Hereditary classes of line graphs

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ABSTRACT. We investigate the connections between families of graphs closed under (induced) subgraphs and their forbidden (induced) subgraph characterizations. In particular, we discuss going from a forbidden subgraph characterization of a family $\mathbb{P}$ to a forbidden induced subgraph characterization of the family of line graphs of members of $\mathbb{P}$ in the most general case. The inverse problem is considered too.

1 Introduction

The first tentative discussion of the connection between families of graphs closed under (induced) subgraphs and their forbidden (induced) subgraph characterizations can be found in [1].

We basically follow the standard terminology of [2]. All graphs will be undirected and finite without any loops or multiple lines.

Denote by $\mathbb{G}$ the set of all graphs distinguished up to isomorphism. A subset $\mathbb{P} \subseteq \mathbb{G}$ is called a *class of graphs* (or a *graph-theoretic property*). A class $\mathbb{P}$ is called *hereditary* if $\text{ISub}(G) \subseteq \mathbb{P}$ for any graph $G \in \mathbb{P}$, where $\text{ISub}(G)$ denotes the set of all induced subgraphs of G. Define a partial order "$\leq$" on $\mathbb{G}$: $H \leq G$ if and only if $H \in \text{ISub}(G)$. For a set $\mathbb{Z} \subseteq \mathbb{G}$ we put $\text{FIS}(\mathbb{Z}) = \{G\colon \text{ISub}(G) \cap \mathbb{Z} = \emptyset\}$ – a class of all graphs which is defined by the set $\mathbb{Z}$ of forbidden induced subgraphs. The following simple but important statement is well known.

Proposition 1. (i) *A class $\mathbb{P}$ is hereditary if and only if $\mathbb{P} = FIS(\mathbb{Z})$ for some set $\mathbb{Z} \subseteq \mathbb{G}$.*

(ii) *The minimal (by inclusion) set $\mathbb{Z}$ satisfying* (i) *is unique and it coincides with the set of minimal elements of the partially ordered set* $(\mathbb{G}-\mathbb{P}, \leq)$.

Similarly, a class $\mathbb{P}$ is *strong-hereditary* if $\mathrm{Sub}(G) \subseteq \mathbb{P}$ for any graph $G \in \mathbb{P}$, where $\mathrm{Sub}(G)$ denotes the set of all subgraphs of G. Define one more partial order "$\ll$" on $\mathbb{G}$: $H \ll G$ if and only if $H \in \mathrm{Sub}(G)$. For a set $\mathbb{Z} \subseteq \mathbb{G}$ we put $\mathrm{FS}(\mathbb{Z}) = \{G: \mathrm{Sub}(G) \cap \mathbb{Z} = \emptyset\}$ – a class of all graphs which is defined by the set $\mathbb{Z}$ of forbidden subgraphs. The following statement is similar to Proposition 1.

Proposition 2. (i) *A class $\mathbb{P}$ is strong-hereditary if and only if $\mathbb{P} = FS(\mathbb{Z})$ for some set $\mathbb{Z} \subseteq \mathbb{G}$.*

(ii) *The minimal (by inclusion) set $\mathbb{Z}$ satisfying* (i) *is unique and it coincides with the set of minimal elements of the partially ordered set* $(\mathbb{G} - \mathbb{P}, \ll)$.

Recall that the line graph of a graph G is denoted by $L(G)$. For a class $\mathbb{P} \subseteq \mathbb{G}$ we put $L(\mathbb{P}) = \{L(G): G \in \mathbb{P}\}$.

Proposition 3. *If $\mathbb{P}$ is a strong-hereditary class, then $L(\mathbb{P})$ is a hereditary class.*

Proof: We check that $\mathrm{ISub}(H) \subseteq L(\mathbb{P})$ for any graph $H \in L(\mathbb{P})$. Since any induced subgraph of a graph can be obtained by a removal some points of the graph, it is sufficient to show that $H - v \in L(\mathbb{P})$ for any point v of the graph H. Since $H \in L(\mathbb{P})$, then $H = L(G)$ where $G \in \mathbb{P}$. Let a line e of the graph G corresponds to a point v of the graph H. Since a class $\mathbb{P}$ is strong-hereditary, then $G - e \in \mathbb{P}$. Obviously, $L(G - e) = H - v$, i.e. $H - v \in L(\mathbb{P})$. This completes the proof. □

However, the converse is false. For example, for class $\mathbb{P} = \{P_3, K_2\}$ we have $L(\mathbb{P}) = \{K_2, K_1\}$, i.e. $L(\mathbb{P})$ is hereditary, but $\mathbb{P}$ is not strong-hereditary. However, some analogue of the converse statement can be obtained. for any class of line graphs $\mathbb{P}$ we define the complete inverse image as $L^{-1}(\mathbb{P}) = \{G: L(G) \in \mathbb{P}\} \cup \{\overline{K}_n : n \geq 1\}$.

