On the complexity of coloring areflexive h-ary relations with given permutation group

by

L. Haddad, P. Hell and E. Mendelsohn

ABSTRACT

The following problem, known as the Strong Coloring Problem for the group G (SCP_G) is investigated for various permutation groups G. Let G be a subgroup of S_h the symmetric group on $\{0,\ldots,h-1\}$. An instance of SCP_G is an h-ary areflexive relation ρ whose group of symmetry is G and the question is "does ρ have a strong h-coloring"? Let $m \geq 3$ and D_m be the Dihedral group of order m. We show that SCP_{D_m} is polynomial for $m = 4$, and NP-complete otherwise. We also show that the Strong Coloring Problem for the wreath product of H and K is in P whenever both SCP_H and SCP_K are in P. This, together with the algorithm for D_4 yields an infinite new class of polynomially solvable cases of SCP_G.

1. PRELIMINARIES

Let A be a finite set. An h-ary relation ρ on A is said to be areflexive if for every $(x_0,\ldots,x_{h-1}) \in \rho$ and all $0 \leq i < j \leq h-1$, we have $x_i \neq x_j$.

Let S_h be the symmetric group on $\underline{h} := \{0,\ldots,h-1\}$, and let $\pi \in S_h$. For an h-ary relation ρ, let

$$\rho^{(\pi)} := \{(x_{\pi(0)},\ldots,x_{\pi(h-1)}) : (x_0,\ldots,x_{h-1}) \in \rho\}.$$

We say that ρ is *symmetric* (respectively *asymmetric*) with respect to π if $\rho = \rho^{(\pi)}$ (respectively $\rho \cap \rho^{\pi} = \emptyset$).

Let ρ be an h-ary relation on a finite set A and suppose that there exists a subgroup G of S_h such that ρ is symmetric with respect to each $\pi \in G$ and asymmetric with respect to each $\pi \in S_h \setminus G$. Then we say that G is the *symmetry group* of ρ. Note that if ρ admits a symmetry group, then such a group is unique. If G is the symmetry group of ρ, then we define the *model* of ρ to be the h-ary relation $M_\rho := \{(\pi(0),\ldots,\pi(h-1)) : \pi \in G\}$ on the set $\underline{h} = \{0,\ldots,h-1\}$.

We also define a *strong h-coloring* of ρ to be a map $\phi : A \to \underline{h}$ which is a relational homomorphism from ρ to M_ρ (i.e. for every $(x_0,\ldots,x_{n-1}) \in \rho$, $(\phi(x_0),\ldots,\phi(x_{h-1})) \in M_\rho$). Note that a strong h-coloring is a surjective map.

Examples.

(1) Let $\rho = (A, E)$ be a simple graph. Then $h = 2$, $G = S_2$, and ρ has $M_\rho = \{(0,1),(1,0)\}$ and ρ has a strong 2-coloring if and only if ρ is bipartite.

(2) Let $h = 3$ and $G =< (0\ 1\ 2) >$ the subgroup of S_3 generated by the cycle $(0\ 1\ 2)$. Then G is the symmetry group of the ternary relation $\rho := \{(0,1,2),(2,0,1),(1,2,0),(3,0,4),(4,3,0),(0,4,3),(2,5,4),(4,2,5),(5,4,2)\}$ on the set $\underline{6} = \{0,1,2,3,4,5\}$. Here $M_\rho = \{(0,1,2),(2,0,1),(1,2,0)\}$ and $\phi : \underline{6} \to \underline{3}$ defined by $\phi(0) = \phi(5) = 0$, $\phi(1) = \phi(4) = 1$, $\phi(2) = \phi(3) = 2$ is a strong 3-coloring of ρ.

Now the motivation for studying h-ary areflexive relations that admit a strong h-colouring is their link to maximal partial subalgebras. This is discussed in [4] and, in more detail, in [6].

Let $h \geq 2$ and consider a subgroup G of S_h operating on the set $\underline{h} := \{0, \ldots, h-1\}$. Then the *Strong Coloring Problem* for G (SCP_G) is stated as follows:

Instance *An h-ary areflexive relation ρ on the finite set A whose symmetry group is G.*

Question *Does ρ have a strong h-coloring?*

Note that for every (fixed) symmetry group G, the SCP_G is in NP. The study of its complexity has been started in [4]. We recall the following:

Theorem 1.1 [4] *If G is a regular subgroup of S_h, then the SCP_G is in P.* □

An obvious consequence of this is that for any abstract group, G, there is a permutation group G', $G \cong G'$ and SCP_G is in P. Note that Theorem 1.1 extends to semi-regular subgroups of S_h, i.e., with the property that for every $i \neq j$, there exists at most one $\alpha \in G$ such that $\alpha(i) = j$.

The strong coloring problem is closely related to the H-coloring problem for graphs.

Let H be a fixed graph. An H-coloring of a graph E is a graph homomorphism $\phi \in \mathrm{Hom}(E, H)$, that means a mapping $\phi : V(E) \to V(H)$ such that $(\phi(x), \phi(y))$ is an edge of H whenever (x, y) is an edge of E. The

H-coloring problem for graphs is stated as follows:

Instance *A graph E.*

Question *Does there exist an H-coloring of E?*

The complexity of the H-coloring problem has been studied by several authors. In particular J. Nesetril together with the second author of the present paper proved the following.

Theorem 1.2. [8] *The H-coloring problem (for undirected graphs) is in P if H is bipartite and is NP-complete if H is not bipartite.* □

For directed graphs, the situation is more complex, cf[1,2]; only recently a conjecture classifying hard and easy instances of the H-coloring has been proposed [1]. Now Maurer et al. [9] have shown that the H-colouring problem is in P when H is a directed cycle or a transitive tournament, but that $C_{n,1}$-coloring is NP-complete for any odd integer $n > 2$. Also it has been shown [7] that if T is a tournament that contains at least two cycles then the T-coloring problem is NP-complete and if T has at most one cycle then the T-coloring problem is in P. Moreover a similar result has been shown for semicomplete digraphs [2].

Returning to the SCP_G, if $0 \leq i < j \leq h-1$ then the binary relation

$$R_{ij} := \{(\pi(i), \pi(j)) : \pi \in G\}$$

is called a biorbit of G on $\underline{h}$. We have

Theorem 1.3. [4] *Let $0 \leq i < j \leq h-1$ and let G be a subgroup of S_h operating on $\underline{h}$. If the R_{ij}-coloring problem for graphs is NP-complete then so is the SCP_G.* □

A direct consequence of this result is

Corollary 1.4. [4] *Let $n > 1$ and G be an n-fold transitive subgroup of S_h, then the SCP_G is NP-complete.* □

Comparing this result with Theorem 1.1, it is natural to expect that there are no transitive but not regular subgroups G of S_h for which the strong coloring problem can be decided in polynomial time. We shall construct an infinite family of such groups in section 3. In the following section, we show that among all dihedral groups of degree $m \geq 4$, only D_4 is such that the SCP_{D_4} is in P. The financial support of NSERC Canada operating grants and the Connaught Grants of the University of Toronto are gratefully acknowledged.

