precise by interpreting the recursively defined number tables as the output of certain cellular automata (CA). For a broad class of CA it has been proven [11] that the long time evolution can generate fractal sets, whose properties can be understood by means of hierarchical iterated function systems. We use these results to show that the mentioned number tables ($mod\, p^\nu$) induce fractal sets which are homeomorphic to a universal fractal set denoted by $\mathcal{S}_{p^\nu}$ which we call Sierpinski triangle ($mod\, p^\nu$).

1 Introduction

The non-zero entries of some classical number tables, binomial coefficients, Gaussian q-binomials, Stirling numbers of first and second kind, and the Eulerian numbers, modulo a given natural number all generate fractal patterns (cf. [4], [9], [14], [18], [19], [22], [23], [25], [28]), which have rather intricate self-similarity features. For example, the non-zero binomial coefficients ($mod\, 2$) represented in a plane lattice generate a self-similar pattern which resembles the Sierpinski triangle.

The problem of describing the self-similar features of these number tables was considered by many authors. In particular M. Sved and J. Pitman

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introduced a "hierarchical" terminology to obtain an exact description by which they were able to prove adopted versions of Lucas' theorem for the Gaussian q-binomials and the Stirling numbers of first and second kind. In [9] pattern formation within the binomial coefficients modulo prime powers was studied from the point of view of hierarchical iterated function systems. It turned out that a combination of ideas from dynamical systems theory and elementary number theory due to E.E. Kummer [15] allowed a rather complete discussion of the hierarchical self-similarity features within the binomials. As a geometrical model for the pattern formation within the binomials a generalization of the Sierpinski triangle was introduced. This is the Sierpinski triangle modulo a prime power. Figure 1 shows the examples for 2^ν where $\nu = 1, 2, 3$. Experiments with Gaussian and Stirling number tables modulo prime powers have shown that patterns arise which are identical with or strongly related to the Sierpinski triangle modulo a prime power. Figure 2 shows a geometrical representation of the non-zero entries in the Gaussian and Stirling number tables modulo 2^2, as a typical example.

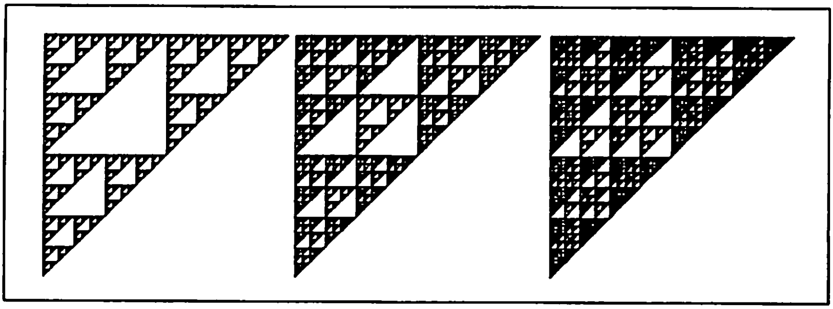

Figure 1. From left to right - graphical representation of the Sierpinski

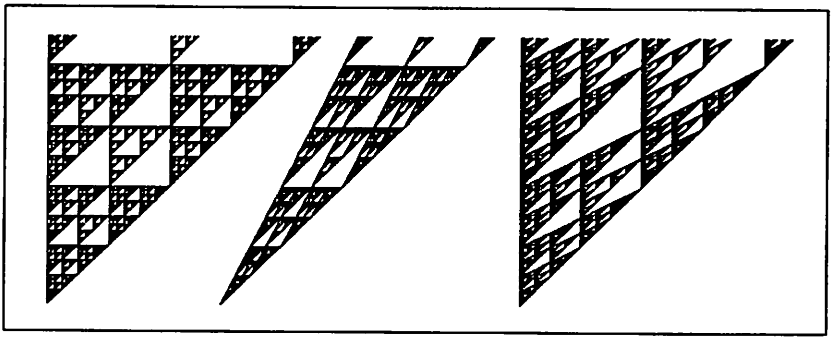

Figure 2. From left to right - graphical representation of the Gaussian 3-binomials and Stirling numbers of 1st and 2nd kind modulo 2^2. In each case we plotted about 1100 rows from bottom to top.

Our goal is to develop a language which is suitable for a mathematical discussion of the pattern formation in these number tables and to establish a theory through which it will become transparent why the binomial, the Gaussian and Stirling number tables generate essentially the same fractal structures.

Our approach will be guided by [11]. It will be crucial that we consider the number tables as output of appropriate cellular automata. A cellular automaton (CA) evolves in discrete time steps in a discrete space according to a local transformation rule (recursion formula) (cf. [29]) which generate for each spatial location one of finitely many states. Fractal aspects (rescaling, limits, Hausdorff dimension, self-similarity features) in the evolution of CA have been studied in [26], [27], [10], and [11]. Building on these ideas we will prove that properly rescaled geometrical representations of the Gaussian q-binomials and Stirling numbers of first kind modulo a prime power p^ν have limits and that these limits are homeomorphic to the Sierpinski triangle modulo p^ν, denoted by $\mathcal{S}_{p^\nu}$. Corresponding patterns of the Stirling numbers of second kind modulo a prime p turn out to be homeomorphic to $\mathcal{S}_p$. In fact this characterization should extend to the case of a prime power, but we have no result for this more general case at this point yet.

2 Preliminaries

Let $\mathbb{N}$ be the set of natural numbers including zero, $\mathbb{Z}$ be the set of integers and $\mathbb{R}$ be the field of real numbers. Let R denote a commutative finite ring with $1 \neq 0$ and $R[X]$ the ring of all polynomials with coefficients in R. For a natural number m let $\mathbb{Z}_m$ be the ring of all residue classes $(mod\, m)$. The ring of all polynomials with coefficients in $\mathbb{Z}$ is denoted by $\mathbb{Z}[X]$.

Let $A : R[X] \to R[X]$ be a map and $a(X) \in R[X]$. Then $A(a)(X) = \sum_j a_j X^j$ is a polynomial and we define for $i \in \mathbb{Z}$

$$A(a)(X)_i := a_i,$$

if i is negative or exceeds the degree of $A(a)(X)$ we define a_i to be zero.

Let d_∞ be the maximum-metric on $\mathbb{R}^2$, and let $\|\cdot\|_\infty$ denote the induced norm on $\mathbb{R}^2$. For every subset $A \subset \mathbb{R}^2$ and $\epsilon > 0$ we define the set $(A)_\epsilon := \{x \in \mathbb{R}^2 \mid \exists a \in A \text{ with } d_\infty(a,x) < \epsilon\}$. The set of all non-empty compact subsets of $\mathbb{R}^2$ equipped with the Hausdorff metric h is denoted by $\mathcal{H}(\mathbb{R}^2)$ where for $A, B \in \mathcal{H}(\mathbb{R}^2)$

$$h(A,B) := \inf\{\epsilon > 0 \mid A \subset (B)_\epsilon \text{ and } B \subset (A)_\epsilon\}.$$

$(\mathcal{H}(\mathbb{R}^2), h)$ is a complete metric space (cf. [16], [7]).

In this note we consider several recursively defined number sequences. First we present their definitions and then will represent them as linear cellular automata.