Proposition 4. *If a class of line graphs $\mathbb{P}$ is hereditary, then its complete inverse image $L^{-1}(\mathbb{P})$ is strong-hereditary.*

Proof: Let $G \in L^{-1}(\mathbb{P})$ and $H \in \mathrm{Sub}(G)$. It is necessary to show that $H \in L^{-1}(\mathbb{P})$, i.e. $L(H) \in \mathbb{P}$ or $H = \overline{K}_n$. If $H = \overline{K}_n$, then there is nothing to prove. Let $H \neq \overline{K}_n$. A subgraph H can be obained from G by a removal of some lines and isolates. For any $e \in EG$ we have $L(G-e) = L(G)-v$, where v is the point of $L(G)$ corresponding to e. Since $L(G) - v \in \mathrm{ISub}(L(G))$ and the class $\mathbb{P}$ is hereditary, then $L(G) - v \in \mathbb{P}$ and $L(G - e) \in \mathbb{P}$, i.e. $G - e \in L^{-1}(\mathbb{P})$. Clearly, $L(G)$ is not changed when we remove any isolate from G. This completes the proof. □

The following problem arises from Proposition 3: for any strong-hereditary class $\mathbb{P} = \mathrm{FS}(\mathbb{Z})$ it is necessary to find a set $\mathbb{Z}'$ such that $L(\mathbb{P}) = \mathrm{FIS}(\mathbb{Z}')$. If

$\mathbb{Z} = \emptyset$, then $\mathrm{FS}(\mathbb{Z}) = \mathbb{G}$ and in this case the set $\mathbb{Z}'$ was found by L.W.Beineke and N. Robertson (Theorem 8.4 in [2]):

Theorem 1. *$L(\mathbb{G}) = FIS(\mathbb{BR})$ where $\mathbb{BR}$ is the set of graphs $G_1, \ldots, G_9$ shown in Figure 1.*

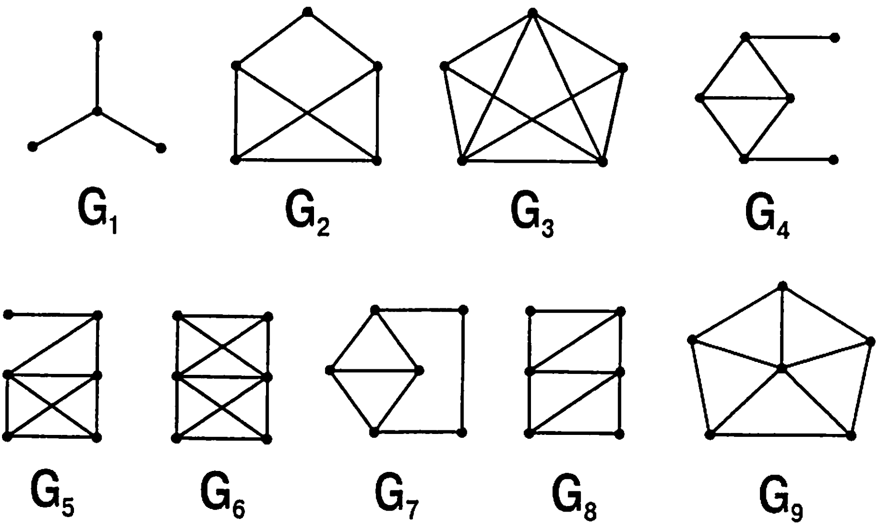

Figure 1. Set $\mathbb{BR}$ of minimal non-line graphs

2 Main results

Components of a graph which are isomorphic to either K_3 or $K_{1,3}$ we call *special*. Graphs G and G' are called *S-equivalent* if there is a bijection φ between their components such that $\varphi(K) \cong K$ for any non-special component K and $\varphi(K) \in \{K_3, K_{1,3}\}$ for any special component K. Denote by $S(G)$ the set of all graphs which are S-equivalent to G. Obviously, $|S(G)| = s + 1$, where $s \geq 0$ is the number of special components of the graph G.

A graph G is called *well-presented in a set* $\mathbb{Z} \subseteq \mathbb{G}$ if every graph $G' \in S(G)$ contains some subgraph $H' \in \mathbb{Z}$. So "G is well-presented in $\mathbb{Z}$" means that no graph S-equivalent to G is in $\mathrm{FS}(\mathbb{Z})$. We denote by $[G]$ the set of minimal (with respect to the partial order "$\ll$") graphs which are well-presented in $\mathbb{Z}$ and contain G as a subgraph. For example, if $G = C_5 \cup K_3$ and $\mathbb{Z} = \{C_5 \cup K_3, C_4 \cup K_{1,3}\}$, then the set $[G] = \{H_1, \ldots, H_7\}$ is shown in Figure 2 (in this case G itself is not well-presented in $\mathbb{Z}$). In particular, notice that $[G] = \{G\}$ for any graph $G \in \mathbb{Z}$ without special components. Accordingly, we denote by $[\mathbb{Z}]$ the set of minimal elements (with respect to the partial order "$\ll$") of $\cup[G]$, where the union is taken over all graphs $G \in \mathbb{Z}$.

