Toughness and Perfect Matchings in Graphs

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ABSTRACT. Let G be a graph with even order p and let k be a positive integer with $p \geq 2k+2$. It is proved that if the toughness of G is at least k, then the subgraph of G obtained by deleting any $2k-1$ edges or $2k$ vertices has a perfect matching. Furthermore, we show that the results in this paper are best possible.

1 Introduction

The graphs considered in this paper will be finite, connected, undirected, and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The connectivity and edge-connectivity of G are denoted by $\kappa(G)$ and $\lambda(G)$, respectively. Notations and definitions not given in this paper can be found in [1].

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Let S be a vertex cutset of graph G and let $c(G-S)$ denote the number of components in $G-S$. Then if G is not complete, the *toughness* of G is defined to be $\min \frac{|S|}{c(G-S)}$ where the minimum is taken over all vertex cutsets S of G. Whereas we define the toughness of K_n to be $+\infty$ for all n. We denote the toughness of G by $t(G)$. We will also say that graph G is *k-tough* if $t(G) \geq k$. This parameter was introduced by Chvátal [2] who noted that every 1-tough graph with an even number of vertices has a perfect matching. Enomoto et al. [3] proved that every k-tough graph with $|V(G)| \geq k+1$ and $k|V(G)|$ even has a k-factor. Furthermore Liu [4] proved that every k-tough ($k \geq 2$) graph has a k-factor containing any given edge.

Let k and p be positive integers with $k \leq \frac{1}{2}(p-2)$ and let G be a graph with p vertices having a perfect matching. Then G is said to be *k-extendable* if every matching of size k in G can be extended to a perfect matching. If $|V(G)| = p$, then we say that the *order* of G is p. Let G be a graph with even order p and let k be positive integer with $p \geq 2k+2$. Plummer [6] proved that if $t(G) > k$, then G is k-extendable. In this paper we show that if $t(G) > k$, then the subgraph G' obtained from G by deleting any $2k-1$ edges or $2k$ vertices has a perfect matching. Furthermore we show that the results in this paper are best possible.

Let $o(G-S)$ denote the number of odd components of $G-S$. To prove the main results we need the following theorems.

Theorem 1.1. (Tutte's Theorem) *A graph G has a perfect matching if and only if for any proper subset $S \subseteq V(G)$*

$$o(G-S) \leq |S|$$

Theorem 1.2. [2] *If G is not complete, then $\kappa(G) \geq 2t(G)$.*

Theorem 1.3. [6] *Let G be a graph with even order p and let k be a positive integer with $p \geq 2k+2$. If $t(G) > k$, then G is k-extendable.*

2 Main results

A graph G is called *n-edge-deletable* if the deletion of any n edges of $E(G)$ results in a graph with a perfect matching. Clearly, if G is n-edge-deletable, G must have even order and G is also r-edge-deletable for any integer $r < n$.

We call a graph G *n-vertex-deletable* if the deletion of any n vertices of $V(G)$ results in a graph with a perfect matching. Notice that the 2-vertex-deletable graphs are also called bicritical graphs and the 1-vertex-deletable graphs are called factor-critical graphs in [5].

Let us start by investigating the edge-depletability of complete graphs.

Theorem 2.1. *Let G be a complete graph with even order n. Then G is $(n-2)$-edge-deletable.*

Proof: It is well known that the edge set of a complete graph of even order n can be decomposed into $n-1$ disjoint perfect matchings. If we delete $n-2$ edges from G, then the remaining graph still has a perfect matching. So G is $(n-2)$-edge-deletable. □

Let G be a complete graph of order n. Clearly, if $n-1$ edges incident with a vertex are deleted, then G has no perfect matching. So Theorem 2.1 is best possible.

Note that for any graph H, we have $c(H-e) \leq c(H)+1$ for any edge e of H. We will use this fact in the proof.