The binomials are given by the well-known recursion formula

$$\binom{0}{0} = 1, \text{ and } \binom{0}{k} = 0 \text{ for } k \neq 0$$

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}. \tag{2.1}$$

The Gaussian q-binomials are an extension of the ordinary binomials (cf. [3], p. 140). They are explicitly defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdot \ldots \cdot (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdot \ldots \cdot (q - 1)} \tag{2.2}$$

with $k, n \in \mathbb{N}$ and $q > 1$. If q is a prime power, the Gaussian q-binomial $\begin{bmatrix} n \\ k \end{bmatrix}_q$ provides the number of k-dimensional subspaces of a n-dimensional vector space over the Galois field $GF(q)$ of order q (cf. [3], p. 128). These numbers can also be recursively generated (cf. [3], p. 161):

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1 \text{ for all } n \in \mathbb{N}, \text{ and } \begin{bmatrix} 0 \\ k \end{bmatrix}_q = 0 \text{ for } k \neq 0$$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{2.3}$$

The Stirling numbers of 1st kind $s(n, k)$ are given by (cf. [3], p. 159)

$$s(0,0) = 1, \text{ and } s(0,k) = 0 \text{ for } k \neq 0$$

$$s(n+1, k) = s(n, k-1) - n\, s(n, k). \tag{2.4}$$

They are also uniquely determined by the equations (cf. [3], p. 165)

$$[X]_n = \sum_{k=0}^{n} s(n,k) X^k, \tag{2.5}$$

where $[X]_i := X(X-1) \cdot \ldots \cdot (X - i + 1) \in \mathbb{Z}[X]$, here n is a positive integer and $i > 0$. These numbers are in some sense dual to the Stirling numbers of 2nd kind $S(n, k)$ (cf. [3], p. 165), whereby the coefficient $S(n, k)$ provides the number of possible partitions of a set N with n elements into k non-empty sets. They are produced by the recursion formula (cf. [3], p. 156)

$$S(0,0) = 1, \text{ and } S(0,k) = 0 \text{ for } k \neq 0$$

$$S(n+1, k) = S(n, k-1) + k\, S(n, k). \tag{2.6}$$

For all $n \in \mathbb{N}$ we have (cf. [3], p. 165)

$$X^n = \sum_{k=0}^{n} S(n,k) [X]_k. \tag{2.7}$$

3 Fractal Rescaled Evolution Patterns of CA

When one considers a certain recursively defined table $\{F(n,k)\}_{n,k}$ of natural numbers like one of the above, and all numbers $\{F(n_0,k)\}_k$ are known for a row, say n_0, then by means of the recursion all numbers of the next row $\{F(n_0+1,k)\}_k$ can be computed. Of course this can also be done for the sequence of remainders $\{F(n,k)\,(mod\,m)\}_{n,k}$ where m is a natural number. By interpreting n as time and the number $F(n,k)\,(mod\,m)$ as the state of the "cell" located at the site (k,n) in a square lattice we have essentially described a cellular automaton. A cellular automaton evolves in discrete time steps (row by row in dimension one) and proceeds by a local transition rule, which is the recursion formula of $\{F(n,k)\,(mod\,m)\}_{n,k}$ here (cf. [29]). Let us be more precise. We define

Definition 3.1 *Let $r(X) \in R[X]$ be a non-trivial polynomial. Then a linear cellular automaton (LCA) with states in R is a map $A = A(r) : R[X] \to R[X]$ which is defined by $a(X) \mapsto r(X) \cdot a(X)$ for any $a(X) \in R[X]$.*

If $A = A(r)$ is a LCA, then the polynomial $r(X) \in R[X]$ is called the local transition rule of the LCA.

Example 3.1 *To produce the binomials $(mod\,m)$ by a CA $A = A(r)$ we consider the polynomial $r(X) = 1 + X \in \mathbb{Z}_m[X]$. Let A^0 be the identity map on $Z_m[X]$ and $A^{t+1} = A(A^t)$ the (t+1)st-composite of A. Then for every $t \in \mathbb{N}$ the polynomial $A^t(\delta) = \sum_{k=0}^{t} a_k X^k$, where $\delta(X) \equiv 1 \in \mathbb{Z}_m[X]$, it is $a_k = \binom{t}{k}\,(mod\,m)$.*

Now we fix a geometrical representation for our number tables. We define a graphical representation for polynomials in $R[X]$.

Definition 3.2 *Let $a(X) = \sum_{i=0}^{d} a_i X^i \in R[X]$ be a polynomial and $\phi : \mathbb{Z} \to \mathbb{Z}$ a mapping. The map $G_\phi : R[X] \to \mathcal{H}(\mathbb{R}^2) \cup \{\emptyset\}$ defined by*

$$G_\phi(a) \quad := \quad \bigcup_{a_i \neq 0} I(i, \phi(i))$$

is called ϕ-graphical representation of $R[X]$, where $I(k,n) := \{(x,y) \in \mathbb{R}^2 \mid k \le x \le k+1 \text{ and } n \le y \le n+1\}$.

Now it is possible to represent the geometrical structure of each set of the orbit $\{A^t(\delta)\}_{t\in\mathbb{N}}$, $\delta(X) \equiv 1 \in R[X]$, as an infinitely growing pattern ("black-and-white image").

Definition 3.3 *Let $A = A(r)$ be an LCA with states in R and $\delta(X) \equiv 1 \in R[X]$. Furthermore, let $\phi : \mathbb{Z} \to \mathbb{Z}$ be a map and let $\rho \in \mathbb{N}$, $\rho > 0$ be a* *constant. The set*

$$X(A,\mu,\phi,\rho) \;:=\; \bigcup_{t=0}^{\mu-1} G_\phi(A^t(\delta)) + (0,\rho t)$$

is called μ-th orbit representation of A (w.r.t. ϕ and ρ).

A way to obtain an "overview" of this growing pattern is by applying certain rescalings at appropriate horizontal lines. S. J. Willson first introduced this idea (cf. [26], [27]). For $\mu = p^l$, p a prime number, he rescaled the set $X(A,p^l,\phi,\rho)$ with $\phi \equiv 0$ and $\rho = 1$, by the factor p^{-l} which yields a convergent sequence $\{p^{-l}X(A,p^l,\phi,\rho)\}_{l\in\mathbb{N}}$ in $\mathcal{H}(\mathbb{R}^2)$. We need a more general result and that is why we introduced the parameters ϕ and ρ. Using the terminology from [11] we define

Definition 3.4 *An m-Fermat LCA is a LCA $A = A(r)$ with states in R satisfying the property $A^{mt}(\delta)(X) = A^t(\delta)(X^m)$ for all $t \in \mathbb{N}$.*

Remark 3.1 *Let p be a prime number, and ν a natural number, and $r_1(X)$, $r_2(X) \in \mathbb{Z}[X]$. Furthermore, let $r_1(X) \equiv r_2(X) \,(\mathrm{mod}\, p^\nu)$. Then it follows that $r_1(X)^p \equiv r_2(X)^p \,(\mathrm{mod}\, p^{\nu+1})$ (cf. [27]). As a consequence we obtain for $r(X) \in \mathbb{Z}_{p^\nu}[X]$ the equation $r(X)^{p^{\nu-1}pt} = r(X^p)^{p^{\nu-1}t}$ for all $t \in \mathbb{N}$. Thus $A(s)$ with $s(X) = r(X)^{p^{\nu-1}}$ is p-Fermat.*

Theorem 3.1 *Let $A = A(r)$ be an m-Fermat LCA with states in R. Let d be the degree of $r(X)$. Furthermore, let $\rho \in \mathbb{N}$, $\rho > 0$ a constant and $\phi : \mathbb{Z} \to \mathbb{Z}$ be a map such that there is a constant C with $\mid \phi(ml + j) - m\phi(l) \mid < C$ for all $l \in \mathbb{Z}$ and $0 \le j \le (m-1)d$. Then the sequence*

$$\left\{ \frac{1}{m^\mu} X(A, m^\mu, \phi, \rho) \right\}_{\mu\in\mathbb{N}}$$

converges in $\mathcal{H}(\mathbb{R}^2)$.

To prove theorem 3.1 we need the following lemma.

Lemma 3.1 *Let $A = A(r)$ be a LCA with states in R and let d be the degree of $r(X)$. Let $\mu \in \mathbb{N}$ with $\mu \ge 2$ and let $\phi : \mathbb{Z} \to \mathbb{Z}$ be a map. If $j \in \mathbb{Z}$ and $1 \le t \le m-1$, such that $I(j, \phi(j) + \rho t) \subset X(A,\mu,\phi,\rho)$, then there is an i, such that $-d \le i \le 0$ and*

$$I(j+i, \phi(j+i) + \rho(t-1)) \subset X(A, m, \phi, \rho),$$

where $I(n,k)$ is defined as in definition 3.2.