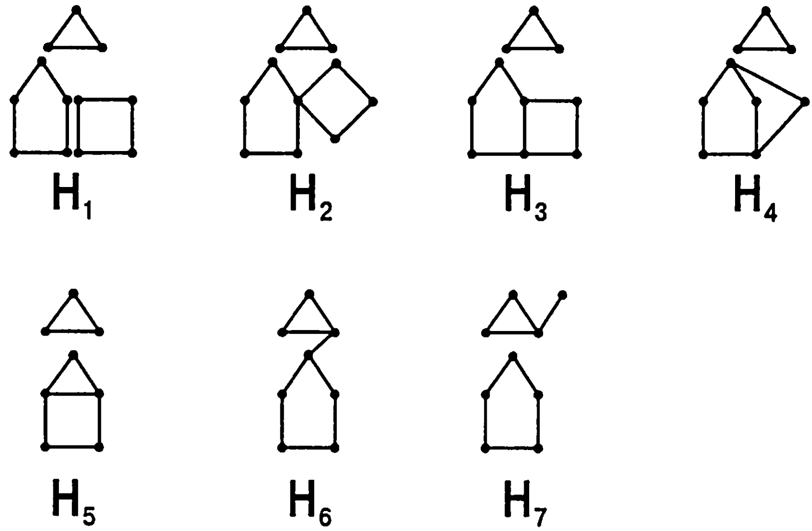

Figure 2. Example of the set $[G]$

Further, we denote by G^x the set of minimal (with respect to the partial order "$\ll$") graphs without isolates which contain G as a subgraph. For example, Figure 3(a) shows the set $\overline{K_7}^x = \{H_8, \ldots, H_{12}\}$. Finally, $\mathbb{Z}^x$ is the set of minimal elements of $\cup G^x$, where the union is taken over all graphs $G \in \mathbb{Z}$.

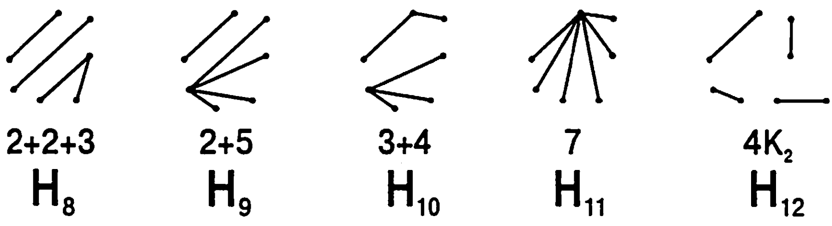

Figure 3a. Set $\overline{K_7}^x$

Theorem 2. *If* $\mathbb{P} = FS(\mathbb{Z})$, *then* $L(\mathbb{P}) = FIS(\mathbb{Z}')$ *where* $\mathbb{Z}' = \mathbb{BR} \cup L([\mathbb{Z}^x])$.

Proof: Firstly we prove the inclusion

$$L(\mathbb{P}) \subseteq \mathrm{FIS}(\mathbb{Z}'). \quad (1)$$

By Theorem 1 for any graph $H \in L(\mathbb{P})$ we have

$$\mathrm{ISub}(H) \cap \mathbb{BR} = \emptyset. \quad (2)$$

Now we show that

$$\mathrm{ISub}(H) \cap L([\mathbb{Z}^x]) = \emptyset. \tag{3}$$

Suppose that (3) is not true, i.e. H contains an induced subgraph $H' \in L([\mathbb{Z}^x])$. Clearly, $H' = L(F')$ for some graph $F' \in [\mathbb{Z}^x]$.

Lemma 1. *If $F' \in [\mathbb{Z}^x]$, then F' does not contain isolates.*

Proof of Lemma 1: Suppose there is an isolate $u \in VF'$. By the definition of $[\mathbb{Z}^x]$, F' contains some subgraph $J \in \mathbb{Z}^x$. There are no isolates in J (by the definition of $\mathbb{Z}^x$). So $J \in \mathrm{Sub}(F' - u)$. Further, F' is well-presented in $\mathbb{Z}^x$, i.e. any graph $F'' \in S(F')$ contains some subgraph $J_1 \in \mathbb{Z}^x$. As in J, there are no any isolates in J_1. Therefore, $J_1 \in \mathrm{Sub}(F'' - u)$. This means that the graph $F' - u$ is well-presented in $\mathbb{Z}^x$. Since $[\mathbb{Z}^x]$ includes only minimal graphs (with respect to the partial order "$\ll$"), we arrive at a contradiction. Lemma 1 is proved. □

Now let $H = L(G)$ where $G \in \mathbb{P}$. Clearly, there is a subgraph G' of G such that $H' = L(G')$. We can consider G' as a graph without any isolates.

So $H' = L(F') = L(G')$ and there are no isolates in both graphs F' and G'. By the theorem of Whitney (Theorem 8.3 in [2]), graphs F' and G' are S-equivalent. But F' is well-presented in $\mathbb{Z}^x$. This means, by the definition of $[\mathbb{Z}^x]$, that G' has a subgraph $G_1 \in \mathbb{Z}^x$. Further, by the definition of $\mathbb{Z}^x$, there is a subgraph $G_2 \in \mathbb{Z}$ of G_1. Since $G_2 \in \mathrm{Sub}(G_1) \subseteq \mathrm{Sub}(G') \subseteq \mathrm{Sub}(G)$, G contains a forbidden subgraph $G_2 \in \mathbb{Z}$. We arrive at a contradiction with the condition $G \in \mathbb{P} = \mathrm{FS}(\mathbb{Z})$. Thus (3) is proved.