2. THE DIHEDRAL GROUP

Let $D_4 = (D_4, \underline{4})$ be the Dihedral group of degree 4 and order 8. Let ρ be a 4-ary areflexive relation on A whose symmetry group is D_4. Then the *Model* M of ρ is the 4-ary relation

$$M := \{(\pi(0), \pi(1), \pi(2), \pi(3)) : \pi \in D_4\}$$

on the set $\underline{4}$.

Notation. If λ is a 4-ary relation and $0 \leq i < j \leq 3$, then let λ_{ij} denote the binary relation defined by

$$\lambda_{ij} = pr_{ij}(\lambda) := \{(x_i, x_j) : (x_0, x_1, x_2, x_3) \in \lambda\}.$$

Hence

$$M_{01} = M_{12} = M_{23} = M_{30} = \{(0,1), (1,0), (1,2), (2,1), (2,3), (3,2), (0,3), (3,0)\}$$

which can be viewed as the undirected edges of a four-cycle. Also

$$M_{02} = M_{13} = \{(0,2), (2,0), (1,3), (3,1)\}$$

which can be viewed as an undirected graph with two disjoint edges. Hence M_{ij} is an bipartite graph for all $0 \leq i < j \leq 3$.

Moreover note that $\rho_{01} = \rho_{12} = \rho_{23} = \rho_{30}$, $\rho_{02} = \rho_{13}$ and $\rho_{ij} = \rho_{ji}$ for all $0 \leq i < j \leq 3$.

Lemma 2.1. *The relation ρ has a strong 4-coloring if and only if both ρ_{01} and ρ_{02} are bipartite.*

Proof. ($\Rightarrow$) Let ψ be a strong 4-coloring of ρ. Then for all $0 \leq i < j \leq 3$, $(\psi(x), \psi(y)) \in M_{ij}$, for all $(x, y) \in \rho_{ij}$.

Since M_{ij} is a bipartite graph, we get that ρ_{ij} is a bipartite graph as well for all $0 \leq i < j \leq 3$.

($\Leftarrow$) We assume that both ρ_{01} and ρ_{02} are bipartite. Let $\phi : A \to \{0,1\}$ and $\psi : A \to \{0,1\}$ be 2-colorings of (the undirected bipartite graphs) ρ_{01} and ρ_{02} respectively. Define the map $f : A \to \{0,1,2,3\}$ by setting $f(x) := 2\psi(x) + \phi(x)$. Let $(x_0, x_1, x_2, x_3) \in \rho$. Since each of x_0x_1, x_2x_3 and x_0x_3 is an edge of ρ_{01}, we have that $\phi(x_1) \neq \phi(x_0) \neq \phi(x_3)$ and $\phi(x_2) \neq \phi(x_3)$. So suppose (without loss of generality) that $\phi(x_0) = \phi(x_2) = 0$, $\phi(x_1) = \phi(x_3) = 1$ and $\psi(x_0) = 0$. If $\psi(x_1) = 0$, then $\psi(x_2) = \psi(x_3) = 1$ and $f(i) = i$ for $i = 0,1,2,3$. Else $\psi(x_1) = \psi(x_2) = 1$, $\psi(x_3) = 0$ and $(f(x_0), f(x_1), f(x_2), f(x_3)) = (0,3,2,1)$. In either case $(f(x_0), f(x_1), f(x_2), f(x_3)) \in M$. Thus $f \in \text{Hom}(\rho, M)$ and ρ is strongly 4-colorable. □

Corollary 2.2. *The SCP_{D_4} is in P.* □

The situation is quite different if $m > 4$. We will distinguish the two cases i) m has an odd factor and ii) $m = 2^n$ for some $n > 2$. For the first case we have the following:

Fact 2.3. *Let $m = nt$ with n and t two positive integers. Then R_{0t} is an undirected graph with a cycle of length n.*

Proof. The fact that R_{ot} is an undirected graph is straightforward. Clearly

$$(0,t) \in R_{ot} = \{(\pi(0), \pi(t)) : \pi \in D_m\},$$

and if $\alpha \in D_m$ is the rotation of the regular m-gon by $\frac{360}{n}$ degrees, then $(\alpha(0), \alpha(t)) = (t, 2t) \in R_{0t}$. Thus $\{(0,t), (t,2t), \ldots, ((n-1)t, 0)\} \subseteq R_{0t}$, proving that R_{0t} has a cycle of length n. □

Combining this Fact with Theorem 1.3 we deduce

Lemma 2.4. *If m has an odd prime factor, then the SCP_{D_m} is NP-complete.* □

Assume now that $m = 2^n > 4$ is a power of 2. Then we have

Lemma 2.5. *The $SCP_{D_{2^n}}$ is NP-complete for any fixed $n \geq 3$.*

Proof. We reduce the 3-vertex-colorability problem for graphs to the

$SCP_{D_{2^n}}$, the NP-completeness of graph 3-vertex coloring is well very well known (e.g., see [3] p.84).

Let $G = (V, E)$ be a simple t-vertex s-edge graph which is an instance of 3-vertex-coloring and assume $V = \{1, \ldots, t\}$. We define a 2^n-ary areflexive relation whose symmetry group is D_{2^n} as follows:

(1) There is a one "central" vertex x which lies in t otherwise disjoint 2^n-tuples $\underset{\sim}{x}_i := (x, x_{i1}, x_{i2}, \ldots, x_{i,2^n-1}),\ i = 1, \ldots, t$.

(2) there are t further 2^n-tuples $\underset{\sim}{y}_i := (x_{i1}, y_{i1}, \ldots, y_{i,2^n-1}), i = 1, \ldots, t$, where $y_{i\ell} \neq y_{i'\ell'}$ whenever $i \neq i'$ or $\ell \neq \ell'$,

(3) for every edge $e = \{i, j\} \in E(G)$, the vertices y_{i1} and y_{j1} are connected by a sequence of three 2^n-tuples

$$\underset{\sim}{z}_e := (y_{j1}, z_{e1}, z_{e2}, \ldots, z_{e,2^n-1}),$$
$$\underset{\sim}{u}_e := (z_{e3}, u_{e1}, u_{e2}, \ldots, u_{e,2^n-1}),$$
$$\underset{\sim}{v}_e := (u_{e4}, v_{e1}, v_{e2}, y_{i1}, v_{e4}, \ldots, v_{e,2^n-1}),$$

where the new $3 \,.\, 2^n - 4$ vertices $z_{e1}, \ldots, z_{e,2^n-1}, u_{e1}, \ldots, u_{e,2^n-1}, v_{e1}, \ldots, v_{e,2^n-1}$ are pairwise distinct (and for every edges $e \neq e'$, the set $\{z_{e1}, \ldots, v_{e,2^n-1}\}$ and $\{z_{e'1}, \ldots, v_{e',2^n-1}\}$ are disjoint).