Proof: Note that $A^t(j) \neq 0$. Now it is obvious that there must be an i with $-d \leq i \leq 0$ and $A^{t-1}(i+j) \neq 0$. Otherwise $A^t(j)$ would be zero.

Proof: (Theorem 3.1) Let $X_\mu := X(A, m^\mu, \phi, \rho)$. For the existence of $\lim X_\mu$ it is sufficient to show that

$$h\left(X_\mu, \frac{1}{m} X_{\mu+1}\right) \leq \text{const}$$

for all $\mu \in \mathbb{N}$.

Choose a $\mu \in \mathbb{N}$. Let $I(l, \phi(l) + \rho t) \subset X_\mu$ with $l \in \mathbb{Z}$ and $t \in \mathbb{N}$, $0 \leq t \leq m^\mu - 1$. Then $A^{mt}(\delta)(X) = A^t(\delta)(X^m)$ provides $I(ml, \phi(ml) + \rho mt) \subset X_{\mu+1}$. Since especially $|\frac{1}{m} \phi(ml) - \phi(l)| < C$ it follows

$$I(l, \phi(l) + \rho t) \subset \left(\frac{1}{m} I(ml, \phi(ml) + \rho mt)\right)_{C+1}.$$

Therefore $X_\mu \subset \left(\frac{1}{m} X_{\mu+1}\right)_{C+1}$.

Now let $I(ml + j, \phi(ml + j) + \rho(mt + q)) \subset X_{\mu+1}$, where $l \in \mathbb{Z}$, $j \in \{0, \ldots, m-1\}$, $t \in \mathbb{N}$, $0 \leq t \leq m^\mu - 1$ and $q \in \{0, \ldots, m-1\}$. We apply lemma 3.1 q times and obtain the existence of an i' with $-qd \leq i' \leq 0$, such that

$$I(ml + j + i', \phi(ml + j + i') + \rho mt) \subset X_{\mu+1}.$$

Because of $A^{mt}(\delta)(X) = A^t(\delta)(X^m)$ there exists an $i \in \mathbb{Z}$ with $mi = ml + j + i'$. Now we have $I(mi, \phi(mi) + \rho mt) \subset X_{\mu+1}$ and $I(i, \phi(i) + \rho t) \subset X_\mu$. We use the estimate for ϕ and get

$$\frac{1}{m} I(ml + j, \phi(ml + j) + \rho(mt + q)) \subset (I(i, \phi(i) + \rho t))_{\tilde{C}},$$

where $\tilde{C} := \max\{d, C + \rho(m-1)\} + 1$. Finally, this leads to

$$\frac{1}{m} X_{\mu+1} \subset \left(\frac{1}{m} X_\mu\right)_{\tilde{C}}$$

This means that $\lim_{\mu \to \infty} X_\mu$ exists.

Remark 3.2 *i) The limit of the above sequence is denoted by $X_\infty(A, \phi, \rho)$ and is called the rescaled evolution pattern of the LCA A (w.r.t ϕ and ρ).*
ii) $X_\infty(A, \phi, \rho)$ is in general a fractal set (cf. [10], [11]).
iii) If $\phi \equiv 0$ and $\rho = 1$, we denote the limit simply by $X_\infty(A)$.
iv) If ϕ is bounded, we have $X_\infty(A, \phi, \rho) = X_\infty(A)$.

We will now see that related LCA possess related rescaled evolution patterns, i.e. limits in $\mathcal{H}(\mathbb{R}^2)$ which are the same up to affine homeomorphisms. We first specify what kind of relationships will be considered (cf. [11], p. 24).

Definition 3.5 *Let $A = A(r)$ and $B = B(s)$ be LCA with states in R, and $\alpha \in \mathbb{N}$, $\alpha \geq 2$. We say that A is the α-th inner power of B if $r(X) = s(X^\alpha)$ holds. We say that A is the α-th (outer) power of B if $r(X) = s(X)^\alpha$ holds.*

Our first result compares the rescaled evolution patterns of CA which are related as inner powers.

Proposition 3.1 *Let the m-Fermat LCA $A = A(r)$ be the α-th inner power of the m-Fermat LCA $B = B(s)$ both with states in R. Furthermore, let $\phi : \mathbb{Z} \to \mathbb{Z}$ a bounded map and $\rho \in \mathbb{N}$, $\rho > 0$ a constant. Then we have*

$$X_\infty(A, \phi, \rho) \quad = \quad F_\alpha(X_\infty(B, \phi, \rho)),$$

where $F_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, t) \mapsto (\alpha x, t)$.

Proof: Because of theorem 3.1 both limits $X_\infty(A, \phi, \rho)$ and $X_\infty(B, \phi, \rho)$ exist. The assumption

$$A^t(\delta)(X) = B^t(\delta)(X^\alpha)$$

yields the equation $G_\phi A^t(\delta)(X) = G_\phi B^t(\delta)(X^\alpha)$.

It is easy to see that for all $a(X) \in R[X]$

$$h\left(G_\phi a(X^\alpha), F_\alpha G_\phi a(X)\right) \leq 2C + \alpha$$

where $|\phi(i)| \leq C$ for all $i \in \mathbb{Z}$. Therefore,

$$h(G_\phi A^t(\delta), F_\alpha G_\phi B^t(\delta)) = h(G_\phi B^t(\delta)(X^\alpha), F_\alpha G_\phi B^t(\delta)(X)) \leq 2C + \alpha.$$

Finally $h(X(A, m^\mu, f, \varrho),\ F_\alpha X(B, m^\mu, f, \varrho)) \leq 2C + \alpha$ for all $\mu \in \mathbb{N}$.

A similar result holds for CA which are related as (outer) powers.

Proposition 3.2 *Let the m-Fermat LCA $A = A(r)$ be the α-th (outer) power of the m-Fermat LCA $B = B(s)$ both with states in R. Furthermore, let $\phi : \mathbb{Z} \to \mathbb{Z}$ be a bounded map and let $\rho \in \mathbb{N}$, $\rho > 0$ be a constant. Then we have*

$$\lim_{\mu \to \infty} \frac{1}{\alpha m^\mu} X(B, \alpha m^\mu, \phi, \rho) \quad = \quad F_{\alpha^{-1}}(X_\infty(A, \phi, \rho)),$$

where $F_{\alpha^{-1}} : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, t) \mapsto (\alpha^{-1} x, t)$.

Proof: Let $X_\mu := X(A, m^\mu, \phi, \rho)$ and $\tilde{X}_\mu := X(B, \alpha m^\mu, \phi, \rho)$. We have to show that

$$h\left(\frac{1}{\alpha}\tilde{X}_\mu, F_{\alpha^{-1}}(X_\mu)\right) \leq \textit{const.}$$

Furthermore, let C be a constant such that $|\phi(i)| \leq C$ for all i. Let $I(l, \phi(l) + \rho t) \subset \tilde{X}_\mu$, where $l \in \mathbb{Z}$, $t = n\alpha + q$ mit $n \in \mathbb{N}$, $0 \leq n \leq m^\mu - 1$ and $q \in \mathbb{N}$, $0 \leq q \leq \alpha - 1$.

