Enumeration of 2-(21,6,3) Designs with Automorphisms of Order 7 or 5*

Stoyan Kapralov,
Department of Mathematics, Technical University, Gabrovo, Bulgaria
Svetlana Topalova,
Institute of Mathematics, Bulgarian Academy of Sciences, Bulgaria

Abstract

All nonisomorphic 2-(21,6,3) designs with automorphisms of order 7 or 5 were found, and the orders of their groups of automorphisms were determined. There are 33 nonisomorphic 2-(21,6,3) designs with automorphisms of order 7 and 203 with automorphisms of order 5.

1 Introduction

A 2-(v, k, λ) design is a set of k-element subsets (*blocks*) of a set of v elements (*points*), such that each pair of points is contained in exactly λ blocks.

Let b denote the number of the blocks of the design, and r – the number of blocks in which a given point is contained. An incidence matrix of the design is a binary matrix of v rows and b columns which contains a 1 in the ith row and jth column iff the ith point is contained in the jth block.

An automorphism of the design is called a permutation of the points that transforms the blocks into blocks.

According to [1] only one nonisomorphic 2-(21,6,3) design had been constructed [2] up to this work. It has an incidence matrix consisting of two circulants of order 21, and its automorphism group order is 63. As it is shown below, a 2-(21,6,3) design cannot have an automorphism of a prime order greater than 7. The aim of this note is to enumerate all nonisomorphic 2-(21,6,3) designs possessing automorphisms of order 7 or 5.

*This work was partially supported by the Bulgarian National Science Fund under Contracts No. I-407/1994 and I-506/1995.

2 Designs with automorphisms of order 7

If φ is an automorphism of a prime order p of a 2-(v,k,λ) design, such that $p > max(k,\lambda)$ then $p < r$, or $p|v$, and φ can fix at most $(v-1)/(k-1)$ points ([3],chapter1). It follows that a 2-(21,6,3) design cannot possess an automorphism of a prime order greater than 7, and that an automorphism of order 7 of a 2-(21,6,3) design cannot fix any points or blocks.

Let D be a 2-(21,6,3) design with an automorphism φ of order 7 fixing no points and no blocks. Without loss of generality we can assume that φ acts as follows:
φ=(1,2,3,4,5,6,7)(8,9,10,11,12,13,14)(15,16,17,18,19,20,21) on the points,
φ=(1,2,3,4,5,6,7)(8,9,10,11,12,13,14)...(36,37,38,39,40,41,42) on the blocks.
Then the incidence matrix of D is of the form:

$$\begin{pmatrix} B_{1,1} & B_{1,2} & B_{1,3} & B_{1,4} & B_{1,5} & B_{1,6} \\ B_{2,1} & B_{2,2} & B_{2,3} & B_{2,4} & B_{2,5} & B_{2,6} \\ B_{3,1} & B_{3,2} & B_{3,3} & B_{3,4} & B_{3,5} & B_{3,6} \end{pmatrix}$$

where $B_{i,j},\ i=1,2,3,\ j=1,2,\dots,6$ are circulant matrices of order 7.

Let $n_{i,j},\ i=1,2,3,\ j=1,2,\dots,6$ be equal to the number of 1's in a row of $B_{i,j}$.The following equations hold for the matrix $N=(n_{i,j})_{3x6}$

$$\sum_{j=1}^{6} n_{i,j} = 12, \qquad i=1,2,3 \tag{1}$$

$$\sum_{j=1}^{6} n_{i,j}^2 = 30, \qquad i=1,2,3 \tag{2}$$

$$\sum_{j=1}^{6} n_{i_1,j}n_{i_2,j} = 21, \qquad 1 \le i_1 < i_2 \le 3. \tag{3}$$

It follows from (1) and (2) that the rows of N must be permutations of (0,2,2,2,3,3), (1,1,1,3,3,3) or (1,1,2,2,2,4). It was found by computer that there are two nonisomorphic matrices with such rows for which (3) is also true.

N_1	N_2
2 2 2 3 3 0	2 2 2 1 1 4
2 2 2 0 3 3	2 2 2 1 4 1
2 2 2 3 0 3	2 2 2 4 1 1

After replacement with circulants in the above given matrices, 33 nonisomorphic 2-(21,6,3) designs were obtained, and the orders of their automorphism groups were computed. None of them is a multiple of 49, and thus each design can be obtained by only one of the matrices N_1 and N_2. None of these designs possess automorphisms of order 5.