For every 2^n-tuple $(x_0, \ldots, x_{2^n-1})$ constructed above, take the set $\{(x_{\pi(0)}, \ldots, x_{\pi(2^n-1)}) : \pi \in D_{2^n}\}$ and let ρ be the obtained 2^n-ary relation. Note that $|\rho| = 2^{n+1}(2t + 3s)$. Clearly D_{2^n} is the symmetry group of ρ. In the sequel the operations $+$ and $-$ are the addition and subtraction mod 2^n.

Claim 1. *For every strong 2^n-coloring of ρ, there exists a color c such that the vertices y_{i1} receive colors from the set $\{c-2, c, c+2\}$ for $i = 1, \ldots, t$.*

Proof. Let the vertex x receive color c. Then each x_{i1} is colored $c-1$, c or $c+1$ and thus y_{i1} receives color $c-2$, c or $c+2$ for all $i = 1, \ldots, t$.

Claim 2. *For every strong 2^n-coloring of ρ and every $e = \{i,\ j\} \in E$, the vertices y_{i1} and y_{j1} receive different colors.*

Proof. Let $\phi : V(\rho) \to \{0, \ldots, 2^n - 1\}$ be a strong 2^n-coloring of ρ and put $c := \phi(x) \in \{0, \ldots, 2^n - 1\}$. Therefore $\{\phi(x_{i1}), \phi(x_{j1})\} \subseteq \{c-1, c+1\}$

and thus $\{\phi(y_{i1}),\phi(y_{j1})\} \subseteq \{c-2,c,c+2\}$. We show that $\phi(y_{i1}) \neq \phi(y_{j1})$. Assume $\phi(y_{j1}) = c$. Then $\phi(z_{e3}) \in \{c-3,c+3\}$ and thus $\phi(u_{e4}) \in \{c \pm 1, c \pm 7\}$. This means that $\phi(y_{i1}) \in \{c-2,c,c+2\} \cap \{c \pm 2, c \pm 4, c \pm 10\}$. As $c \in \{0,\dots,2^n-1\}$ where $n \geq 3$, we see that c does not belong to the set on the right hand side proving that $\phi(y_{i1}) \neq \phi(y_{j1})$. The proofs are similar for $\phi(y_{j1}) = c+2$ or $c-2$.

Claim 3. *If all vertices $y_{11},\dots,y_{t1}$ are assigned arbitrary colors from $c-2$, c, $c+2$, so that y_{i1} and y_{j1} receive different colors whenever $\{i,j\} \in E$, then there exists a strong 2^n-coloring ψ of ρ that extends this assignment.*

Proof. Let $\phi : \{y_{11},\dots,y_{t1}\} \to \{c-2,c,c+2\}$ be such an assignment. Set $\psi(x) := c$ and let $i \in V(G)$. Choose $\psi(x_{i1})$ such that $c \pm 1 = \psi(x_{i1}) = \phi(y_{i1}) \pm 1$ (note that such a choice is unique if $\phi(y_{i1}) \neq c$) and extend ψ to the vertices $x_{i2},\dots,x_{i,2^n-1},y_{i2},\dots,y_{i,2^n-1}$ so that the integers $c,\psi(x_{i1}),\psi(x_{i2}),\dots,\psi(x_{i,2^n-1})$ and $\psi(x_{i1}),\psi(y_{i1}),\phi(y_{i2}),\dots,\psi(y_{i,2^n-1})$ are consecutive mod 2^n. Let $j \in V(G)$ form an edge with i and set $e := \{i,j\} \in E$. We extend ϕ to the vertices of the 2^n-tuples $\underset{\sim}{z_e}$, $\underset{\sim}{u_e}$ and $\underset{\sim}{v_e}$ as follows. Since $\phi(y_{i1}) \neq \phi(y_{j1})$ we have 6 different but similar cases. Assume $(\phi(y_{i1}),\phi(y_{j1})) = (c-2,c)$. Then put

$$(\psi(x),\psi(x_{i1}),\psi(x_{i2}),\dots,\psi(x_{i,2^n-1})) = (c,c-1,c-2,\dots,c-2^n+1),$$
$$(\psi(x_{i1}),\phi(y_{i1}),\psi(y_{i2}),\dots,\psi(y_{i,2^n-1})) = (c-1,c-2,\dots,c),$$
$$(\psi(x),\psi(x_{j1}),\psi(x_{j2}),\dots,\psi(x_{j,2^n-1})) = (c,c+1,c+2,\dots,c+2^n-1)$$
$$(\psi(x_{j1}),\phi(y_{j2}),\psi(y_{j3}),\dots,\psi(y_{j,2^n-1})) = (c+1,c,c-1,\dots,cc-2^n+2).$$

Moreover for $\ell = 1,\dots,2^n-1$, set

$$\psi(z_{e\ell}) := c-\ell, \psi(u_{e\ell}) := c+\ell-3, \psi(v_{e\ell}) := c-\ell+1.$$

It is easy to check that the three sequences of positive integers 1) $\phi(y_{j1}),\psi(z_{e1}),\dots,\psi(z_{e,2^n-1})$; 2) $\psi(z_{e3}),\psi(u_{e1}),\dots,\psi(u_{e,2^n-1})$ and 3) $\psi(u_{e4}),\psi(v_{e1}),\psi(v_{e2}),\phi(y_{i1}),\psi(v_{e4}),\dots,\psi(v_{e,2^n-1})$ are consecutive mod 2^n. As mentioned above the five remaining cases are similar. Since ρ contains vertex disjoint copies of the three 2^n-tuples $\underset{\sim}{z_e}$, $\underset{\sim}{u_e}$ and $\underset{\sim}{v_e}$ for every edge $e \in E$, Claim 3 follows.

We turn to the proof of the Lemma. Let $f : V(G) \to \{0,2,4\}$ be a 3-coloring of G. Set $\phi(y_{i1}) = f(i)$ for all $i \in V(G)$. Then $\phi(y_{i1}) \neq \phi(y_{j1})$ whenever $e = \{i,j\} \in E(G)$. By Claim 3 the assignment $\phi : \{y_{11},\dots,y_{t1}\} \to \{0,2,4\}$ can be extended to a strong 2^n-coloring of ρ. Conversely let

$\psi : V(\rho) \to \{0, \dots, 2^n - 1\}$ be a strong 2^n-coloring of ρ. Let $\psi(x) = c$. Then by Claims 1 and 2 the vertices y_{i1} ($i \in V(G)$) receive colors from $\{c-2, c, c+2\}$ such that $\phi(y_{i1}) \neq \phi(y_{j1})$ whenever $\{i, j\} \in E(G)$. Clearly $f : V(G) \to \{c-2, c, c+2\}$ defined by $f(i) := \phi(y_{i1})$ is a 3-coloring of the graph G. As the above construction can be achieved in polynomial time, the Lemma follows. □

Remark. The above proof can also be used to show that the SCP_{D_m} is NP-complete for all $m \neq 3, 4, 5$ and 10.