We apply lemma 3.1 q times. This yields an $i \in \mathbb{Z}$ with $-qd \leq i \leq 0$ such that $I(l + i, \phi(l + i) + \rho n\alpha) \subset \tilde{X}_\mu$. Since $r(X) = s(X)^\alpha$, we have $I(l + i, \phi(l + i) + \rho n) \subset X_\mu$. Then

$$\frac{1}{\alpha} I(l, \phi(l) + \rho(n\alpha + q)) \subset \left(F_{\frac{1}{\alpha}} I(l + i, \phi(l + i) + \rho n)\right)_{\tilde{C}},$$

where $\tilde{C} := \max\{d,\ \rho\alpha + 2C\} + 1$. From this it follows that

$$\frac{1}{\alpha} \tilde{X}_\mu \subset \left(F_{\frac{1}{\alpha}}(X_\mu)\right)_{\tilde{C}}$$

Conversely, let $I(l, \phi(l) + \rho t) \subset X_\mu$. Then $I(l, \phi(l) + \rho\alpha t) \subset \tilde{X}_\mu$, because A is the α-th (outer) power of B. As a consequence we get

$$F_{\frac{1}{\alpha}}(X_\mu) \subset \left(\frac{1}{\alpha} \tilde{X}_\mu\right)_{2C+1},$$

and finally the assertion.

Another important tool for our analysis of the "geometrical structure" of the considered number sequences is the next proposition which deals with the situation that one LCA is the shifted version of another.

Proposition 3.3 *Let $A = A(r)$ be an m-Fermat LCA with states in R and degree d. For $k \in \mathbb{Z}$ let $r_k(X) := X^k r(X) \in R[X]$ and $A_k := A(r_k)$. Furthermore, let $\phi : \mathbb{Z} \to \mathbb{Z}$ be a bounded map and let $\rho \in \mathbb{N}$, $\rho > 0$. Then*

$$X_\infty(A_k, \phi, \rho) = F_{k,\rho}(X_\infty(A, \phi, \rho)),$$

where $F_{k,\rho} : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, t) \mapsto (x + \frac{t}{\rho}\, k, t)$.

Proof: The automaton A_k is m-Fermat, too. This means that it induces a rescaled evolution pattern as well. Now it suffices to show that

$$h\left(X(A_k, m^\mu, \phi, \rho),\ F_{k,\rho}\left(X(A, m^\mu, \phi, \rho)\right)\right) \leq const$$

for all $\mu \in \mathbb{N}$.

We have the equation $A^t(\delta)(X)_l = A_k^t(\delta)(X)_{l+kt}$. Hence $I_1 := I(l, \phi(l) + \rho t)$ is a subset of $X(A, m^\mu, \phi, \rho)$ if and only if $I_2 := I(l + kt, \phi(l + kt) + \rho t)$ is a subset of $X(A_k, m^\mu, \phi, \rho)$. Therefore

$$h\left(F_{k,\rho}(I_1),\ I_2\right) \leq \max\{\frac{1}{\rho}|k|C,\ 2C\} + \tilde{C},$$

where $|\phi(i)| \le C$ for all $i \in \mathbb{Z}$ and $\tilde{C} := \max\{\|x\|_\infty \,|\, x \in F_{k,\rho}(I(0))\} + 1$. This yields the assertion.

At the beginning of this paragraph a first example has shown the connections between LCA and the binomials $mod\, m$. To extend these connections to other number sequences we introduce new types of CA, time- and place-dependent CA.

Definition 3.6 *Let $r_0(X), \dots, r_{\alpha-1}(X) \in R[X]$ be polynomials. Let $(A_T^t)_{t \in \mathbb{N}}$ be a sequence of LCA with states in R, given in the following way. For any $a(X) \in R[X]$ and $t \in \mathbb{N}$ we define*

$$A_T^t(a)(X) \quad := \quad \left(\prod_{\nu=0}^{\alpha-1} r_\nu(X)\right)^n r_0(X) \cdot \ldots \cdot r_{\beta-1}(X)\, a(X),$$

where $t = n\alpha + \beta$ with $n \in \mathbb{N}$ and $0 \le \beta < \alpha$. It shall be denoted by $A_T(r_0, \dots, r_{\alpha-1})$ and we call $A_T = A_T(r_0, \dots, r_{\alpha-1})$ the time-dependent cellular automaton induced by $A(r_0), \dots, A(r_{\alpha-1})$.

A time-dependent automaton applies periodically the local transition rules $r_0(X)$ up to $r_{\alpha-1}(X)$. Which local rule is used in a particular time step depends on the actual time itself. Time-dependent automata also generate fractal rescaled evolution patterns. We use the analog definition of the μ-th orbit representation for the time-dependent automata

$$X(A_T, \mu, \phi, \rho) \quad := \quad \bigcup_{t=0}^{\mu-1} G_\phi(A_T^t(\delta)) + (\phi(t), \rho t).$$

Proposition 3.4 *Let $r_0(X), \dots, r_{\alpha-1}(X) \in R[X]$ be polynomials such that $A(r)$ is m-Fermat where $r(X) := \prod_{\nu=0}^{\alpha-1} r_\nu(X)$. Let $A_T = A_T(r_0, \dots, r_{\alpha-1})$ be a time-dependent automaton. Furthermore, let $\phi : \mathbb{Z} \to \mathbb{Z}$ be a bounded map, and let $\rho \in \mathbb{N}$, $\rho > 0$ be a constant. Then we have*

$$\lim_{\mu \to \infty} \frac{1}{\alpha m^\mu} X(A_T, \alpha m^\mu, \phi, \rho) \quad = \quad F_{\alpha^{-1}}(X_\infty(A(r), \phi, \rho)),$$

where $F_{\alpha^{-1}} : \mathbb{R}^2 \to \mathbb{R}^2$, $(x,t) \mapsto (\alpha^{-1}x, t)$.

Proof: The LCA $A(r)$ can be interpreted as the α-th power of A_T. Then the assertion follows from proposition 3.2.

The next new class of CA are the place-dependent automata.

Definition 3.7 *Let $r_0(X), \dots, r_{\alpha-1}(X) \in R[X]$ be polynomials. The map $A_P = A_P(r_0, \dots, r_{\alpha-1}) : R[X] \to R[X]$ defined for $i \in \mathbb{Z}$ by*

$$A_P(a)(X)_i \quad := \quad A_{\nu(i)}(a)(X)_i,$$

where $\nu(i) \in \{0, \ldots, \alpha - 1\}$ and $\nu(i) \equiv i \, (mod \, \alpha)$, is called the place-dependent automaton induced by $A(r_0), \ldots, A(r_{\alpha-1})$.

Place-dependent automata apply periodically α local rules depending on the location of the cells. For this class we have no general statement about the existence of corresponding limits in $\mathcal{H}(\mathbb{R}^2)$, but we will see that we can solve this problem for special cases.

4 Fractal Patterns of Classical Number Tables

We will now study the number tables modulo a prime power p^ν and will define an appropriate graphical representation by

$$X_{p^\nu}(\{F(n,k)\}, \mu) \;:=\; \bigcup_{n=0}^{\mu-1} \{I(k,n) \mid F(n,k) \not\equiv 0 \, (mod \, p^\nu)\}.$$

We have seen that the ordinary LCA $A = A(r)$ induced by $r(X) = X + 1 \in \mathbb{Z}_{p^\nu}[X]$ produces the binomials $(mod \, p^\nu)$. Remark 3.1 yields that the LCA $B = B(s)$ with $s(X) := r(X)^{p^{\nu-1}}$ is p-Fermat and therefore, with theorem 3.1 the existence of a rescaled evolution pattern $X_\infty(B)$ is guaranteed. In fact the fractal and self-similarity features of these patterns were studied in [9] using hierarchical iterated function systems and elementary number theoretical results due to E.E. Kummer from 1852. We will denote the rescaled evolution pattern by $\mathcal{S}_{p^\nu}$ and call it the Sierpinski triangle $(mod \, p^\nu)$. We will demonstrate that this fractal set $\mathcal{S}_{p^\nu}$ is in some sense universal for all considered number tables when they are considered modulo prime powers.

4.1 Gaussian q-Binomials

Let us start with the Gaussian q-binomials. The following proposition can be easily verified by induction.