The base blocks of the designs obtained from N_1 and N_2 are given in Table 1 and Table 2 respectively. The points of the design are denoted by the numbers from 1 to 21, but to save place their number (between 1 and 7) in the corresponding orbit is given in the tables. The orbits are denoted by the numbers from 0 to 2. So to obtain the point number one has to add to its number in the orbit the orbit number multilpied by 7. (For instance for the first design obtained from N_1, block B_{22} consists of the following points: 1,4,7,15,17,18.) A "t" after the order of the automorphism group denotes a transitive group. The $15th$ design obtained from N_1 is equivalent to the design, presented in [2].

Table1: Base blocks of the designs obtained from N_1.

	B_1	B_8	B_{15}	B_{22}	B_{29}	B_{36}	$\|Aut(D)\|$
No\ Orbit	001122	001122	001122	000222	000111	111222	
1	171716	173526	161434	147134	146235	157235	7
2	171716	174623	162345	147146	146236	146145	21 (t)
3	171716	173634	163457	146126	147135	157347	7
4	171716	171434	163416	146235	147356	146125	7
5	171615	173536	162517	147345	146167	147135	21 (t)
6	171615	173536	152313	146157	157147	157234	21 (t)
7	171716	162415	153723	157134	157245	157235	21
8	171716	162436	153734	157124	157245	137356	21
9	171716	163515	153623	157134	157134	137245	21
10	171716	163525	153617	157235	157134	137346	21
11	171716	162415	153723	157134	137457	157235	21
12	171715	162445	153727	157137	157245	157346	21
13	171715	163545	153627	157137	157134	137356	21
14	171715	162445	153727	157137	137457	157346	21
15	171615	161523	152313	157467	157134	157134	63 (t)
16	171615	163612	153457	157346	157124	157156	21
17	171615	162523	151713	157467	157235	157346	21
18	171615	161523	152313	157467	157134	137346	21
19	171615	161512	152357	157346	157134	137126	21
20	171615	161445	154527	157137	137245	137126	63 (t)
21	171615	163712	151257	157346	137356	137134	63 (t)

Table2: Base blocks of the designs obtained from N_2

	B_1	B_8	B_{15}	B_{22}	B_{29}	B_{36}	$\|Aut(D)\|$
No\Orbit	001122	001122	001122	012222	011112	000012	
1	171716	174623	153635	141457	414672	135724	7
2	171716	173525	152734	411467	214572	135714	7
3	171715	173524	152447	141567	614577	135753	21 (t)
4	171715	173436	153624	641567	113575	135743	14
5	171715	173436	153624	641567	413577	135775	14
6	171715	173436	153624	531567	113577	135745	14
7	171715	173436	153657	531567	113576	135744	14
8	171615	173536	151724	611567	714573	135745	21 (t)
9	171615	173536	151724	511567	714576	135725	21 (t)
10	171716	163636	154612	521467	413672	146723	7
11	171615	163634	154527	151467	514677	146727	21 (t)
12	171615	161523	152313	251467	514674	146747	63 (t)

3 Designs with automorphisms of order 5

Lemma 1: Let α be an automorphism of order 5 of a 2-(21,6,3) design D. If α fixes more than 2 blocks, then a nonfixed point cannot be contained in more than one fixed block.

Proof: Let α fix f points ($f = 1,6,11,16$) and h blocks ($h = 7,12,17,...,37$). Then there are $d = (b-h)/5$ orbits of nonfixed blocks with respect to α, and $d < 8$. Without loss of generality we can assume that α acts as follows:

α = (1,2,3,4,5)...(21-f+1)(21-f+2)...(21) on the points, and

α = (1,2,3,4,5)...(42-h+1)(42-h+2)...(42) on the blocks.

If a nonfixed point is contained in a fixed block, then all the points from the same point orbit with respect to α are also contained in this fixed block. That is why a nonfixed point cannot be contained in more than 3 fixed blocks ($\lambda = 3$).