From (2) and (3) follows that $L(\mathbb{P}) \subseteq \mathrm{FIS}(\mathbb{BR}) \cap \mathrm{FIS}(L([\mathbb{Z}^x])) = \mathrm{FIS}(\mathbb{BR} \cup L([\mathbb{Z}^x])) = \mathrm{FIS}(\mathbb{Z}')$, i.e. (1) is correct.

Now we prove the inverse inclusion

$$\mathrm{FIS}(\mathbb{Z}') \subseteq L(\mathbb{P}). \tag{4}$$

We consider any graph $H \in \mathrm{FIS}(\mathbb{Z}')$. Since $\mathbb{BR} \subseteq \mathbb{Z}'$, then (by Theorem 1) H is a line graph, i.e. $H = L(G)$. We can consider G as a graph without any isolates.

If there is a graph $G_1 \in \mathbb{P}$ which is S-equivalent to G, then $H = L(G_1) \in L(\mathbb{P})$ and (4) is proved. Otherwlse G is well-presented in $\mathbb{Z}$.

Lemma 2. *If G is well-presented in $\mathbb{Z}$ and does not contain any isolates, then G is well-presented in $\mathbb{Z}^x$.*

Proof of Lemma 2: Let $G_1 \in S(G)$. Since G is well-presented in $\mathbb{Z}$, there is a subgraph $F \in \mathbb{Z}$ in G_1. The graph G_1 (as well as G) does not contain any isolates. Hence there is a subgraph $F' \in F^x$ in G_1 such that $F \in \mathrm{Sub}(F')$ and F' has no any isolates. By the definition of $\mathbb{Z}^x$,

F'contains some subgraph $F'' \in \mathbb{Z}^x$. Thus $F'' \in \mathrm{Sub}(F') \subseteq \mathrm{Sub}(G_1)$ and G is well-presented in $\mathbb{Z}^x$. Lemma 2 is proved. □

By Lemma 2, G contains some subgraph $G' \in [\mathbb{Z}^x]$. But then there is an induced subgraph $H' = L(G') \in L([\mathbb{Z}^x])$ of $H = L(G)$. It contradicts the condition $H \in \mathrm{FIS}(\mathbb{Z}') = \mathrm{FIS}(\mathbb{BR}) \cap \mathrm{FIS}(L([\mathbb{Z}^x]))$. The theorem is proved. □

Now we consider some algorithmic problems concerning a construction of $[\mathbb{Z}^x]$. Let G be an arbitrary graph. We denote by $I = I(G)$ the set of all isolates of G and put $N = VG - I$. Fix a subgraph isomorphic to G in a graph $H \in G^x$. Let $D = VH - VG$. Lines of the fixed subgraph G we call *old* and the other lines of H we call *new*. We will consider that the central point of star $K_{1,1}$ is either of its two points.

Proposition 5. *For any graph G of order n and any graph $H \in G^x$ the following statements are correct.*

(i) *The subgraph H' of H, which is induced by new lines, is a disjoint union of stars $K_{1,r}$ $(r \geq 1)$ with all points of every star (possibly, except the central point) belonging to I.*

(ii) $0 \leq |D| \leq 1$.

(iii) *If $|D| = 1$, then $i = |I|$ is odd and $H = \frac{i+1}{2}K_2 \cup G\langle N \rangle$.*

Proof: (i) Suppose that H' contains a path u, v, w, x (possibly, $x = u$). Clearly, the removal of the line vw from H results in a subgraph which has no isolates and contains G as a subgraph. But it contradicts the minimality of the set G^x with respect to the partial order "$\ll$". Thus H' has no paths or cycles of length 3, i.e. all components of H' are stars $K_{1,r}$ $(r \geq 1)$.

Then let a component $K_{1,r}$ have the central point u. Denote by v one of non-central points. Suppose that $v \notin I$ (when $r = 1$ we suppose that $u, v \notin I$, because either of two points u, v can be chosen as central). We consider all possible cases and in every of them we transform H to a subgraph H_1.

Case 1. If $v \in N$ and $r > 1$, then we put $H_1 = H - uv$.

Case 2. If $v \in N$ and $r = 1$, then $u \notin I$ and we put $H_1 = H - uv$ (when $u \in N$) or $H_1 = H - u$ (when $u \in D$).

Case 3. If $v \in D$ and $r > 1$, then $H_1 = H - v$.

Case 4. If $v \in D$ and $r = 1$, then $u \notin I$ and we put $H_1 = H - v$ (when $u \in N$) or $H_1 = H - \{u, v\}$ (when $u \in D$).

Clearly, in every case $H_1 \in \mathrm{Sub}(H) \setminus \{H\}$ has no isolates and contains a subgraph which is isomorphic to G. It contradicts the minimality of the set G^x.