Proposition 4.1 *For $i \in \mathbb{N}$ and $q \in \mathbb{N}\backslash\{0,1\}$ let $r_i(X) = X + q^i$. Then we have for all $n \in \mathbb{N}$*

$$\prod_{i=0}^{n} r_i(X) \;=\; \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q q^{\binom{n+1-k}{2}} X^k.$$

From proposition 4.1 it follows that the Gaussian q-binomials $(mod \, p^\nu)$ can be computed up to certain factors by a time-dependent automaton with local rules $r_i(X) = X + q^i \in \mathbb{Z}_{p^\nu}[X]$ provided that p does not divide q.

Theorem 4.1 *Let p be a prime and let $\nu \in \mathbb{N}$, $\nu > 0$. Let $q \in \mathbb{N}$ with $q \equiv \beta\,(mod p^\nu)$ for $\beta \geq 2$ and such that p does not divide q. Furthermore, let α be the smallest natural number with $q^\alpha \equiv 1 (mod p^\nu)$. Then*

$$\lim_{\mu\to\infty} \frac{1}{\alpha p^\mu} X_{p^\nu}\left(\left\{\begin{bmatrix} n \\ k \end{bmatrix}_q\right\}, \alpha p^\mu\right) = F_{p^{1-\nu}}(S_{p^\nu}),$$

where $F_{p^{1-\nu}} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $(x,t) \mapsto (p^{1-\nu}x, t)$.

Proof: Let $r_i(X) := X + q^i \in \mathbb{Z}_{p^\nu}[X]$ for $i = 0, \dots, \alpha$ and let A_T be the induced time-dependent cellular automaton $A_T(r_0, \dots, r_{\alpha-1})$. Let $r(X) := \prod_{i=1}^{\alpha} r_i(X)$ and $\hat{r}(X) := r(X)^{p^{\nu-1}}$. If α' is the smallest natural number with $q^{\alpha'} \equiv 1\,(mod p)$, it is clear that α' divides α and it follows for $s_1(X) := \prod_{i=1}^{\alpha'} r_i(X)$

$$r(X) \equiv s_1(X)^{\frac{\alpha}{\alpha'}}\ (mod p).$$

Claim: p divides $\begin{bmatrix} \alpha' \\ k \end{bmatrix}_q$ for $0 < k < \alpha'$.

p divides $(q^{\alpha'} - 1)$ where α' is minimal. Therefore, p does not divide any term in the denominator of

$$\begin{bmatrix} \alpha' \\ k \end{bmatrix}_q = \frac{(q^{\alpha'}-1)(q^{\alpha'-1}-1)\cdot\ldots\cdot(q^{\alpha'-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdot\ldots\cdot(q-1)}$$

This proves our claim.

From this it follows that $s_1(X) \equiv s_2(X)\,(mod p)$ where $s_2(X) := X^{\alpha'} + q^{\binom{\alpha'}{2}} \in \mathbb{Z}_{p^\nu}[X]$. Then remark 3.1 provides the congruence

$$\hat{r}(X) = r(X)^{p^{\nu-1}} \equiv s_2(X)^{p^{\nu-1}\frac{\alpha}{\alpha'}}\ (mod p^\nu).$$

Let $s_3(X) := s_2(X)^{p^{\nu-1}}$. We now show
Claim: $\frac{\alpha}{\alpha'} = p^l$ for a $l \in \mathbb{N}$.

Note that $q^{\alpha'} \equiv 1\,(mod p)$. We apply remark 3.1 and get $q^{\alpha' p^{\nu-1}} \equiv 1\ (mod p^\nu)$. Moreover, α divides $\alpha' p^{\nu-1}$, therefore there is an $n_0 \in \mathbb{N}$ such that $\alpha' p^{\nu-1} = n_0\alpha$ and $p^{\nu-1} = n_0\frac{\alpha}{\alpha'}$. Since every natural number possesses a unique factorization as a product of prime powers, there exists an $l \in \mathbb{N}$ with $\frac{\alpha}{\alpha'} = p^l$.

Now $A(s_3)$ is p-Fermat, and thus we get by means of our second claim:

$$\hat{r}(X) \equiv (X^\alpha + q^{\frac{\alpha}{\alpha'}\binom{\alpha'}{2}})^{p^{\nu-1}}\ (mod p^\nu).$$

Since p does not divide q, $A(\hat{r})$ is the α-th inner power of $A(s)$, whereas $s(X) := (X+1)^{p^{\nu-1}} \in \mathbb{Z}_{p^\nu}[X]$. Now proposition 3.1 yields $X_\infty(A(\hat{r})) =$

$F_\alpha(S_{p^\nu})$. Furthermore, we apply proposition 3.4 and get

$$\lim_{\mu\to\infty} \frac{1}{\alpha p^{\mu+\nu-1}} X(A_T, \alpha p^{m+\nu-1}, \phi, \rho) = F_{\frac{1}{\alpha p^{\nu-1}}}(X_\infty(A(\hat{r})))$$

where $\phi \equiv 0$ and $\rho = 1$. Altogether we have

$$\lim_{\mu\to\infty} \frac{1}{\alpha p^{\mu+\nu-1}} X_{p^\nu}\left(\left\{\begin{bmatrix} n \\ k \end{bmatrix}_q\right\}, \alpha p^{\mu+\nu-1}\right) = F_{\frac{1}{p^{\nu-1}}}(S_{p^\nu}),$$

and this is what we had to show.

4.2 Stirling numbers of first kind

The Stirling numbers of 1st kind can be modelled by means of a time-dependent automaton, too. For them we get the next theorem:

Theorem 4.2 *Let p be a prime number and $\nu \in \mathbb{N}$, $\nu > 0$. Then*

$$\lim_{\mu\to\infty} \frac{1}{p^\mu} X_{p^\nu}(\{s(n,k)\}, p^\mu) = F(S_{p^\nu}),$$

where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $(x,t) \mapsto (p^{-\nu}(p-1)x + p^{-1}t, t)$.

Proof: Let $\alpha := p^\nu$ and $\alpha' := p$. Let $r_i(X) := X + i \in \mathbb{Z}_{p^\nu}[X]$ for $i = 0, \dots, \alpha$ and $A_T = A_T(A(r_0), \dots, A(r_{\alpha-1}))$ the induced time-dependent automaton. Then we have with $\delta(X) := 1 \in \mathbb{Z}_{p^\nu}[X]$

$$A_T^n(\delta)(X)_k = s(n,k) \ (mod\, p^\nu).$$

Furthermore, let $r(X) := \prod_{i=1}^{\alpha} r_i(X)$ and $s_1(X) := \prod_{i=1}^{\alpha'} r_i(X)$. We have $s_1(X) \equiv [X]_p \ (mod\, p)$ and also $r(X) \equiv s_1(X)^{\frac{\alpha}{\alpha'}} \ (mod\, p)$.

If $p > 2$, then p divides the Stirling numbers of 2nd kind $S(p,k)$ for $1 < k < p$ (cf. [22]) and with equation 1 we get $X^p \equiv X + [X]_p \ (mod\, p)$. This means $s_1(X) \equiv s_2(X) \ (mod\, p)$ where $s_2(X) = X^{\alpha'} + (p-1)X \in \mathbb{Z}_{p^\nu}[X]$. This is also true in case of $p = 2$.