Suppose there is a nonfixed point that is contained in 2 fixed blocks. The other 4 points from the same point orbit are also contained in these 2 fixed blocks. So the part of the incidence matrix of the design corresponding to this point orbit is of the form:

$$(A_1 \quad A_2 \quad A_3 \quad \dots \quad A_d \quad U^T \quad U^T \quad Z^T \quad Z^T \quad \dots \quad Z^T)$$

where A_i, $i = 1,2,\dots,d$ are circulant matrices of order 5, U=$(1,1,1,1,1)$, Z=$(0,0,0,0,0)$.

Let m_i, $i = 1,2,\dots,d$ be equal to the number of 1's in a row of A_i. Then

$$\sum_{i=1}^{d} m_i = 10 \quad , \quad \sum_{i=1}^{d} m_i^2 = 14 \tag{4}$$

The system (4) has no solution in nonnegative integer numbers if $d < 8$, so it is impossible for a nonfixed point to be contained in 2 fixed blocks.

In a similar way it can be proved that if a nonfixed point is contained in 3 fixed blocks, the following equations hold:

$$\sum_{i=1}^{d} m_i = 9 \quad , \quad \sum_{i=1}^{d} m_i^2 = 9 \tag{5}$$

The system (5) has no solution in nonnegative integer numbers if $d < 9$, so it is impossible for a nonfixed point to be contained in 3 fixed blocks. It follows that if α fixes more than two blocks, a nonfixed point can be contained in no more than one fixed block.

Proposition 1: If α is an automorphism of order 5 of a 2-(21,6,3) design D, then α fixes 1 point and 2 blocks of D, and any nonfixed point is contained in at most 1 fixed block.

Proof:

1. Suppose that α fixes more than one point. Let α fix f points ($f = 6, 11, 16$). As far as $k = 6$ a fixed block consists either of 6 fixed points, or of 1 fixed and 5 nonfixed points. There always must be fixed blocks consisting of 6 fixed points, because a pair of fixed points must be in exactly three blocks ($\lambda = 3$), and the nonfixed blocks cannot contain more than one fixed point (if a nonfixed block contains a pair of fixed points, then this pair is contained in 5 blocks which is impossible).

Let us consider the fixed blocks containing only fixed points. They must form a 2-(f,6,3) design. Let $h' = f(f-1)/10$ be the number of blocks of this design, let h'' be the number of fixed blocks that contain 1 fixed and 5 nonfixed points, and let $l = (21 - f)/5$ be the number of nonfixed point orbits with respect to α. If $h'' > l$, then the points from at least one nonfixed point orbit must be in more than one fixed block, but this is impossible (Lemma 1). It follows that

$$h'' \leq l. \tag{6}$$

As far as $h' + h'' \equiv 2(mod\ 5)$, (6) holds only if $f = h' = 11$ and $h'' = 1$, i.e. α fixes 11 points and 12 blocks. In this case one of the fixed points belongs to 7 fixed blocks, and each of the rest 10 fixed points is contained in 6 fixed blocks (the $2-(11,6,3)$ design is symmetric) and hence in 6 nonfixed blocks. Yet the number of nonfixed blocks in which a point is contained must be divisible by 5, so this is impossible.

Thus α cannot fix more than one point.

2. Suppose that α fixes 1 point and more than 2 blocks. Let α fix h blocks ($h = 7, 12, 17, ..., 37$). In this case any fixed block consists of 5 nonfixed points of one and the same point orbit and one fixed point. As far as there are 4 orbits of nonfixed points, the points from at least one orbit must be contained in more than one fixed block, but this is impossible (Lemma 1). Thus α cannot fix 1 point and more than 2 blocks.

3. Suppose that α fixes 1 point and 2 blocks, and there are nonfixed points which are contained in both fixed blocks. Without loss of generality we can assume that these are the points from the first point orbit, and that α acts as follows:

α = (1,2,3,4,5)(6,7,8,9,10)...(16,17,18,19,20)(21) on the points, and

α = (1,2,3,4,5)(6,7,8,9,10)...(36,37,38,39,40)(41)(42) on the blocks.

Then the incidence matrix of the design is of the kind:

$$\begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & A_{1,5} & A_{1,6} & A_{1,7} & A_{1,8} & U^T & U^T \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & A_{2,5} & A_{2,6} & A_{2,7} & A_{2,8} & Z^T & Z^T \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & A_{3,5} & A_{3,6} & A_{3,7} & A_{3,8} & Z^T & Z^T \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} & A_{4,5} & A_{4,6} & A_{4,7} & A_{4,8} & Z^T & Z^T \\ Z & Z & Z & Z & Z & Z & U & U & 1 & 1 \end{pmatrix}.$$

where $A_{i,j}$, $i = 1,2,3,4$, $j = 1,2,\ldots,8$ are circulant matrices of order 5, $U = (1,1,1,1,1)$, $Z = (0,0,0,0,0)$.