Combining Lemmas 2.1, 2.2 and 2.3 we deduce

Corollary 2.6. *(1) The SCP_{D_4} is in P and (2) if $m > 4$, then the SCP_{D_m} is NP-complete.* □

Note that D_4 is a non regular transitive subgroup of S_4.

We now give an interpretation in terms of graphs of these results. We need the following.

Definition 2.7. Let $H_0 = (V_0, E_0)$ be a subgraph of $H = (V, E)$. We say that H is a (colimit) *mosaic* of H_0 if there are a set Λ and a family of monomorphisms (i.e., injective graph homomorphisms) $i_\alpha : H_0 \to H$, $\alpha \in \Lambda$, that are pairwise distinct (i.e., $\alpha \neq \alpha'$ implies $Imi_\alpha \neq Imi_\alpha$) such that

1) $V = \bigcup_{\alpha \in \Lambda} Im\alpha$ and

2) for every $(a, b) \in E$, there are $(a_0, b_0) \in E_0$ and $\alpha \in \Lambda$ such that $(a, b) = (i_\alpha(a_0), i_\alpha(b_0))$. Moreover we say that H is a *finite mosaic* of H_0 if the set Λ is finite.

Examples 2.8.

1) Every graph is a finite mosaic of itself.

2) Let $m \geq 2$ and ρ be an m-ary areflexive relation on the set A whose symmetry group is D_m. Fix $(a_0, \dots, a_{m-1}) \in \rho$. Let $H_0 = (V_0, E_0)$ be defined by

$$V_0 = \{a_0, \dots, a_{m-1}\} \quad \text{and} \quad (a_i, a_j) \in E_0$$

if and only if either $i = j + 1 \pmod{m}$ or $j = i + 1 \pmod{m}$.

Moreover let $H = (V, E)$ be the graph defined by $V = A$ and $E = pr_{01}(\rho) := \{(x,y) : (x,y,z_2,\ldots,z_{m-1}) \in \rho$ for some $z_2,\ldots,z_{m-1} \in A\}$. Then H is a finite mosaic of H_0, (here $|\Lambda| = |\rho|$).

In the following we assume that the graph $H_0 = (V_0, E_0)$ is finite and we will say mosaic for finite mosaic. The H_0-retraction coloring problem ($H_0 - RCP$) for H_0-mosaics is stated as follows.

Instance *A finite set V and a mosaic graph $H = (V, E)$ of H_0.*

Question *Does there exist a graph homomorphism $\phi : H \to H_0$ such that $\psi \circ i_\alpha$ is an isomorphism for every $\alpha \in \Lambda$ (i.e., a retraction $\phi : H \to H_0$)?*

Now let G be a subgroup of S_h and assume that $(G, \underline{h})$ is realizable as Aut (H_0) for some graph $H_0 = (\underline{h}, E_0)$. Then it is easy to see that the SCP_G is equivalent to the H_0-RCP problem for H_0-mosaics. This can be generalized as follows:

Let G be a subgroup of S_h operating on $\underline{h} = \{0,\ldots,h-1\}$. Then it is well known that there is a connected graph $H_G = (X, E)$ without loops or isolated points with i) $|X| < (h!)^4$, ii) $\underline{h} \subseteq X$ and iii) there is a group isomorphism $\phi : G \to \mathrm{Aut}(H_G)$ such that $\phi(g)|_{\underline{h}} = g$ for all $g \in G$ (see [10]). This allows the stronger result.

Proposition 2.9. *Let G be a subgroup of S_h and the graph H_G be as above. Then there is a polynomial reduction from the SCP_G to the H_G-RCP.*

Proof. Let ρ be an h-ary areflexive relation with vertex set Y whose symmetry group is G. Construct the graph $H_G = (X, E)$ as above. Put $\hat{Y} := \{(x, \underset{\sim}{r}) : x \in X, \underset{\sim}{r} \in \rho\}$ and define an equivalence relation $\equiv$ on the set $\hat{Y}$ by setting

$$(x, \underset{\sim}{r}) \equiv (y, \underset{\sim}{r}') \Longleftrightarrow x = y.$$

Next define a graph $H := (\hat{V}, \hat{E})$ with vertex set $\hat{V} := V/\equiv$ the quotient set of V by the equivalence $\equiv$ and for any equivalence classes $\overline{(x, \underset{\sim}{r})}, \overline{(y, \underset{\sim}{s})} \in \hat{V}$, $(\overline{(x, \underset{\sim}{r})}, \overline{(y, \underset{\sim}{s})}) \in \hat{E}$ if and only if $(x, y) \in E$ and $r = s$. Note that as ρ is an areflexive relation, the equivalence $\equiv$ does not collapse any edge of H_G. Now it is straightforward to check that H is an H_G-mosaic and ρ has a strong h-coloring if and only if there is a retraction $\phi : H \to H_0$. We also note that the only difference between this and the previous construction is the introduction of some extra points to each h-tuple $\underset{\sim}{r}$ which are "bound" to the tuple anyway.

Remark 2.10. 1) The converse of last proposition holds. Indeed let $H_0 = (V(H_0), E(H_0))$ be a graph. Indeed let $H_0 = (V(H_0), E(H_0))$ be a graph. Put $G := \mathrm{Aut}(H_0)$. We show that there is a polynomial reduction from the H_G-RCP to the SCP_G. Let $H = (V(H), E(H))$ be a mosaic of H_0 (hence H is an instance of the H_G-RCP). Hence there are n graph monomorphisms $i_1, \ldots, i_n : V(H_0) \to V(H)$ with properties 1) and 2) of Definition 2.6. For notational ease assume $V(H_0) = \underline{h} = \{0, \ldots, h-1\}$. Define the h-ary areflexive relation ρ with vertex set $V(H)$ by setting

$$\rho := \{(i_j(\pi(0)), \ldots, i_j(\pi(h-1))) : j = 1, \ldots, n, \pi \in G\}$$

Then clearly $\rho = \rho^{(\pi)}$ for all $\pi \in G$. Moreover let $\beta \in S_h \setminus G$ and assume $(x_0, \ldots, x_{h-1}), (x_{\beta(0)}, \ldots, x_{\beta(h-1)}) \in \rho$. Then $(x_0, \ldots, x_{h-1}) = (i_j(\pi(0)), \ldots, i_j(\pi(h-1)))$ and $(x_{\beta(0)}, \ldots, x_{\beta(h-1)}) = (i_{j'}(\pi'(0)), \ldots, i_{j'}(\pi'(h-1)))$ for some $j, j' = 1, \ldots, n$ and some π, $\pi' \in G$. Hence $x_{\beta(t)} = i_{j'}(\pi'(t))$ and thus $x_t = i_{j'}(\pi'(\beta^{-1}(t)))$ for all $t = 0, \ldots, h-1$. As the i_j's are pairwise distinct we see that $j = j'$ and hence $i_j(\pi(t)) = i_j(\pi'(\beta^{-1}(t))$ proving that $\pi = \pi'\beta^{-1}$ and hence $\beta = \pi^{-1}\pi' \in G$, a contradiction. Now let $\phi : V(H) \to V(H_0)$ be an homomorphism. Then clearly ϕ is an H_0-retract iff $\phi \circ i_j \in \mathrm{Aut}(H_0)$ for all $j = 1, \ldots, n$ and this holds if and only if ϕ is a strong h-coloring of ρ. This shows that the $H_0 - RCP$ is polynomial equivalent to the SCP_G.