Let $\hat{r}(X) := r(X)^{p^{\nu-1}}$, then remark 3.1 provides

$$\begin{aligned} \hat{r}(X) &\equiv s_2(X)^{p^{2(\nu-1)}} \ (mod\, p^\nu) \\ &= X^{p^{2(\nu-1)}} \cdot s_3(X)^{p^{\nu-1}} \end{aligned}$$

with $s_3(X) := (X^{\alpha'-1} + (p-1))^{p^{\nu-1}} \in \mathbb{Z}_{p^\nu}[X]$. Note that $A(s_3)$ is p-Fermat, because of remark 3.1, and thus $s_4(X) := s_3(X)^{p^{\nu-1}} = s_3(X^{p^{\nu-1}})$. Now we have

$$\hat{r}(X) = X^{p^{2(\nu-1)}} s_4(X). \tag{4.8}$$

Let $s(X) := (X+1)^{p^{\nu-1}} \in \mathbb{Z}_{p^\nu}[X]$. Then $A(s_4)$ is the $(\alpha'-1)p^{\nu-1}$-th inner power of $A(s)$, and we get by means of proposition 3.1

$$X_\infty(A(s_4)) \quad = \quad F_{(\alpha'-1)p^{\nu-1}}(S_{p^\nu}).$$

For the rescaled evolution pattern of $A(\hat{r})$ we have

$$X_\infty(A(\hat{r})) \quad = \quad F'(X_\infty(A(s_4)))$$

because of 4.2 and proposition 3.3, where $F' : \mathbb{R}^2 \to \mathbb{R}^2$, $(x,t) \mapsto (x + p^{2(\nu-1)}t,\, t)$.

Furthermore, $A(\hat{r})$ is the $\alpha p^{\nu-1}$-th power of A_T. This yields

$$\lim_{\mu\to\infty} \frac{1}{\alpha p^{\mu+\nu-1}} X(A_T, \alpha p^{m+\nu-1}, \phi, \rho) \quad = \quad F_{\frac{1}{\alpha p^{\nu-1}}}(X_\infty(A(\hat{r}))).$$

where $\phi \equiv 0$ and $\rho = 1$. Altogether we have

$$\lim_{\mu\to\infty} \frac{1}{p^{\mu+2\nu-1}} X(A_T, p^{\mu+2\nu-1}, \phi, \rho) \quad = \quad F(S_{p^\nu}),$$

where F is defined as in the statement of the theorem.

4.3 Stirling numbers of second kind

The Stirling numbers of 2nd kind $S(n,k)$ modulo a prime power p^ν are much more difficult, because they are generated by a place-dependent automaton, for which we have no general result concerning the existence of rescaled evolution patterns. Furthermore, we only consider the case of $S(n,k)$ modulo a prime (i.e. $\nu = 1$).

It is necessary to introduce a new graphical representation for the Stirling numbers of 2nd kind, because we do not rescale them at horizontal but diagonal lines. Now for $\mu \in \mathbb{N}$ let

$$\tilde{X}_p(\{S(n,k)\},\mu) := \bigcup_{n=0}^{\mu(p-1)} \{I(k, n + \left[\frac{k}{p}\right]) \mid S(n + \left[\frac{k}{p}\right], k) \not\equiv 0\,(mod\,p)\},$$

where for $a \in \mathbb{R}$ the expression $[a]$ is the integer part of a. If we call the sequence

$$\left\{S(n + \left[\frac{k}{p}\right], k)\right\}_{k\in\mathbb{N}}$$

a staircase-line at n, then $\tilde{X}_p(S(n,k),\mu)$ represents all staircase-lines at 0 up to $\mu(p-1)$. We have the result

Theorem 4.3 *Let $p > 2$ a prime number. Then*

$$\lim_{\mu \to \infty} \frac{1}{p^\mu} \tilde{X}_p(\{S(n,k)\}, p^\mu) = F(S_p),$$

where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $(x,t) \mapsto (px, x + (p-1)t)$.

For the proof of theorem 4.3 we need a result from M. Sved (cf. [22]) from which we deduce a corollary about particular staircase-lines.

Theorem 4.4 (M. Sved) *Let $p > 2$ a prime number and $S(n,k)$ a Stirling number of 2nd kind. Further let*

$$n' := \left[\frac{pn - p\left[\frac{k}{p}\right] - 1}{p-1} \right]$$

and $n' = \sum_{\nu=0}^{h} a_\nu p^\nu$, $k = \sum_{\nu=0}^{h} b_\nu p^\nu$ the p-adic expansion of n' resp. k, where $a_h \neq 0$. Then we have

$$S(n,k) \equiv \binom{\frac{n'-a_0}{p}}{\frac{k-b_0}{p}} S(a_0, b_0) \ (mod p),$$

if p does not divide k. If p divides k, then p divides also $S(n,k)$, except for $n' \equiv -1 \,(mod p^l)$, where p^l is the greatest power of p which divides k. In this case is

$$S(n,k) \equiv \binom{a_h}{b_h}\binom{a_{h-1}}{b_{h-1}} \cdot \ldots \cdot \binom{a_{l+1}}{b_{l+1}}\binom{a_l}{b_l - 1} \ (mod p).$$

Now we consider certain staircase-lines of $\{S(n,k)\,(mod p)\}_{n,k}$.

Corollary 4.1 *Let $p > 2$ a prime number. Then one has for $i, j \in \mathbb{N}$ the congruence*

$$S(j(p-1) + 1 + \left[\frac{k}{p}\right], k) \equiv \binom{j}{i} \ (mod p),$$

if $k = ip + 1$, $1 \leq k \leq jp + 1$ and

$$S(j(p-1) + 1 + \left[\frac{k}{p}\right], k) \equiv 0 \ (mod p)$$

otherwise.

Proof: For $j \in \mathbb{N}$ let $\tilde{n} := j(p-1)+1$ and additionally let $k := ip+s$ with $0 \le s \le p-1$ and $1 \le k \le jp+1$. We want to calculate the remainders of $S(n,k)\,(mod\,p)$ where $n := \tilde{n} + \left[\frac{k}{p}\right]$. We have

$$n' := \left[\frac{pn - p\left[\frac{k}{p}\right] - 1}{p-1}\right] = \left[\frac{p\tilde{n}-1}{p-1}\right]$$
$$= \left[\frac{pj(p-1)+p-1}{p-1}\right] = jp+1.$$

Let $n' = \sum_{i=0}^{n} a_i p^i$, $k = \sum_{i=0}^{n} b_i p^i$, with $a_h \neq 0$, be the corresponding p-adic expansions. Especially we have $a_0 = 1$ and $b_0 = s$. We distinguish several cases:

i) p does not divide k.
Then we have with theorem 4.4

$$S(n,k) \equiv \binom{\frac{n'-a_0}{p}}{\frac{k-b_0}{p}} S(1,s)\ (mod\,p),$$

If $s = 1$, then it follows that $S(1,1) = 1$ and one gets

$$S(n,k) \equiv \binom{j}{i}\ (mod\,p).$$

If $s > 1$, then it follows that $S(1,s) = 0$, which means $S(n,k) \equiv 0\,(mod\,p)$.

ii) p divides k.

We have $n' \equiv 1\ (mod\,p)$, and therefore we get $n' \not\equiv -1\ (mod\,p^l)$ for all $l \in \mathbb{N}$, such that p^l divides k. Thus it follows with theorem 4.4 that $S(n,k) \equiv 0\,(mod\,p)$.

Remark 4.1 *It is also possible to get corollary 4.1 from a result of L. Carlitz (cf. [6]).*

Proof: (Theorem 4.3) The proof consists of the following steps. Corollary 4.1 shows that the staircase-line $(u_k)_{k\in\mathbb{N}}$, $u_k \in \mathbb{Z}_p$, at $j(p-1)+1$ for $j \in \mathbb{N}$ contains in column $ip+1$ the binomial coefficient $\binom{j}{i}\,(mod\,p)$, i.e. $u_{ip+1} = \binom{j}{i}\,(mod\,p)$. All other entries are zero. This means that the Stirling numbers of 2nd kind $(mod\,p)$ include a certain distorted version of the number table of the binomials $(mod\,p)$. Having this, it is not difficult to prove that $\mathcal{S}_p$ is affine homeomorphic to $\lim_{\mu\to\infty} p^{-\mu}\,\tilde{X}_p(\{S(n,k)\}, p^{\mu})$.