Let $m_{i,j}$, $i = 1,2,3,4$, $j = 1,2,\ldots,8$ be equal to the number of 1's in a row of $A_{i,j}$. The following equations hold for the first row of the matrix $M = (m_{i,j})_{4\text{x}8}$

$$\sum_{j=1}^{8} m_{1,j} = 10, \qquad \sum_{j=1}^{8} m_{1,j} = 14. \tag{7}$$

It follows from (7) that the first row of M must be a permutation of (2,2,1,1,1,1,1,1). But as far as each point from the first nonfixed point orbit must be in exactly 3 blocks with the fixed point, the sum of $m_{1,7}$ and $m_{1,8}$ must be equal to 1, which is impossible. So there are no solutions in this case.

The only possibility that remains for α is to fix 1 point and 2 blocks, and each nonfixed point to be contained in at most 1 fixed block. Thus the proof is completed.

◇

Let D be a 2-(21,6,3) design with an automorphism α of order 5. Without loss of generality we can assume that α acts as follows:

α = (1,2,3,4,5)(6,7,8,9,10)...(16,17,18,19,20)(21) on the points, and

α = (1,2,3,4,5)(6,7,8,9,10)...(36,37,38,39,40)(41)(42) on the blocks.

Then the incidence matrix of D is:

$$\begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & A_{1,5} & A_{1,6} & A_{1,7} & A_{1,8} & U^T & Z^T \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & A_{2,5} & A_{2,6} & A_{2,7} & A_{2,8} & Z^T & U^T \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & A_{3,5} & A_{3,6} & A_{3,7} & A_{3,8} & Z^T & Z^T \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} & A_{4,5} & A_{4,6} & A_{4,7} & A_{4,8} & Z^T & Z^T \\ Z & Z & Z & Z & Z & Z & U & U & 1 & 1 \end{pmatrix}$$

where $A_{i,j}$, $i = 1,2,3,4$, $j = 1,2,\ldots,8$ are circulant matrices of order 5, $U = (1,1,1,1,1)$, $Z = (0,0,0,0,0)$.

Let $m_{i,j}$, $i = 1,2,3,4$, $j = 1,2,\ldots,8$ be equal to the number of 1's in a row of $A_{i,j}$. The following equations hold for the matrix $M = (m_{i,j})_{4\text{x}8}$

$$\sum_{j=1}^{8} m_{i,j} = 11, \qquad \sum_{j=1}^{8} m_{i,j}^2 = 19, \quad i = 1,2 \tag{8}$$

$$\sum_{j=1}^{8} m_{i,j} = 12, \qquad \sum_{j=1}^{8} m_{i,j}^2 = 24, \quad i = 3,4 \tag{9}$$

$$\sum_{j=1}^{8} m_{i_1,j} m_{i_2,j} = 15, \qquad 1 \le i_1 < i_2 \le 4. \tag{10}$$

It follows from (8) and (9) that the first two rows of M must be permutations of $(2,2,1,1,1,1,1,0)$, while rows 3 and 4 must be permutations either of $(2,2,2,2,1,1,0,0)$ or of $(3,2,1,1,1,1,1,0)$. It was found by computer that there are 4 nonisomorphic matrices with such rows for which (10) is also true.

M_1	M_2	M_3	M_4
2 0 1 2 2 2 1 1	2 1 1 1 2 2 0 2	2 1 1 1 2 2 0 2	2 3 1 1 1 1 1 1
2 2 1 0 2 2 1 1	2 2 1 2 0 2 1 1	2 1 1 3 1 1 1 1	2 1 1 1 1 3 1 1
1 2 2 2 1 1 0 3	1 3 2 0 2 1 2 1	1 2 2 1 0 3 2 1	1 1 2 2 2 1 3 0
1 2 2 2 1 1 3 0	1 0 2 3 2 1 2 1	1 2 2 1 3 0 2 1	1 1 2 2 2 1 0 3

After replacement with circulants in the above given matrices, and addition of the fixed point and blocks 203 nonisomorphic 2-(21,6,3) designs were obtained. The base blocks of the designs obtained from M_1, M_2, M_3, and M_4 are presented in Tables 3, 4, 5, and 6 respectively. The two fixed blocks B_{41} and B_{42} are not given in these tables.