Theorem 2.2. *Let G be a graph of even order p, where $p \geq 2k+2$. If $t(G) \geq k \geq 1$, then G is $(2k-1)$-edge-deletable.*

Proof: If G is a complete graph, then by Theorem 2.1 G is $(2k-1)$-edge-deletable. Now we assume that G is not a complete graph. Let E' be any subset of $E(G)$ and $|E'| = 2k-1$. Set $G' = G - E'$. By Theorem 1.1 we only need to prove that for any proper subset $S \subsetneq V(G)$

$$o(G'-S) \leq |S|$$

or

$$\frac{|S|}{o(G'-S)} \geq 1. \tag{2.1}$$

Then G' has a perfect matching.

Since $t(G) \geq k$, by Theorem 1.2 $\kappa(G) \geq 2k$. Thus $\lambda(G) \geq \kappa(G) \geq 2k$. Hence G' is connected. We consider two cases.

Case 1. S is not a vertex cutset of G.

In this case $G-S$ is connected. So

$$o(G-S) \leq c(G-S) = 1.$$

When $|S| \geq 2k$, we have

$$\frac{|S|}{o(G'-S)} \geq \frac{|S|}{c(G'-S)} \geq \frac{|S|}{c(G-S)+2k-1} \geq \frac{2k}{2k} = 1.$$

When $|S| = r \leq 2k-1$, $\lambda(G-S) \geq \kappa(G-S) \geq 2k-r$. We have

$$c(G'-S) \leq c(G-S) + 2k-1-(2k-r) = 1+r-1 = r.$$

Thus

$$\frac{|S|}{o(G'-S)} \geq \frac{|S|}{c(G'-S)} \geq \frac{r}{r} = 1.$$

Case 2. S is a vertex cutset of G.

In this case we have $\frac{|S|}{c(G-S)} \geq t(G) \geq k$, that is,

$$|S| \geq c(G-S)k.$$

Thus

$$\frac{|S|}{o(G'-S)} \geq \frac{|S|}{c(G'-S)} \geq \frac{|S|}{c(G-S)+2k-1} \qquad (2.2)$$

Case 2.1. $|S| \geq c(G-S)k+1$

In this case, since

$$c(G-S)k+1-(c(G-S)+2k-1) = c(G-S)(k-1)-2k+2 \geq 2(k-1)-2k+2 = 0$$

by (2.2) we have

$$\frac{|S|}{o(G'-S)} \geq \frac{|S|}{c(G'-S)+2k-1} \geq \frac{c(G-S)k+1}{c(G-S)+2k-1} \geq 1$$

Case 2.2. $|S| = c(G-S)k$.

If $c(G-S) = 2$, we have $|S| = 2k$ and $|V(G-S)|$ is even. Suppose that (2.1) does not hold. Then $o(G'-S) > |S|$ and by parity we have $o(G'-S) \geq |S|+2 = 2k+2$. Thus

$$2k+2 \leq o(G'-S) \leq c(G'-S) \leq c(G-S)+2k-1 = 2+2k-1 = 2k+1.$$

which is impossible. Hence, in this case

$$\frac{|S|}{o(G'-S)} \geq 1.$$

If $c(G-S) \geq 3$ and $k \geq 2$, we have

$$|S| = c(G-S)k \geq c(G-S)+2k-1.$$

By (2.2)

$$\frac{|S|}{o(G'-S)} \geq \frac{|S|}{c(G-S)+2k-1} \geq 1.$$

Now we assume that $c(G-S) \geq 3$ and $k = 1$. We have $|S| = c(G-S)k = c(G-S)$. By simple parity arguments, we can see that $c(G-S) \equiv |S| \equiv o(G-S) \pmod 2$ or $c(G-S) - o(G-S) \equiv 0 \pmod 2$. That is, the number of even components of $G-S$ is even. By noticing $|E'| = 2k-1 = 1$, we have $o(G'-S) \leq o(G-S)+2$ and $c(G'-S) \leq c(G-S)+1$. If $G-S$ has at least two even components, then

$$\frac{|S|}{o(G'-S)} \geq \frac{|S|}{o(G-S)+2} \geq \frac{|S|}{c(G-S)} = 1.$$

If $G-S$ has no even components, then $o(G'-S) = o(G-S) = c(G-S)$. So

$$\frac{|S|}{o(G'-S)} = \frac{|S|}{c(G-S)} = 1.$$

By Theorem 1.1, G' has a perfect matching. Now, we reach the conclusion that G is $(2k-1)$-edge-deletable. □