(ii) Suppose the contrary: there are two different points x and y in D. It follows from (i) that these points are both the central points of stars

$K_{1,r}$ and $K_{1,s}$ (components of H'). If $r > 1$, then remove x from H and construct a star $K_{1,r-1}$ on the set $VK_{1,r} - \{x\}$. The resulting graph H_1 gives a contradiction with the minimality of the set G^x. An analogous contradiction arises when $s > 1$. Therefore $r = s = 1$. Denote by x' (correspondingly, y') the only point of the subgraph H' which is adjacent to x (correspondingly, to y). Then remove points x, y from H and add the new line $x'y'$. As a result we arrive at a contradiction with the minimality of the set G^x again.

(iii) Let $|D| = 1$. Clearly, if there is at least one point of the set $I \cup D$ which in the case of its removal from the graph H does not result in an appearance of an isolate, then we arrive to a contradiction with the minimality of the set G^x. Hence every point of the set $I \cup D$ has degree 1 and is adjacent to a point of the same set. Thus $H = \frac{i+1}{2} K_2 \cup G\langle N \rangle$. The proposition is proved. □

Corollary 1. *If $\mathbb{Z}$ is a finite set, then $\mathbb{Z}^x$ is also a finite set.*

The following method of construction of the set G^x for any graph G follows from Proposition 5. If a graph G is edgeless ($N = \emptyset$), then G^x consists of all graphs of order $n = |VG|$ in which all components are stars $K_{1,r}$ ($r \geq 1$) as well as (when n is odd) of a graph $\frac{n+1}{2} K_{1,1}$. These graphs can be generated by means of an algorithm constructing all partitions of an integer $n = n_1 + n_2 + \cdots + n_k$ with $n_i \geq 2$ (see [5]). An illustration for $n = 7$ is given in Figure 3(a).

Now let G be a graph that is not edgeless. We number points of the set N: $u_1, u_2, \ldots, u_n$, $n = |N|$. We construct all possible partitions of a number $i = |I|$ into non-zero parts. Every part $i_j > 1$ corresponds to a star-component K_{1,i_j-1} with all points belonging to the set I. Every part $i_j = 1$ corresponds to those points of I which are adjacent to points from N. Denote by m the number of parts which are equal to 1. Then we can generate all ordered partitions of the number $m = m_1 + m_2 + \cdots + m_n$ into $n = |N|$ parts $m_j \geq 0$, $j = 1, 2, \ldots, n$. Further, every point $u_j \in N$ we connect with m_j points of the set I ($j = 1, 2, \ldots, n$) so that these m_j points of I have degree 1. When $|I|$ is odd, we need to construct one more graph described in Proposition 5 (iii). In Figure 3(b) the described above method is illustrated for $G = \overline{K_4} \cup K_2$. It is clear that in contrast to an edgeless graph in this case the resulting set can be redundant, so it has to be reduced by removing "superfluous" graphs.

Thus for a finite set $\mathbb{Z}$ we can construct sets G^x ($G \in \mathbb{Z}$), then choose all minimal elements in the union $\cup G^x$ over all $G \in \mathbb{Z}$ with respect to the order "$\ll$" and obtain the set $\mathbb{Z}^x$.