Let A_P be the place-dependent automaton induced by $A(r_i)$ with $r_i(X) := X + i \in \mathbb{Z}_p[X]$ for $i = 0, \ldots, p-1$. This automaton satisfies

$$A_P^n(\delta)(X)_k = S(n,k)\ (mod\,p).$$

Now let $s(X) := X + 1 \in \mathbb{Z}_p[X]$, $\tilde{s}(X) := s(X^p)$. Then Corollary 4.1 provides for $\delta_1(X) := X$

$$A^j(\tilde{s})(\delta_1)(X)_k = S(j(p-1) + 1 + \left[\frac{k}{p}\right], k) \ (mod\, p).$$

Let $\phi : \mathbb{Z} \to \mathbb{Z}$ be defined by $k \mapsto \left[\frac{k+1}{p}\right]$, and let $H : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $(x,t) \mapsto (x+1, t+1)$. Then we have that

$$H\left(X(A(\tilde{s}), p^\mu, \phi, p-1)\right) \subset \tilde{X}_p(\{S(n,k)\}, p^\mu).$$

Now let $I(l,t) \subset \tilde{X}_p(\{S(n,k)\}, p^\mu)$, i.e. $A^t_P(\delta)(X)_l \neq 0$. Since all place-dependent local transformation rules $A(r_i)$ are linear and of degree 1, there exists $I(l',t') \subset H\left(X(A(\tilde{s}), p^\mu, \phi, p-1)\right)$ with $I(l,t) \subset \left(I(l',t')\right)_p$. Therefore we can conclude that

$$h\left(H\left(X(A(\tilde{s}), p^\mu, \phi, p-1)\right), \tilde{X}_p(\{S(n,k)\}, p^\mu)\right) \leq p-1.$$

Now we consider the rescaled evolution patterns. To apply theorem 3.1 for $A(\tilde{s})$ it is necessary to check if ϕ fulfills the condition of the theorem. It is easy to see that the following estimate for $j \in \mathbb{Z}$ and $-(p-1)p \leq i \leq 0$ holds

$$|\,\phi(pj - i) - p\phi(j)\,| \leq p.$$

Therefore $X_\infty(A(\tilde{s}), \phi, p-1)$ exists, and for all $\mu \in \mathbb{N}$ we have

$$h\left(H\left(X(A(\tilde{s}), p^\mu, \phi, p-1)\right), X(A(\tilde{s}), p^\mu, \phi, p-1)\right) \leq const.$$

Hence

$$\lim_{m \to \infty} \frac{1}{p^\mu} \tilde{X}_p(\{S(n,k)\}, p^\mu) \ = \ X_\infty(A(\tilde{s}), \phi, p-1)$$

In the next step we eliminate the dependence from ϕ to clarify the global geometric structure of the Stirling numbers. We claim that

$$X_\infty(A(\tilde{s}), \phi, p-1) \ = \ F'(X_\infty(A(\tilde{s}))),$$

where $F' : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $(x,t) \mapsto (x, \frac{1}{p}x + (p-1)t)$.

This can be easily seen. First we have

$$I(l,t) \subset X(A(\tilde{s}), p^\mu) \text{ if and only if } I(l,t') \subset X(A(\tilde{s}), p^\mu, \phi, p-1),$$

where $t' = (p-1)t + \left[\frac{l+1}{p}\right]$. Further $h\left(F'(I(l,t)),\, I(l,t')\right) \leq 2$. Hence we get $h\left(F'(X(A(\tilde{s}), p^\mu),\, X(A(\tilde{s}), p^\mu, \phi, p-1)\right) \leq 2$, which proves the claim.

Finally we note that the LCA $A(\tilde{s})$ is the p-th inner power of $A(s)$ and then proposition 3.1 yields

$$\lim_{\mu\to\infty} \frac{1}{p^\mu} X_p(\{S(n,k)\}, p^\mu) = F(S_p),$$

where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $(x,t) \mapsto (px, \frac{1}{p}x + (p-1)t)$. This proves the theorem.

Remark 4.2 *It is also possible to extend theorem 4.3 to the case of $p = 2$. Then we have $\lim_{\mu\to\infty} \tilde{X}_2(S(n,k), 2^\mu) = F(S_2)$, where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $(x,t) \mapsto (2x, \frac{1}{2}x + t)$.*

5 Growth Rate of non-trivial States

Let V be a finite set with a fixed (distinguished) element $v_0 \in V$. We call the elements of V states and in particular v_0 the trivial state. Let $\{F(n,k)\}_{n,k\in\mathbb{N}}$ be a number table in V. For a given non-trivial $v \in V$ we consider the sequence

$$N(\mu, v) := card\{\ (n,k) \mid \exists 0 \le n \le \mu - 1,\ k \in \mathbb{N} \text{ such that } F(n,k) = v\}$$

for $\mu \in \mathbb{N}$. We also count the total number of non-trivial states by defining

$$N(\mu) := \sum_{v\in V, v\neq v_0} N(\mu, v).$$

Definition 5.1 *A non-trivial state v appearing in the sequence $\{F(n,k)\}_{n,k}$ has a power law growth rate α if the limit*

$$\lim_{\mu\to\infty} \frac{\log N(\mu, v)}{\log \mu}$$

exists and is equal to α.
The total number of non-trivial states has a power law growth rate β in the sequence $\{F(n,k)\}_{n,k}$ if the limit

$$\lim_{\mu\to\infty} \frac{\log N(\mu)}{log \mu}$$

exists and is equal to β.

If the sequence $\{F_A(n,k)\}_{n,k\in\mathbb{N}}$ is obtained as the orbit of any initial configuration with finitely many non-trivial states under iteration of an m-Fermat automaton A, then the total number of non-trivial states has the same power law growth rate β and that is equal to the box-counting dimension $dim_B\ X_\infty(A)$ (cf. [7]) of the rescaled evolution pattern $X_\infty(A)$ of A (cf. [11]), i.e

$$\lim_{\mu\to\infty} \frac{\log N(m^\mu)}{m \log \mu} = dim_B\ X_\infty(A).$$

Results like that were first proved by S. J. Willson (cf. [27]) for LCA with states in $\mathbb{Z}_{p^\nu}$. He also showed that the power law growth rate is independent from ν.

Example 5.1 *Let $F(n,k) := \binom{n}{k}\ (mod\, p^\nu)$, then the power law growth rate of all non-trivial states of this sequence is $\log_p \frac{p(p+1)}{2}$ where p is a prime number (cf. also [14]). This is the box-counting dimension of S_{p^ν} which is independent from ν (cf. [9], corollary 3.2).*

The result mentioned above combined with the theorems 4.1, 4.2, 4.3 and corollary 3.2 in [9] yield the following corollary.

Corollary 5.1 *Let p be a prime number and $\nu \in \mathbb{N}$, $\nu > 0$. Let q be a natural number as in theorem 4.1. Then the Gaussian q-binomials $(mod\, p^\nu)$, the Stirling numbers of first kind $(mod\, p^\nu)$ and second kind $(mod\, p)$ all have the same growth rate of the total number of non-zero states and that is equal to $\log_p \frac{p(p+1)}{2}$*

Remark 5.1 *For the Stirling numbers of first kind the case $\nu = 1$ has been proven in [5] as well.*

We will now examine the growth rate of each non-trivial state appearing in the recursively defined number sequences. To do that we present another approach to prove a corresponding result from S. J. Willson (cf. [27], Corollary 7.9) for LCA with states in $\mathbb{Z}_p$.

Let p be a prime number and $r(X) \in \mathbb{Z}_p[X]$ be a polynomial. Then the orbit $\{A^n(\delta)\}_{n\in\mathbb{N}}$ of $\delta(X)$ under iteration of $A = A(r)$ generates a number sequence $\{F_A(n,k)\}_{k,n\in\mathbb{N}}$ where $F_A(n,k) = A^n(\delta)(X)_k$ for all k and n. Furthermore, we define for each non-zero element v of $\mathbb{Z}_p$ and for all $\mu \in \mathbb{N}$

$$X^{(v)}(A,\mu) := \{\ (n,k) \mid \exists 0 \le n \le \mu - 1,\ k \in \mathbb{N} \text{ such that } F_A(n,k) = v\}.$$

A non-trivial state $v \in \mathbb{Z}_p$ is called accessible (with respect to A) if there exist $k, n \in \mathbb{N}$ such that $F_A(n,k) = v$, hence, $X^{(v)}(A,\mu) \ne \emptyset$ for all sufficiently large $\mu \ge \mu_0$.