2) Let H_0 be the undirected cycle of length $m \geq 2$, i.e., $H_0 = (V(H_0), E(H_0))$ where $V(H_0) = \{0, \ldots, m-1\}$ and $\{i, j\} \in E(H_0)$ if and only if $|i - j| = 1 \pmod m$. Note that here $\mathrm{Aut}(H_0) = D_m$. Then combining Corollary 2.5 with Proposition 2.8 we deduce that the H_{D_m}-RCP is NP-complete if $m \neq 2$ and $m \neq 4$. Moreover according to Theorem 1.2 the H_0-coloring problem for graphs is in P whenever m is even. This shows that the situation is quite different in the H_0-coloring problem for graphs if we ask the H_0-coloring to be moreover a retraction.

3. THE WREATH PRODUCT

In this section we seek to construct infinitely many new transitive groups (not covered under Theorem 1.1) with polynomial strong coloring problem. Let the group H act on the set $\underline{h}$ and the group K act transitively on $\underline{k}$. Assume moreover that both SCP_H and SCP_K are in P. Then it is easy to see that the same holds for the $SCP_{H \times K}$ where the direct product $H \times K$ acts on $\underline{h} \times \underline{k}$ (see below). However $H \times K$ is not transitive on $\underline{h} \times \underline{k}$. In this section we show that the SCP_G is in P where $G = H \wr K$ is the wreath product of H and K. Note that G acts transitively on $\underline{h} \times \underline{k}$ whenever both H and K are transitive. However if $|H| > 1$ and $k > 1$ then G is not regular.

Let $h \geq 1$, $k \geq 1$ be two integers, H a permutation group acting on the set $\underline{h} = \{0, \ldots, h-1\}$, K a transitive permutation group on the set $\underline{k} = \{0, \ldots, k-1\}$. We take k copies of H and $\underline{h} = \{0, \ldots, h-1\}$, say $H_0, \ldots, H_{k-1}$, $X_0, \ldots, X_{k-1}$ where

$$X_i = \{(t, i) : 0 \leq t \leq h-1\} \text{ for all} i = 0, \ldots, k-1.$$

Then the action of $F := H_0 \times \cdots \times H_{k-1}$ on $X := X_0 \cup \cdots \cup X_{k-1}$ is defined by

$$[\alpha_0, \ldots, \alpha_{k-1}](t, i) := (\alpha_i(t), i)$$

for all $[\alpha_0, \ldots, \alpha_{k-1}] \in F$ and $(t, i) \in X$.

Let $G = H \wr K$ be the *wreath product* of H and K (in that order). Then G is a subgroup of $S_{(\underline{h} \times \underline{k})}$ (the symmetric group on $\underline{h} \times \underline{k}$) and every permutation in G is written uniquely as $[\alpha_0, \ldots, \alpha_{k-1}, \beta]$ where $\alpha_t \in H$ for $t = 0, \ldots, k-1$ and $\beta \in K$. The action of G on $\underline{h} \times \underline{k}$ is defined by

$$[\alpha_0, \ldots, \alpha_{k-1}, \beta](t, i) := (\alpha_i(t), \beta(i))$$

for all $0 \leq t \leq h-1$ and $0 \leq i \leq k-1$. It is known [11] that

$$\deg G = k \cdot \deg H \quad \text{and} \quad |G| = |H|^k \cdot |K|.$$

For notational ease we will consider the following total order defined on $\underline{h} \times \underline{k}$ by

$$(t, i) \leq (s, j) \Longleftrightarrow t + ih \leq s + jh.$$

Let ρ be an hk-ary areflexive relation with vertex set A and whose symmetry group is $H \wr K$. Put

$$\lambda = pr_{(0,0),(1,0)\ldots(h-1,0)}\rho := \big\{ (x_{(0,0)}, x_{(1,0)}, \ldots, x_{(h-1,0)}) : (x_{(0,0)}, \ldots, x_{(h-1,0)}, x_{(0,1)}, \ldots, x_{(h-1,k-1)}) \in \rho, \text{ for some } x_{(0,1)}, \ldots, x_{(h-1,k-1)} \in A \big\}$$

i.e., λ consists of the first h coordinates of ρ. Clearly λ is an h-ary areflexive relation on A.

We have

Lemma 3.1. *Let ρ and λ be as defined above. Then*

1) For every $i = 0, \dots, k-1$

$$\lambda = pr_{(0,i)(1,i)\dots(h-1,i)}\rho := \{(x_{(0,i)}, x_{(1,i)}, \dots, x_{(h-1,i)}) : (x_{(0,0)}, \dots, x_{(h-1,0)}, \dots, x_{(0,k-1)}, \dots, x_{(h-1,k-1)}) \in \rho\}$$

2) If there exists a permutation $\alpha \in S_h \setminus H$ for which $(a_0, \dots, a_{h-1})$ and $(a_{\alpha(0)}, \dots, a_{\alpha(h-1)}) \in \lambda$, then ρ does not have a strong hk-coloring.

Proof. 1) Fix $i \in \{0, \dots, k-1\}$. Since K is transitive on $\underline{k}$, there is a $\beta \in K$ such that $\beta(i) = 0$. Thus for every $[\alpha_0, \dots, \alpha_{k-1}] \in F$ and every $0 \le t \le h-1$,

$$[\alpha_0, \dots, \alpha_{k-1}, \beta](t,i) = (\alpha_i \mid (t), 0).$$

In particular for $\alpha_i = 1$ the identity on $\underline{h}$,

$$[\alpha_0, \dots, \alpha_{i-1}, 1, \alpha_{i+1}, \dots, \alpha_{k-1}, \beta](t,i) = (t, 0).$$

Since $\sigma = [\alpha_0, \dots, \alpha_{i-1}, 1, \alpha_{i+1}, \dots, \alpha_{k-1}, \beta] \in H \wr K$, we have that $\rho = \rho^{(\sigma)}$. Let $(a_{(0,0)}, \dots, a_{(h-1,0)}) \in \lambda$. Then $(a_{(0,0)}, \dots, a_{(h-1,k-1)}) \in \rho$ for some $a_{(0,1)}, \dots, a_{(h-1,k-1)} \in A$, which implies that

$$(a_{\sigma(0,0)}, \dots, a_{\sigma(h-1,k-1)}) \in \rho,$$

and thus

$$(a_{(0,0)}, \dots, a_{(h-1,0)}) = (a_{\sigma(0,i)}, \dots, a_{\sigma(h-1,i)}) \in pr_{(0,i)\dots(h-1,i)}\rho,$$

proving that $\lambda \subseteq pr_{(0,i)\dots(h-1,i)}\rho$. The converse is similar.