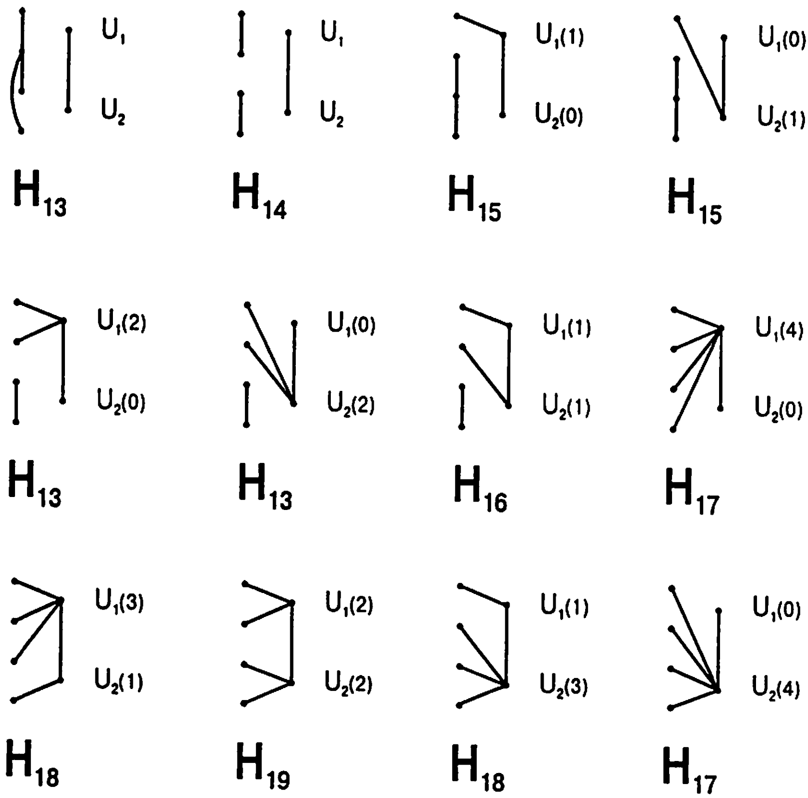

Figure 3b Set $(\overline{K}_4 \cup K_2)^x$

Constructing the $[\mathbb{Z}]$ is more complicated since the set $[G]$ does not depend on a graph G only, but on a set $\mathbb{Z}$ too. As the next example shows the set $[G]$ can turn out infinite. Let $\mathbb{Z} = \{G_1, G_2, \dots\}$, where $G_1 = C_4 \cup K_{1,3}$ and $G_i = C_{i+3} \cup K_3$ $(i \geq 2)$. The graph G_1 is not well-presented in $\mathbb{Z}$ because the replacement of its only special component $K_{1,3}$ by component K_3 results in graph $C_4 \cup K_3$ which does not contain any subgraph G_i $(i = 1, 2, \dots)$. But all graphs $H_i = C_4 \cup C_{i+3} \cup K_{1,3}$ $(i = 2, 3, \dots)$ are well-presented in $\mathbb{Z}$. We check the minimality of graphs H_i $(i \geq 2)$ with respect to the partial order "$\ll$". There are only three proper maximal subgraphs of H_i: $H_i^1 = P_4 \cup C_{i+3} \cup K_{1,3}$, $H_i^2 = C_4 \cup P_{1+3} \cup K_{1,3}$, and $H_i^3 = C_4 \cup C_{i+3} \cup P_2 \cup K_1$ (every of them is obtained from the graph H_i by removing one line). It is clear that none of these graphs has triangles and so they have no subgraphs G_i $(i \geq 2)$. Graphs H_i^1 and H_i^3 have no subgraph G_1 either. The graph H_i^2 has G_1 as a subgraph, but it is not well-presented in $\mathbb{Z}$, because graph $C_4 \cup P_{i+3} \cup K_3$ obtained from H_i^2 by replacing component $K_{1,3}$ with K_3 does not contain any graph from $\mathbb{Z}$ as a subgraph.

So all graphs H_i ($i \geq 2$) are well-presented in $\mathbb{Z}$ and minimal with respect to the partial order "$\ll$". Thus $[G_1] \supseteq \{H_2, H_3, \dots\}$ and $[G_1]$ is an infinite set.

However, the next statement is valid.

Proposition 6. *Let $\mathbb{Z} = \{G_1, G_2, \dots, G_k\}$ is a finite set of graphs. Then for any graph $G \in \mathbb{Z}$ the set $[G]$ is finite.*

Proof: If a graph G is well-presented in $\mathbb{Z}$ then $[G] = \{G\}$ and the proof is finished.

Suppose graph G is not well-presented in $\mathbb{Z}$. We show that the order of any graph $H \in [G]$ is bounded above. This implies the finiteness of $[G]$. Denote by N the number of special components of H. Let $q = \max_{1 \leq i \leq k} |VG_i|$.

Firstly, suppose that $N \geq 2q$. Let $H_1 \in S(H)$ and every special component of H_1 is isomorphic to K_3. By the definition of $[G]$, there is some subgraph $G_{i_1} \in \mathbb{Z}$ in H_1. Clearly, G_{i_1} has common points at most with $|VG_{i_1}| \leq q$ components K_3 of H_1. Similarly, let $H_2 \in S(H)$ and every special component of H_2 is isomorphic to $K_{1,3}$. As before we can find a subgraph $G_{i_2} \in \mathbb{Z}$ in H_2. Clearly, G_{i_2} has common points at most with $|VG_{i_2}| \leq q$ components $K_{1,3}$ of H_2. Since any graph $H_3 \in S(H)$ has at least either q components K_3 or q components $K_{1,3}$, then H_3 contains correspondingly either the subgraph G_{i_1} or the subgraph G_{i_2}.

Since $|VG| \leq q$, there is a special component of H which has no common points with the subgraph G. If we remove this special component from H, then the resulting graph H' will contain the subgraph G and the number of special components in H' will be $N - 1 \geq 2q - 1$. Consider any graph $H_4 \in S(H')$ and corresponding to it graphs $H_4 \cup K_3$, $H_4 \cup K_{1,3} \in S(H)$. It is clear that H_4 contains at least q identical special components. Suppose that H_4 contains q components K_3. Then $H_4 \cup K_3$ and $H_4 \cup K_{1,3}$ also contain at least q components K_3, so they have the subgraph $G_{i_1} \in \mathbb{Z}$. Clearly, H_4 contains the subgraph G_{i_1} since G_{i_1} has common points with at most q components K_3 of $H_4 \cup K_3$ and $H_4 \cup K_{1,3}$. Similarly, in the case when there are at least q components $K_{1,3}$ in H_4, we obtain that H_4 has the subgraph $G_{i_2} \in \mathbb{Z}$. So any graph $H_4 \in S(H')$ contains either the subgraph $G_{i_1} \in \mathbb{Z}$ or the subgraph $G_{i_2} \in \mathbb{Z}$ and therefore H' is well-presented in $\mathbb{Z}$. Since H' contains the subgraph G, then $H' \in [G]$. But H' is a subgraph of H. This contradicts the minimality of members of $[G]$ with respect to the partial order "$\ll$". Hence $N \leq 2q - 1$.