The order of the automorphism group of each design is either 5 or 10. It is 10 for designs 12 and 15 from Table 3, design 70 from Table 4, and designs 12 and 21 from Table 6. It is 5 for the rest of the designs. The orders of the automorphism groups are not divisible by 25, and thus each design can be generated by only one of the matrices M_1, M_2, M_3, or M_4.

Table 3: Base blocks of the designs obtained from M_1

	B_1	B_6	B_{11}	B_{16}	B_{21}	B_{26}	B_{31}	B_{36}
n\o	001123	112233	012233	002233	001123	001123	013334	012224
1	151542	141515	531525	151345	143442	142535	411241	521241
2	151534	141514	131434	151513	143435	142521	521451	531241
3	151442	151515	431524	152524	142345	141431	121451	131241
4	151433	151514	121523	152545	142345	141421	131241	541241
5	151423	151514	121434	152414	141545	141331	531451	531451
6	151434	151514	131412	151535	141335	144521	341451	531241
7	151422	151514	421434	154513	142352	141323	121451	441241
8	151435	141515	131534	151425	141513	143422	141241	511241
9	151452	141515	431523	152425	141554	141243	141241	411241
10	151435	141514	131524	152523	144555	143421	121451	411241
11	151415	141514	321514	152523	141541	141235	131451	311241
12	151442	141515	131413	151245	142312	141234	551241	321241
13	151455	141514	131434	152314	141552	144523	421451	241241
14	151455	141514	111423	152314	142311	143452	241451	341241
15	151423	141414	521534	154515	141521	143432	531241	451241

Table 4: Base blocks of the designs obtained from M_2

	B_1	B_6	B_{11}	B_{16}	B_{21}	B_{26}	B_{31}	B_{36}
n\o	001123	011222	012233	011333	002233	001123	122334	001234
1	151535	315145	141515	414124	151423	143552	314241	142151
2	151525	315145	131515	214135	153523	142452	414251	145411
3	151531	315145	131415	114345	151524	143551	214141	142321
4	151523	315145	521415	514134	154525	142443	314341	143141
5	151531	315145	241415	514124	151535	142451	214151	143421
6	151531	315145	231415	214245	151524	142441	514451	143551
7	151531	315124	521515	214124	153534	142413	515241	145521
8	151513	315124	521515	314235	153424	143535	214231	144141
9	151535	315124	411514	214345	151535	142445	514121	145321
10	151535	114145	521514	314124	151323	142331	514121	143541
11	151552	314145	421514	414245	153545	142342	314151	143531
12	151542	314145	521415	114134	152545	142343	515131	143121
13	151554	314145	251414	414124	151312	142354	315231	143451
14	151424	415145	231515	215124	151412	141431	414351	143421
15	151412	415145	231515	215124	152414	141431	314231	143421
16	151442	415145	241515	115135	151325	142414	314121	143521
17	151424	515145	241514	315124	153512	143524	114451	144111
18	151442	415145	441514	215134	153545	141352	314231	142441
19	151442	415145	431415	315135	151445	141435	415141	145241
20	151433	515145	241514	314234	151325	143452	114121	143141
21	151432	515145	241514	214125	152525	143412	314231	142241
22	151433	415145	131514	514123	152415	143453	314131	144451
23	151444	415145	431514	114123	151435	142342	514121	141351
24	151445	415145	241514	114124	152523	143415	514341	145241
25	151444	415145	241514	114235	152512	143414	514231	145241
26	151424	415145	231514	114135	153512	143444	214451	143521
27	151423	415145	231514	114124	153515	143443	214341	143521
28	151423	415145	231514	214123	153525	143454	214121	144421
29	151435	515145	531415	214135	152423	142343	315131	141141
30	151431	415145	211415	314235	151423	143455	415241	141441
31	151431	515145	131414	114345	151325	142353	415121	143321
32	151435	515145	411414	314145	152435	143452	315451	143451
33	151441	415145	551414	214145	152435	142341	315451	144121
34	151441	415145	251414	214145	151425	141524	315451	144121
35	151442	515145	521415	214135	152335	142343	314151	144141
36	151425	515145	421415	414124	153435	142342	114231	143551
37	151411	515124	311514	214345	153545	141253	315241	143411
38	151445	415124	511514	214125	151323	143452	315351	144311
39	151441	415124	531514	214234	152434	144514	115251	141531
40	151443	415124	531514	214234	152414	144514	115121	141531
41	151455	415124	411514	114145	151423	144522	315241	143351
42	151412	415124	221514	114123	152545	142332	315131	143241
43	151411	415124	251514	514234	152414	143412	215341	141511
44	151442	515124	241514	214125	151214	142312	214151	144411
45	151444	415124	431514	114123	151524	142324	514451	141351
46	151443	415124	431514	114145	151513	142325	514151	141351
47	151441	415124	251514	514234	151214	143412	214341	141511
48	151445	415124	211514	414125	151214	142352	514231	144111
49	151454	115124	241415	314345	152325	142351	115131	141421
50	151454	515124	141415	214345	152325	142351	115131	141421
51	151424	515124	141415	214345	154513	141211	315141	143251