2) Let $\phi : A \to \underline{h} \times \underline{k}$ be a strong hk-coloring of ρ and assume that $(a_0, \dots, a_{h-1}) \in \lambda$ and $(a_{\alpha(0)}, \dots, a_{\alpha(h-1)}) \in \lambda$ for some $\alpha \in S_h$. We show that $\alpha \in H$.

Clearly $(a_0, \dots, a_{h-1}) = (x_{(0,0)}, \dots, x_{(h-1,0)})$ for some $(x_{(0,0)}, \dots, x_{(h-1,k-1)}) \in \rho$. It is easy to see that $(a_{\alpha(0)}, \dots, a_{\alpha(h-1)}) = (x_{(\alpha(0),0)}, \dots, x_{(\alpha(h-1),0)})$. It follows that

$$(x_{(\alpha(0),0)}, \dots, x_{(\alpha(h-1),0)}, y_{(0,1)}, \dots, y_{(h-1,k-1)}) \in \rho,$$

for some $y_{(0,1)}, \dots, y_{(h-1,k-1)} \in A$.
Since ϕ is a strong hk-coloring of ρ, we have that

$$(\phi(x_{(0,0)}), \dots, \phi(x_{(h-1,k-1)})) = (\sigma(0,0), \dots, \sigma(h-1,k-1)) \quad (1)$$

and

$$(\phi(x_{(\alpha(0),0)}), \dots, \phi(x_{(\alpha(h-1),0)}), \phi(y_{(0,1)}), \dots, \phi(y_{(h-1,k-1)})) = (\sigma'(0,0), \dots, \sigma'(h-1,0), \sigma'(0,1), \dots, \sigma'(h-1,k-1)) \quad (2)$$

for some $\sigma = [\alpha_0, \ldots, \alpha_{h-1}, \beta]$ and $\sigma' = [\alpha'_0, \ldots, \alpha'_{h-1}, \beta']$ both elements of $H \wr K$.

From (1) we deduce

$$\big(\sigma(\alpha(0), 0), \ldots, \sigma(\alpha(h-1, 0))\big) = \big(\phi(x_{(\alpha(0),0)}), \ldots, \phi(x_{(\alpha(h-1),0)})\big),$$

and combining with (2) we obtain

$$\big(\sigma(\alpha(0), 0), \ldots, \sigma(\alpha(h-1), 0)\big) = \big(\sigma'(0,0), \ldots, \sigma'(h-1, 0)\big).$$

Thus $\sigma(\alpha(t), 0) = \sigma'(t, 0)$ for all $0 \leq t \leq h-1$ which implies

$$\big(\alpha_0(\alpha(t)), \beta(0)\big) = \big(\alpha'_0(t), \beta'(0)\big), \quad \text{for all } 0 \leq t \leq h-1.$$

This shows that

$$\alpha_0(\alpha(t)) = \alpha'_0(t) \qquad \text{for all } t \in \underline{h},$$

and thus $\alpha = \alpha_0^{-1}\, \alpha'_0 \in H$. □

Remark 3.2.

1) It is easy to see that the property
$(a_0, \ldots, a_{h-1}) \in \lambda$ and$(a_{\alpha(0)}, \ldots, a_{\alpha(h-1)}) \in \lambda$ imply $\alpha \in H$
can be checked in polynomial time in $\mid \rho \mid$ and $\mid H \wr K \mid = \mid H \mid^k \cdot \mid K \mid$.

2) The above property combined with 1) of Lemma 3.1 gives that H is the symmetry group of λ. In this case the model of λ is the h-ary relation

$$M_\lambda = \big\{(\alpha(0), \ldots, \alpha(h-1)) : h \in H\big\}$$

on the set $\underline{h}$. Then we have

Lemma 3.3. *Let ρ and λ be as above. If ρ has a strong hk-coloring, then λ has a strong h-coloring.*

Proof. Let $\phi : A \to \underline{h} \times \underline{k}$ be a strong hk-coloring of ρ. Define the map $e_1 : \underline{h} \times \underline{k} \to \underline{h}$ by setting

$$e_1(t, i) := t \quad \text{for all} \quad (t, i) \in \underline{h} \times \underline{k}.$$

We show that $\theta = e_1 \circ \phi : A \to \underline{h}$ is a strong h-coloring of λ. Indeed let $(a_0, \ldots, a_{h-1}) \in \lambda$. Then

$$(a_0,\dots,a_{h-1}) = (x_{(0,0)},\dots,x_{(h-1,0)})$$

for some $(x_{(0,0)},\dots,x_{(h-1,0)},x_{(0,1)},\dots,x_{(h-1,k-1)}) \in \rho$. Now

$$(\phi(x_{(0,0)}),\dots,\phi(x_{(h-1,k-1)})) = (\sigma(0,0),\dots,\sigma(h-1,k-1))$$

for some $\sigma = [\alpha_0,\dots,\alpha_{k-1},\beta] \in H \wr K$. Here

$$(\sigma(0,0),\dots,\sigma(h-1,0)) = ((\alpha_0(0),\beta(0)),\dots,(\alpha_0(h-1),\beta(0))),$$

therefore

$$\begin{aligned}(\theta(a_0),\dots,\theta(a_{h-1})) &= (e_1 \circ \phi(x_{(0,0)}),\dots,e_1 \circ \phi(x_{(h-1,0)})) \\ &= (e_1(\alpha_0(0),\beta(0)),\dots,e_1(\alpha_0(h-1),\beta(0))) \\ &= (\alpha_0(0),\dots,\alpha_0(h-1)),\end{aligned}$$

where $\alpha_0 \in H$. This shows that $\theta \in \mathrm{Hom}(\lambda, M_\lambda)$ and thus is a strong h-coloring of λ. □

We need to construct a k-ary areflexive relation R whose symmetry group is K and that admits a strong k-coloring if ρ has a strong hk-coloring.