Further, let $S(H) = \{H^1, \dots, H^{N+1}\}$. Since $H \in [G]$, any $H^i \in S(H)$ contains a subgraph $G^i \in \mathbb{Z}$ ($i = 1, 2, \dots, N + 1$). Denote by U the set of all points of H which are not included in any special component. It is clear that $U \subseteq VH^i$ for every $i = 1, 2, \ldots, N+1$. Let $U \cap VG^i = U^i$ and $X = \cup_{i=1}^{N+1} U^i$.

Denote by S the number of special components of the induced subgraph $H\langle X \rangle$. It is clear that $S \leq \frac{1}{3}|X|$ and none of these components is a special component in H (by the definition of U). So for any such a component R a point $x_R \in U \setminus X$ which is adjacent to at least one of points of this component can be found. Add into the set X all such points x_R and denote by Y the resulting set.

By the construction, the induced subgraph $H\langle Y \rangle$ has no special components. From the minimality of members of $[G]$ it follows that $U = Y$. Then

$$
\begin{aligned}
|VH| &= |U| + |VH \setminus U| \leq |Y| + 4N \leq |X| + \frac{1}{3}|X| + 4N \\
&= \frac{4}{3}|X| + 4N \leq \frac{4}{3}\sum_{i=1}^{N+1} |U^i| + 4N \leq \frac{4}{3}\sum_{i=1}^{N+1} q + 4N = \frac{4}{3}q(N+1) + 4N \\
&= (\frac{4}{3}q + 4)N + \frac{4}{3}q \leq (\frac{4}{3}q + 4)(2q-1) + \frac{4}{3}q = \frac{8}{3}q^2 + \frac{28}{3}q - 4,
\end{aligned}
$$

i.e. the order of H is bounded above. The proposition is proved. □

Corollary 2. *If $\mathbb{P} = FS(\mathbb{Z})$ and $\mathbb{Z}$ is a finite set, then $L(\mathbb{P}) = FIS(\mathbb{Z}')$ and $\mathbb{Z}'$ is also a finite set.*

Now we consider special cases of Theorem 2.

Corollary 3. *Let $\mathbb{P} = FS(\mathbb{Z})$ with all graphs from $\mathbb{Z}$ being connected and non-trivial. Then $L(\mathbb{P}) = FIS(\mathbb{Z}')$, where $\mathbb{Z}'$ is one of following sets:*

(i) $\mathbb{BR} \cup L(\mathbb{Z})$ *if* $K_3, K_{1,3} \notin \mathbb{Z}$;

(ii) $\{K_3, K_{1,3}\} \cup L(\mathbb{Z} - \{K_3, K_{1,3}\})$ *if* $K_3, K_{1,3} \in \mathbb{Z}$;

(iii) $\{K_{1,3}, K_2 + \overline{K}_2\} \cup L(\mathbb{Z} - \{K_3\})$ *if* $K_3 \in \mathbb{Z}$ *and* $K_{1,3} \notin \mathbb{Z}$;

(iv) $\{K_{1,3}, K_1 + (K_1 \cup K_2), K_2 + \overline{K}_2, K_4\} \cup L(\mathbb{Z} - \{K_{1,3}\})$ *if* $K_{1,3} \in \mathbb{Z}$ *and* $K_3 \notin \mathbb{Z}$.

Proof: Without loss of generality we consider that $\mathbb{Z}$ is minimal with respect to the partial order "$\ll$". By Theorem 2, $\mathbb{Z}' = \mathbb{BR} \cup L([\mathbb{Z}^x])$. We construct the set $[\mathbb{Z}^x]$. Since there are no graphs with isolates in $\mathbb{Z}$ it follows that $\mathbb{Z}^x = \mathbb{Z}$. Then all graphs from $\mathbb{Z}$, with the exception of K_3 and $K_{1,3}$ in Cases (iii) and (iv) respectively, are well-presented in $\mathbb{Z}$. We obtain that $\mathbb{Z}' = \mathbb{BR} \cup L(\mathbb{Z})$ in Cases (i) and (ii) (notice that in (ii) the set $\mathbb{Z}'$ is written in a reduced form owing to the graphs $G_2 - G_9$ from $\mathbb{BR}$ containing induced subgraph K_3).

Further, we find $[K_3] = \{K_1 + (K_1 \cup K_2)\}$, $L([K_3]) = \{K_2 + \overline{K}_2\}$ in Case (iii) and $[K_{1,3}] = \{T, K_1 + (K_1 \cup K_2), K_{1,4}\}$, where T is the tree with degree sequence $(3,2,1,1,1)$, $L([K_{1,3}]) = \{K_1 + (K_1 \cup K_2), K_2 + \overline{K}_2, K_4\}$ in Case (iv). As above the set $\mathbb{BR}$ can be reduced in both Cases (iii) and (iv) to only the graph G_1 owing to the presence of induced subgraph $K_2 + \overline{K}_2$ in the graphs $G_2 - G_9$. The corollary is proved. □

Denote by $\mathbb{A}$, $\mathbb{B}$, $\mathbb{TF}$, $\mathbb{BD}_k$, $\mathbb{N}_k$ and $\mathbb{D}_k$ sets of acyclic graphs, bipartite graphs, triangle-free graphs, graphs with bounded density $w(G) \leq k$, graphs with bounded order $p \leq k$, and graphs with bounded maximum degree $\Delta(G) \leq k$ respectively.