Table 4: continuing.

	B_1	B_6	B_{11}	B_{16}	B_{21}	B_{26}	B_{31}	B_{36}
n\o	001123	011222	012233	011333	002233	001123	122334	001234
52	151433	415124	511415	114145	151524	143425	415251	143311
53	151413	415124	551415	114345	153425	142355	115141	143141
54	151454	415124	541415	114345	152325	142351	115131	141421
55	151424	415124	541415	114345	154513	141211	315141	143251
56	151424	415124	541415	114234	154535	141211	315241	143251
57	151432	415124	211415	114123	151514	141224	215131	143531
58	151432	415124	211415	114125	151513	141224	215141	143531
59	151433	415124	211415	414245	151525	142325	315341	144511
60	151442	415124	221415	314345	151235	143444	215241	143551
61	151523	315145	121514	114245	141415	152441	314341	143531
62	151531	415145	531415	114345	141513	152444	214351	143421
63	151532	315145	451414	114145	144525	152454	414231	143511
64	151522	315124	121514	214124	144512	153534	414341	142151
65	151553	315124	531514	514125	141214	152434	114151	141351
66	151553	315124	531514	514125	142324	151335	514451	143131
67	151535	315124	511514	314125	141524	153542	414151	143111
68	151542	415124	311415	114145	141525	152415	415131	143311
69	151553	315124	511415	214145	141225	152435	315241	144111
70	151424	115145	531515	415125	143513	151335	214351	143421
71	151425	515145	331515	315124	141412	152441	514351	143531
72	151422	115145	131514	315235	142415	152524	214121	142321
73	151412	415145	241514	215235	143523	151435	314231	141441
74	151431	415145	131415	115245	142512	153553	315241	142331
75	151432	115124	141514	415124	143415	152424	114451	141551
76	151432	115124	351514	315235	141523	152555	214121	142541
77	151415	515124	321514	315235	141523	152442	514151	141441
78	151454	415124	131514	115235	142345	153522	114121	142211

Table 5: Base blocks of the designs obtained from M_3

	B_1	B_6	B_{11}	B_{16}	B_{21}	B_{26}	B_{31}	B_{36}
n\o	001123	012233	012233	011123	001333	001222	122334	001234
1	151435	111514	131412	114522	152234	142123	114351	143531
2	151425	141414	131512	114543	152234	141235	115241	145151
3	151534	141415	121524	113522	142123	153124	515351	143311
4	151432	111514	131525	114525	143123	152235	514121	142241
5	151412	151415	131535	114534	141234	142235	115351	152311
6	151423	141415	121525	114514	141123	143235	215351	152311
7	151425	141514	121435	514533	153123	141125	114121	143351
8	151523	151415	131524	513531	142145	153124	115351	143331
9	151422	141415	121524	514544	143124	153235	415341	141231
10	151432	141514	131412	514543	141135	145125	114121	153241
11	151425	141414	121525	514512	143234	141345	514231	153321
12	151422	111415	141524	414553	153235	145234	214341	143341
13	151424	151514	141412	414521	143123	152124	215141	144351
14	151424	151514	131423	414533	144125	153235	115131	145211
15	151412	151415	131535	414553	142124	154234	314121	141321
16	151433	141515	131423	414553	141124	154124	115241	145351
17	151523	111415	141535	413545	142234	143235	215351	153311
18	151524	141415	131513	413542	145125	144235	115141	153321
19	151514	141415	131513	413515	142234	141235	215131	154321
20	151414	151415	121513	414512	143125	144235	315351	153321

Table 5: continuing.