First recall that a subrelation C of an h-ary areflexive relation μ is a connected component of μ if the the h-regular hypergraph $H_C := \{\{a_0,\dots,a_{h-1}\} : (a_0,\dots,a_{h-1}) \in C\}$ is a connected component of the h-regular hypergraph $H_\mu := \{\{x_0,\dots,x_{h-1}\} : (x_0,\dots,x_{h-1}) \in \mu\}$. Let λ be the h-ary relation defined above and $C_1,\dots,C_n$ be its connected components. Define a k-ary relation R with vertex set $\mathcal{C} := \{C_1,\dots,C_n\}$ by setting
$R := \{(C_{j_0},\dots,C_{j_{k-1}}) \in \mathcal{C}^k :$ There are k h-tuples $\underset{\sim}{x}_i = (x_{(0,i)},\dots,x_{(h-1,i)}) \in C_{j_i}$ ($i = 0,\dots,k-1$) and such that $(x_{(0,0)},\dots,x_{(h-1,0)},\dots,x_{(0,k-1)},\dots, x_{(h-1,k-1)}) \in \rho\}$. We have:

Lemma 3.4. *Let ρ and R be as above. If ρ has a strong hk-coloring then*
i) R is a k-ary areflexive relation whose group of symmetry is K and
ii) there exists a relational homomorphism $\psi \in Hom(R, \{(\beta(0),\dots,\beta(k-1)) : \beta \in K\})$.

Remark. According to i) the Model of the relation R is the k-ary relation $M_R = \{(\beta(0),\dots,\beta(k-1)) : \beta \in K\}$ and thus ii) means that R has a strong k-coloring.

Proof. (of Lemma 3.4.) Let $\phi : A \to \underline{h} \times \underline{k}$ be a strong hk-coloring of ρ.

i) First note that the map $e_2 \circ \phi : A \to \underline{k}$ (where $e_2(x,y) = y$) satisfies $e_2 \circ \phi(x_{(0,i)}) = \cdots = e_2 \circ \phi(x_{(h-1,i)})$ for any $(x_{(0,i)}, \ldots, x_{(h-1,i)}) \in \lambda$ (follows from the definitions of the wreath product and the strong coloring of an areflexive relation). Hence $e_2 \circ \phi(x) = e_2 \circ \phi(y)$ whenever the vertices x, y belong to a same component C_i of λ. Assume that R is not areflexive. Thus there exists an hk-tuple $\big(a_{(0,0)}, \ldots, a_{(h-1,k-1)}\big) \in \rho$ such that the vertices $a_{(0,i)}, \ldots, a_{(h-1,i)}, a_{(0,j)}, \ldots, a_{(h-1,j)}$ (with $i \neq j$) belong to a same component of λ. But this implies that $e_2 \circ \phi(a_{(0,i)}) = \cdots = e_2 \circ \phi(a_{(h-1,i)}) = e_2 \circ \phi(a_{(0,j)}) = \cdots = e_2 \circ \phi(a_{(h-1,j)})$ and hence the hk-tuple $\big(\phi(a_{(0,0)}), \ldots, \phi(a_{(h-1,k-1)})\big)$ does not belong to $M_\rho = \{(\sigma(0,0), \ldots, \sigma(h-1, k-1)) : \sigma \in H \wr K\}$, a contradiction. Next we show that K is the symmetry group of R. Note that the equality $R = R^{(\beta)}$ for all $\beta \in K$ follows from the definition of R. Indeed let $(C_{j_0}, \ldots, C_{j_{k-1}}) \in R$ and $\beta \in K$. Denote C_{j_i} by D_i for $i = 0, \ldots, k-1$. Then $(D_0, \ldots, D_{k-1}) \in R$ implies that there are kh-tuples $\underset{\sim}{x}_i = (x_{(0,i)}, \ldots, x_{(h-1,i)})$ with $x_{(t,i)} \in D_i$ for all $i = 0, \ldots, k-1$ and $t = 0, \ldots, h-1$ and such that

$$\big(x_{(0,0)}, \ldots, x_{(h-1,k-1)}\big) \in \rho \tag{1}$$

As $\beta \in K$, the permutation $\sigma := [1, 1, \ldots, 1, \beta] \in H \wr K$ and so $\big(x_{\sigma(0,0)}, \ldots, x_{\sigma(h-1,k-1)}\big) = \big(x_{(0,\beta(0))}, \ldots, x_{(h-1,\beta(k-1))}\big) \in \rho$ where $x_{(t,\beta(i))} \in D_{\beta(i)}$ for $i = 0, \ldots, k-1$. Hence $\big(D_{\beta(0)}, \ldots, D_{\beta(k-1)}\big) \in R$. This shows that $R^{(\beta)} \subseteq R$. Thus $\big(R^{(\beta)}\big)^{\beta^{-1}} \subseteq R^{(\beta^{-1})}$ which implies that $R^{(\beta)} = R$ for all $\beta \in K$. The proof of $R \cap R^{(\delta)} = \phi$ for all $\delta \in S_h \setminus K$ is similar to the proof of R is an areflexive relation. This establishes i).

ii) Define $\psi : C \to \{0, \ldots, k-1\}$ by setting $\psi(C_i) = e_2 \circ \phi(a)$ where $a \in C_i$. According to the above proof ψ is well defined. Let $(D_0, \ldots, D_k) \in R$ and $x_{(0,0)}, \ldots, x_{(h-1,k-1)}$ be as in (1). Then there is a $\sigma = [\alpha_0, \ldots, \alpha_{k-1}, \beta] \in H \wr K$ such that $\big(\phi(x_{(0,0)}), \ldots, \phi(x_{(h-1,k-1)})\big) = \big(\sigma(0,0), \ldots, \sigma(h-1, k-1)\big)$. Therefore

$$\begin{aligned}(\psi(D_0), \ldots, \psi(D_{k-1})) &= \big(e_2 \circ \phi(x_{(0,0)}), \ldots, e_2 \circ \phi(x_{(0,k-1)})\big)\\ &= \big(e_2(\alpha_0(0), \beta(0)), \ldots, e_2(\alpha_{k-1}(0), \beta(k-1))\big)\\ &= \big(\beta(0), \ldots, \beta(k-1)\big).\end{aligned}$$

This shows that ψ is a strong k-coloring of the relation R. □

Theorem 3.5. *Let ρ, λ and R be as in Lemmas 3.1 and 3.3. Then ρ has a strong hk-coloring if and only if 1) λ (respectively R) is an areflexive relation whose symmetry group is H (respectively K), 2) λ has a strong h-coloring and 3) R has a strong k-coloring.*

Proof. ($\Rightarrow$) Lemmas 3.3 and 3.4

($\Leftarrow$) Assume λ and R as in the statement. Let θ be a strong h-coloring of λ and ψ be a strong k-coloring of R. We define a map $\phi : A \to \underline{h} \times \underline{k}$ as follows. Let $x \in A$. Then $x = x_{(t,i)}$ for some $(x_{(0,0)}, \ldots, x_{(t,i)}, \ldots, x_{(h-1,k-1)}) \in \rho$. As $\underset{\sim}{x}_j = (x_{(0,j)}, \ldots, x_{(h-1,j)}) \in \lambda$ for all $j = 0, \ldots, k-1$ and θ is a strong k-coloring of λ, there are permutations $\alpha_j \in H$ $(j = 0, \ldots, k-1)$ such that

$$\big(\theta(x_{(0,j)}), \ldots, \theta(x_{(h-1,j)})\big) = (\alpha_j(0), \ldots, \alpha_j(h-1)).$$