Proposition 5.1 *Let A be a LCA with states in $\mathbb{Z}_p$ and $v \in \mathbb{Z}_p$ an accessible state. Then*

$$\lim_{\mu\to\infty} \frac{1}{p^\mu} X^{(v)}(A, p^\mu) = X_\infty(A),\ \ \mu \ge \mu_0$$

with respect to the Hausdorff-metric.

Proof: We make use of the fact that a sequence of compact subsets $\{A_\mu\}_{\mu\in\mathbb{N}}$ in a given compact metric space X converges with respect to the Hausdorff-metric if and only if its limes inferior and limes superior coincide, i.e. $\underline{\lim} A_\mu = \overline{\lim} A_\mu$ (cf. [16]) where these sets are defined in the following way:

$$\begin{aligned}\overline{\lim} A_\mu &:= \{\, x \in X \mid \exists\, \{n_i\}_{i\in\mathbb{N}},\ n_i \in \mathbb{N},\ n_i < n_{i+1} \text{ for all } i, \\ &\qquad \text{and } x_i \in A_{n_i},\ i \in \mathbb{N} \text{ such that } \lim_{i\to\infty} x_i = x \,\} \\ \underline{\lim} A_\mu &:= \{\, x \in X \mid \exists x_i \in A_i \text{ for all } i \in \mathbb{N} \text{ s.t. } \lim_{i\to\infty} x_i = x \,\}.\end{aligned}$$

Let A be an LCA with states in $\mathbb{Z}_p$ and let $v \in \mathbb{Z}_p$ be an accessible state. First we prove that the sequence $\{\frac{1}{p^\mu} X^{(v)}(A, p^\mu)\}$ converges with respect to the Hausdorff-metric. Let $Y_\mu := \frac{1}{p^\mu} X^{(v)}(A, p^\mu)$, then it is sufficient to show that

$$\overline{\lim}\, Y_\mu \subset \underline{\lim}\, Y_\mu. \tag{5.9}$$

Let $(x, y) \in \overline{\lim}\, Y_\mu$, i.e. there exits an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ and points $(x_k, y_k) \in X^{(v)}(A, p^{n_k})$ such that $(x, y) = \lim_{k\to\infty} \frac{1}{p^{n_k}}(x_k, y_k)$.

We have $F_A(x_k, y_k) = v$ where $0 \le x_k \le p^{n_k} - 1$. Now define the sequence $(\bar{x}_k, \bar{y}_k) \in \mathbb{N} \times \mathbb{N}$ by setting $\bar{x}_{n_k} := x_k$, $\bar{y}_{n_k} := y_k$ and $\bar{x}_k := x_i\, p^{\nu_k}$, $\bar{y}_k := y_i\, p^{\nu_k}$ where $n_i < k < n_{i+1}$ and $k = n_i + \nu_k$.

Then $(\bar{x}_\mu, \bar{y}_\mu) \in X^{(v)}(A, p^\mu)$ for all $\mu \in \mathbb{N}$ and $(x_0, y_0) = \lim_{\mu\to\infty} \frac{1}{p^\mu}(x_\mu, y_\mu)$ which proves equation 1. Hence $\{Y_\mu\}_{\mu\in\mathbb{N}}$ converges. Let $X^{(v)}_\infty(A) := \lim_{\mu\to\infty} Y_\mu$.

Let v_1 and v_2 be accessible states for the automaton A, i.e. $F_A(n_1, k_1) = v_1$ and $F_A(n_2, k_2) = v_2$ with n_i, $k_i \in \mathbb{N}$ for $i = 1, 2$. Then the state $v_1 v_2$ is also accessible (cf. [27]). We go on and prove the inclusion

$$X^{(v_1)}_\infty(A) \subset X^{(v_1 v_2)}_\infty(A). \tag{5.10}$$

Let $(x, y) \in X^{(v_1)}_\infty(A)$ and $(x_\mu, y_\mu) \in X^{(v_1)}(A, p^\mu)$ for $\mu \in \mathbb{N}$ such that $(x, y) = \lim_{\mu\to\infty} \frac{1}{p^\mu}(x_\mu, y_\mu)$. Let $l \in \mathbb{N}$ be a natural number such that $max\{d\, k_2,\, n_2\} < p^l$ where $d = deg\ r(X)$. Then

$$(\bar{x}_\mu, \bar{y}_\mu) := (p^l x_\mu + n_2, p^l y_\mu + k_2) \in X^{(v_1 v_2)}(A, p^{\mu+l})$$

and hence

$$\lim_{\mu\to\infty} \frac{1}{p^{\mu+l}}(\bar{x}_\mu, \bar{y}_\mu) = (x, y) \in X^{(v_1 v_2)}_\infty(A).$$

The set Z of all accessible states with respect to A forms a multiplicative group. Then the inclusion 1 implies $X^{(u)}_\infty(A) = X^{(v)}_\infty(A)$ for allaccessible states u and v. Obviously, we have

$$X'(A,p^\mu) = \bigcup_{v \in Z} X^{(v)}(A,p^\mu),$$

where $X'(A,p^\mu) := \{\ (n,k) \mid F_A(n,k) \neq 0, 0 \leq n < p^\mu\ \}$. It is known (cf. [11]) that

$$X_\infty(A) = \lim_{\mu \to \infty} X'(A,p^\mu).$$

Finally, we get the statement of the proposition from

$$X_\infty(A) = \bigcup_{u \in Z} X_\infty^{(u)}(A) = X_\infty^{(v)}(A)$$

for all $v \in Z$.

Corollary 5.2 (S.J. Willson) *Let $A = A(r)$ be an LCA with states in $\mathbb{Z}_p$ and v be an accessible state. The power law growth rate of the state v in the sequence $\{F_A(n,k)\}_{k,n \in \mathbb{N}}$ is equal to the box-counting dimension of the rescaled evolution pattern $X_\infty(A)$.*

Corollary 5.3 *Let p be a prime number and $q \in \mathbb{N}$ a natural number as in theorem 4.1. The power law growth rate of a non-zero state appearing in the Gaussian q-binomials (mod p), the Stirling numbers of first or second kind (mod p) is equal to $\log_p \frac{p(p+1)}{2}$.*

6 Open Questions

We have already mentioned one open question - the conjecture that the patterns generated by the Stirling numbers of second kind modulo a prime power p^ν is also homeomorphic to the Sierpinski triangle modulo p^ν.

Out of reach of our method is the deciphering of the self-similar features of the Eulerian numbers because they are modelled by a time- and place-dependent cellular automata for which we have no general result concerning the existence of rescaled evolution patterns. Nevertheless, experiments with the Eulerian numbers showed that the induced patterns are resembling the geometric structure of S_{p^ν} as well.

In [2] it was shown that the double sequences of Gaussian q-binomials and Stirling numbers of first kind ($mod\ p^\nu$) are p-automatic, i.e. they are generated by a 2-dimensional finite p-automaton (cf. [1] and [20] for the definitions and further references). We do not know whether the Stirling numbers of second kind modulo a given prime power are also p-automatic. It was also shown in [2] that the binomial coefficients ($mod\ m$), where m is not a prime power, are not k-automatic for any k. Is this also true for the Gaussian q-binomials and Stirling numbers ?

We did not analyze the distribution of the residues ($mod\ p^{\nu}$) in the number tables we considered. This question is connected with multifractal measures (cf. [8] for the definitions) on the corresponding rescaled evolution sets. For the binomial coefficients modulo a prime we refer to [8] and for a class of CA to [12].

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