	B_1	B_6	B_{11}	B_{16}	B_{21}	B_{26}	B_{31}	B_{36}
n\o	001123	012233	012233	011123	001333	001222	122334	001234
21	151534	141415	121524	313542	153123	143235	515351	145111
22	151522	131415	121524	313524	154345	143123	414241	142211
23	151435	151514	131423	314541	155235	141123	214451	142551
24	151532	151514	131513	313552	145345	153124	514231	142221
25	151533	151515	131523	313542	145124	153235	514351	142251
26	151524	151514	121423	313555	143145	153134	215241	142551
27	151432	111414	131525	314551	143345	155124	515341	142511
28	151432	111414	131525	314545	143345	155124	515341	142341
29	151433	151415	141524	314542	141125	151124	315251	143311
30	151432	111515	151424	314524	143134	144125	114231	152241
31	151533	151514	131435	213531	153125	142123	214121	141521
32	151523	151415	131513	213555	153124	142235	215341	141151
33	151423	141415	131513	214554	141245	151235	215341	145551
34	151523	111415	131534	213512	142235	143234	214141	153211
35	151514	141415	131513	213545	145125	141235	215351	153321
36	151423	111414	141535	214543	145125	141125	214231	152451
37	151423	111514	131423	214511	144125	145234	214131	152251
38	151543	141415	521513	113531	153235	145125	514451	142351
39	151514	141415	521412	113541	153235	142123	315141	145321
40	151443	151415	521513	114522	152245	142123	414231	141311
41	151443	151415	521412	114522	152245	142123	315241	141141
42	151451	131414	541423	114535	152124	141123	215451	142231
43	151435	131514	541514	114544	152123	142235	514231	141131
44	151431	131415	521513	114555	151234	143135	415241	142211
45	151511	131415	531523	113554	142135	153234	214241	145321
46	151454	121414	521512	114521	144235	152234	114231	143311
47	151435	131514	511513	114513	143345	144235	114451	152321
48	151424	131415	511525	114553	143245	144123	414451	152421
49	151414	121415	531425	114552	141125	143123	315351	152331
50	151531	141415	521525	413532	154234	141125	514241	142351
51	151442	151415	531525	414553	153134	141123	214121	145321
52	151412	151414	531423	414521	152145	144123	115141	143351
53	151444	141415	521513	414515	155235	143125	514231	142311
54	151442	141415	521525	414512	155245	142135	415341	143221
55	151442	141415	521524	414555	154134	145123	514341	141341
56	151433	131415	541425	414555	154134	141145	315341	142351
57	151442	121415	541513	414555	153345	144123	314351	143311
58	151515	131414	531512	413532	145235	153234	214231	144311
59	151424	141415	521513	414521	143125	155245	115141	142311
60	151433	131415	531513	414515	145145	154245	115241	141531
61	151433	151415	511423	414514	144235	143345	215241	153111
62	151421	151414	541412	414543	142123	141345	415241	153341
63	151432	151514	531523	414544	145235	141235	314451	153351
64	151542	131415	541534	313551	153124	145235	515251	142111
65	151442	111415	541513	314555	152245	144123	514231	143541
66	151423	111415	521434	314514	154124	141345	115241	145111
67	151422	111415	521423	314512	154124	141345	115351	145141
68	151422	111415	521423	314523	154245	141123	115351	145421
69	151432	111415	521523	314522	154124	141123	514241	145541
70	151431	151415	531524	314523	151234	141125	114141	143321
71	151533	151514	521525	313552	145125	153245	314231	142251

Table 5: continuing.