Moreover let the vertices $x_{(0,i)}, \ldots, x_{(h-1,i)}$ belong to the connected component D_i of λ $(i = 0, \ldots, k = 1)$. Therefore $(D_0, \ldots, D_{k-1}) \in R$ which implies that

$$\big(\psi(D_0), \ldots, \psi(D_{k-1})\big) = \big(\beta(0), \ldots, \beta(k-1)\big),$$

for some $\beta \in K$. Set $\phi(x) := (\alpha_i(t), \beta(i)) \in \underline{h} \times \underline{k}$. Clearly $\phi(x) = [\alpha_0, \ldots, \alpha_{k-1}, \beta](t, i)$ (where $\alpha_0, \ldots, \alpha_{k-1}$ are the above permutations). We have:

Claim *ϕ is well defined.*

Proof. Let $x = x_{(t,i)} = y_{(s,j)}$ for some $(x_{(0,0)}, \ldots, x_{(h-1,k-1)})$, $(y_{(0,0)}, \ldots, y_{(h-1,k-1)}) \in \rho$ and some $t, s \in \underline{h}$, $i, j \in \underline{k}$. Then

$$\big(\theta(x_{(0,i)}), \ldots, \theta(x_{(h-1,i)})\big) = \big(\alpha_i(0), \ldots, \alpha_i(h-1)\big) \text{ and}$$
$$\big(\theta(y_{(0,j)}), \ldots, \theta(y_{(h-1,j)})\big) = \big(\alpha'_j(0), \ldots, \alpha'_j(h-1)\big),$$

where the permutations α_i and $\alpha'_j \in H$ are such that $\alpha_i(t) = \alpha'_j(s)$. Moreover let the vertices $y_{(0,i)}, \ldots, y_{(h-1,i)}$ belong to the connected component D'_i of λ $(i = 0, \ldots, k-1)$. Thus $(D'_0, \ldots, D'_{k-1}) \in R$ which implies that $(\psi(D'_0), \ldots, \psi(D'_{k-1})) = (\beta'(0), \ldots, \beta'(k-1))$ for some $\beta' \in K$. Now the equality $x_{(t,i)} = y_{(s,j)}$ gives $D_i = D'_j$ and thus $\beta(i) = \beta'(j)$. Therefore

$$\begin{aligned}[\alpha_0, \ldots, \alpha_{k-1}, \beta](t, i) = (\alpha_i(t), \beta(i)) &= (\alpha'_j(s), \beta'(j)) \\ &= [\alpha'_0, \ldots, \alpha'_{k-1}, \beta'](s, j).\end{aligned}$$

This shows that $\phi(x_{(t,i)}) = \phi(y_{(s,j)})$ and completes the proof of our claim.

We show that ϕ is an hk-coloring of ρ. Let $(x_{(0,0)}, \ldots, x_{(h-1,k-1)}) \in \rho$ and denote by D_i the connected component of x containing the vertices $x_{(0,i)}, \ldots, x_{(h-1,i)} (i = 0, \ldots, k-1)$. Thus there are $\alpha_0, \ldots, \alpha_{k-1} \in H$ and $\beta \in K$ such that

$$\big(\theta(x_{(0,i)}), \ldots, \theta(x_{(h-1,i)})\big) = \big(\alpha_i(0), \ldots, \alpha_i(h-1)\big)(i = 0, \ldots, k-1),$$

and

$$\big(\psi(D_0), \ldots, \psi(D_{k-1})\big) = \big(\beta(0), \ldots, \beta(k-1)\big).$$

Put $\sigma := [\alpha_0, \ldots, \alpha_{k-1}, \beta]$. Then clearly $\sigma \in H \wr K$ and
$$(\phi(x_{(0,0)}), \ldots, \phi(x_{(h-1,k-1)})) = (\sigma(0,0), \ldots, \sigma(h-1, k-1)),$$
proving that ϕ is a strong hk- coloring of ρ. □

Corollary 3.6. *Let H and K be two permutation groups acting respectively on $\underline{h}$ and $\underline{k}$ and assume that K is transitive. Let $G = H \wr K$ be the wreath product of H and K. If both the SCP_H and SCP_K are in P, then so is the SCP_G.* □

For $t \geq 2$ let C_t denote the cycle permutation group acting on $\underline{t} = \{0, \ldots, t-1\}$. As C_t is regular we have by Theorem 1.1 that the SCP_{C_t} is in P.

Corollary 3.7. *Let $n, m \geq 2$ and $G := C_n \wr C_m$. Then the SCP_G is in P.* □

Note that $C_n \wr C_m$ acting on $\underline{n} \times \underline{m}$ is transitive but not regular and has degree $n \cdot m$.

REFERENCES

[1] J. Bang-Jensen, P. Hell, The effect of two cycles on the complexity of colorings by directed graphs, *Discrete Applied Math.* 26 (1990) 1–23.

[2] J. Bang-Jensen, P. Hell, G. MacGillivray, The complexity of colorings by semi-complete digraphs, *SIAM. J. Discrete Math. 1* No. 3 (1988) 281–298.

[3] M.R. Garey, D.S. Johnson, Computers and intractability. A guide to the theory of NP-completeness (W.H. Freeman, San Francisco, 1979).

[4] L. Haddad, V. Rödl, On the complexity of coloring areflexive relations. *Ars Combinatoria* **33** (1992), pp 217–225.

[5] L. Haddad, I.G. Rosenberg, Critère général de complétude pour les algèbres partielles finies, *C.R. Acad. Sci. Paris* t. 304, Série I, n. 17 (1987), 507–509.

[6] L. Haddad, I.G. Rosenberg, Maximal partial clones determined by areflexive relations. *Discrete Appl. Math.* 24, (1989) 133–143.

[7] P. Hell, G. MacGillivray, The complexity of Tournament Colorings of Digraphs, *LCCR Tech Report*, SFU 87-12.

[8] P. Hell, J. Nesetril, On the complexity of H-coloring, *J. Combin. Theory*

Ser. B, 48 (1990) 92–110.

[9] H.A. Maurer, J.H. Sudborough, E. Welzl, On the complexity of the general coloring problem, *Information and Control* 51 (1981), 123–145.

[10] E. Mendelsohn, On a technique for representing semigroups as endomorphism semigroups of graphs with given properties, *Semigroup Forum* 4, (1972), 283- 294.

[11] D.S. Passman, Permutation Groups (Benjamin 1968).

[12] I.G. Rosenberg, Personal communications.

Département de Mathématiques et Informatique, Collège Militaire Royal du Canada, Kingston, ON, K7K 5L0

School of Computing Sciences, S.F.U. Burnaby, B.C., V5A 1S6

Department of Mathematics, University of Toronto, Toronto, ON, M1C 1A4