Chvátal [2] has proved that if $t(G) \geq 1$ then G has a perfect matching. For $k=1$, Theorem 2.2 can be stated as follows: if $t(G) \geq 1$ then for any given edge e there exists a perfect matching in G avoiding e. So Theorem 2.2 is slightly stronger than Chvátal's result.

Let $G = K_{2k+2} - e$ where K_{2k+2} is a complete graph of $2k+2$ vertices and $e = uv$ is any edge of K_{2k+2}. It is easy to see that

$$t(G) = \frac{2k}{2} = k.$$

Let E' be the set of edges incident with u in G. Then $G - E'$ has no perfect matching and $|E'| = 2k$. In this sense Theorem 2.2 is best possible.

The condition $t(G) \geq k$ in Theorem 2.2 is sufficient but not necessary. Let G be a graph as shown in Figure 1. Clearly $t(G) = \frac{1}{2}$ and G is $(n-2)$-edge-deletable if n is even.

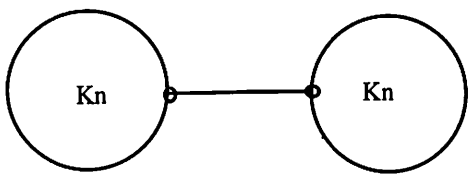

Figure 1

Theorem 2.3. *Let G be a graph with even order p and let k be a positive integer with $p \geq 2k+2$. Then if $t(G) > k$, the graph G is $2k$-vertex-deletable.*

Proof: Let $S = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$ be a subset of $V(G)$. If $x_i y_i \notin E(G)$, then join x_i and y_i by an edge. Denote the resulting graph by G'.

Since $t(G') \geq t(G) > k$, G' is k-extendable by Theorem 1.3. Hence there is a perfect matching M in G' containing edges $x_1y_1, x_2y_2, \ldots, x_ky_k$. $M - \{x_1y_1, x_2y_2, \ldots, x_ky_k\}$ is a perfect matching of $G - S$. That is, G is $2k$-vertex-deletable. □

Remarks: 1. In general, every $2k$-vertex-deletable graph must be k-extendable, but a k-extendable graph may not be $2k$-extendable. Under the condition of $t(G) > k$, in light of Theorem 2.3, we see that k-extendibility is equivalent to $2k$-vertex-depletability.

2. The condition $t(G) > k$ in Theorem 2.3 cannot be replaced by $t(G) \geq k$ (see the example in [6]).

References

[1] J.A. Bondy and U.S.R. Murty, *Graphs with applications*, Macmillan Press Ltd., 1976.

[2] V. Chvátal, Tough graphs and Hamiltonian circuits, *Discrete Math.* **5** (1973), 215–228.

[3] H. Enomoto, B. Jackson, P. Katerinis and A. Saito, Toughness and the existence of k-factors, *J. Graph Theory* **9** (1985), 87–95.

[4] G. Liu, Toughness and k-covered graphs, *Acta Math. Appl. Sinica*, **15** (1992), 397–402.

[5] L. Lovász and M. D. Plummer, *Matching Theory*, North-Holland, Amsterdam, 1986.

[6] M. D. Plummer, Toughness and matching extension in graphs, *Discrete Math.* **72** (1988), 311–320.

[7] Q.L. Yu, Characterizations of various matching extensions in graphs, *Australasian Journal of Combin.* **7** (1993), 55–64.