n\o	B_1 001123	B_6 012233	B_{11} 012233	B_{16} 011123	B_{21} 001333	B_{26} 001222	B_{31} 122334	B_{36} 001234
72	151431	111415	551423	314515	143245	153125	515131	145331
73	151432	111415	541513	314542	141123	153245	515351	145341
74	151433	111514	541525	314542	145125	153134	314231	141351
75	151443	151415	531512	314542	143245	151235	515351	141311
76	151442	151415	531513	314543	143234	151235	515241	141331
77	151421	141415	541523	314555	143235	151345	114241	141541
78	151542	121415	541513	313543	142123	143135	515351	153341
79	151445	141414	521423	314554	141345	145123	215351	155121
80	151532	111414	521512	213521	153245	143235	515341	142341
81	151532	141415	521513	213545	141145	153245	515351	145531
82	151524	141415	521513	213543	141134	145345	414451	153421
83	151543	141415	521513	213531	143123	144135	515251	152311
84	151525	131415	511513	213554	144125	143135	215131	153431
85	151442	111415	511513	214532	143123	144135	415351	152341
86	151443	151415	551513	214534	142345	141135	315351	152311
87	151431	131415	551513	214534	144345	145135	115351	152431
88	151421	131415	421412	414531	155134	143345	115251	142251
89	151422	151415	441412	314514	153245	141345	215241	145541

Table 6: Base blocks of the designs obtained from M_4

n\o	B_1 001123	B_6 000123	B_{11} 012233	B_{16} 012233	B_{21} 012233	B_{26} 011123	B_{31} 012224	B_{36} 013334
1	151423	135524	151415	141424	121535	314543	241451	441451
2	151432	135444	131514	151412	531423	414513	411451	111351
3	151432	135453	131415	131524	541414	414551	311451	111451
4	151532	135225	151415	141524	521513	313555	251351	121451
5	151432	135125	111514	131415	541513	214541	231351	351451
6	151443	135122	131415	121424	411524	114535	541451	411451
7	151554	135425	141414	131423	411525	113523	231451	531451
8	151423	135444	131415	141535	411524	514531	111351	411451
9	151522	135443	111514	131415	421523	413525	131351	421351
10	151444	135123	151415	141423	411535	414531	431451	521351
11	151433	135245	111514	131423	441535	314541	151351	531451
12	151443	135525	121415	131424	421525	314552	541451	441451
13	151441	135312	151514	121434	431513	314554	441351	131451
14	151423	135152	111514	131412	431535	314515	211351	131451
15	151522	135241	111514	131415	421423	213514	311451	411351
16	151422	135542	111415	121514	421423	214513	341451	351351
17	151424	135153	131514	131423	311525	514511	211351	521451
18	151425	135552	111514	141423	351415	314522	321451	511351
19	151423	135444	131415	111524	341535	214531	111351	411451
20	151413	135244	111514	131423	341525	214542	421351	551451
21	151513	135251	141415	531434	411514	413525	221451	421351

The 236 designs are well distinguished by two invariants suggested by Tonchev [3, Chapter 1]. For each block P the characteristics $(n_0, n_1, ..., n_{39})$ and $(m_0, m_1, ..., m_{39})$ were found , where n_i $(i = 0, 1, ..., 39)$ is the number of pairs (Q, R) of blocks different from P, and such that there are exactly i other blocks having at least one common point with each of the blocks

P, Q, R, and m_j ($j = 0, 1, \ldots, 39$) is the number of pairs (Q, R) of blocks different from P, and such that there are exactly j other blocks having at least two common points with each of the blocks P, Q, R.

Acknowledgements

The authors would like to thank the anonymous referee for the appropriate remarks concerning the proof of Proposition 1.

Note Added in Proof

The referee himself has independently considered 2-(21,6,3) designs and their automorphism groups. The referee points out that:

1. There are 6176 non-isomorphic tactical configurations corresponding to possible 2-(21,6,3) designs with an automorphism of order 3 having no fixed points and no fixed blocks, and 6144 of them give rise to exactly 109314 non-isomorphic designs.
2. There are at least 8203 non-isomorphic 2-(21,6,3) designs with a trivial automorphism group, at least 69 having an automorphism of order 2 only, and a further 8804 that have an automorphism of order 3 fixing at least 3 points.

These observations of the referee will appear in a future paper.

References

[1] R. Mathon, A. Rosa, Tables of Parameters of BIBDs with $r \leq 41$ Including Existence, Enumeration and Resolvability Results, *Ars Combinatoria* 30, 1990, 65-96.

[2] M. Hall Jr., Combinatorial Theory, Blaisdell, Waltham, Mass. 1967

[3] V. D. Tonchev, Combinatorial structures and codes, Kliment Ohridski University press, Sofia 1988.