Corollary 4.

(i) $L(\mathbb{A}) = FIS(K_{1,3}, K_2 + \overline{K}_2, C_n, n \geq 4)$;

(ii) (see [4]) $L(\mathbb{B}) = FIS(K_{1,3}, K_2 + \overline{K}_2, C_{2n+1}, n \geq 2)$;

(iii) (see [3,4]) $L(\mathbb{TF}) = FIS(K_{1,3}, K_2 + \overline{K}_2)$;

(iv) $L(\mathbb{BD}_k) = FIS(\mathbb{BR} \cup L(K_{k+1}))$;

(v) $L(\mathbb{N}_1) = FIS(K_1) = \emptyset$;
$L(\mathbb{N}_2) = FIS(\overline{K}_2, K_2) = \{K_1\}$;
$L(\mathbb{N}_k) = FIS(\mathbb{BR} \cup L([\overline{K}_{k+1}])), k \geq 3$;

(vi) $L(\mathbb{D}_1) = FIS(K_2)$;
$L(\mathbb{D}_2) = FIS(K_{1,3}, K_1 + (K_1 \cup K_2), K_2 + \overline{K}_2, K_4)$;
$L(\mathbb{D}_k) = FIS(\mathbb{BR} \cup \{K_{k+1}\}), k \geq 3$.

Proof: It is enough to use Corollary 3 (or Theorem 2 in Case (v)) and known characterisations (or definitions): $\mathbb{A} = \mathrm{FS}(C_n, n \geq 3)$, $\mathbb{B} = \mathrm{FS}(C_{2n+1}, n \geq 1)$, $\mathbb{TF} = \mathrm{FS}(K_3)$, $\mathbb{BD}_k = \mathrm{FS}(K_{k+1})$, $\mathbb{N}_k = \mathrm{FS}(\overline{K}_{k+1})$, and $\mathbb{D}_k = \mathrm{FS}(K_{1,k+1})$. The proof is finished. □

Now we investigate a problem arising from Proposition 4: from a given hereditary class $\mathbb{P} = \mathrm{FIS}(\mathbb{Z})$ of line graphs find a set $\mathbb{Z}'$ such that $L^{-1}(\mathbb{P}) = \mathrm{FS}(\mathbb{Z}')$.

We divide a set $\mathbb{Z}$ into two parts: $\mathbb{Z} = \mathbb{Z}_{\mathbb{BR}} \cup \mathbb{Z}_L$, where $\mathbb{Z}_{\mathbb{BR}}$ contains all non-line graphs and $\mathbb{Z}_L$ contains all line graphs of the set $\mathbb{Z}$.

Let $L^{(-1)}(G) = \{H \colon L(H) = G \text{ and } H \text{ has no isolates}\}$. Correspondingly, $L^{(-1)}(\mathbb{Z}) = \cup L^{(-1)}(G)$ over all graphs $G \in \mathbb{Z}$.

Proposition 7. *If $\mathbb{P} = FIS(\mathbb{Z})$ is an arbitrary hereditary class of line graphs, then $L^{-1}(\mathbb{P}) = FS(\mathbb{Z}')$ where $\mathbb{Z}' = L^{(-1)}(\mathbb{Z}_L)$.*

Proof: Firstly, we prove the inclusion $L^{-1}(\mathbb{P}) \subseteq \mathrm{FS}(\mathbb{Z}')$. Let $G \in L^{-1}(\mathbb{P})$, i.e. $G = \overline{K}_n$ or $L(G) = H \in \mathbb{P}$. Suppose the opposite: there is a subgraph

$G' \in \mathbb{Z}' = L^{(-1)}(\mathbb{Z}_L)$ in G. Then there is an induced subgraph $L(G') \in \mathbb{Z}_L \subseteq \mathbb{Z}$ in H. We arrive at a contradiction with the condition $H \in \mathbb{P} = \text{FIS}(\mathbb{Z})$.

Now prove the inverse inclusion: $\text{FS}(\mathbb{Z}') \subseteq L^{-1}(\mathbb{P})$. Suppose the opposite: there is a graph $G \in \text{FS}(\mathbb{Z}')$ but $G \notin L^{-1}(\mathbb{P})$. By the definition of $L^{-1}(\mathbb{P})$, we have $G \neq \overline{K}_n$ and $H = L(G) \notin \mathbb{P}$, i.e. there is a forbidden induced subgraph $H' \in \mathbb{Z}$ in H. But H' is a line graph (as an induced subgraph of a line graph). So $H' \in \mathbb{Z}_L$. There is a subgraph G' without any isolates of G corresponding to the induced subgraph H' of H. Clearly, $G' \in L^{(-1)}(\mathbb{Z}_L) = \mathbb{Z}'$. It is a contradiction with the condition $G \in \text{FS}(\mathbb{Z}')$. The proof is finished. □

We use Proposition 7 for the class $\mathbb{S} = \text{FIS}(2K_2, C_4, C_5)$ of split graphs. So the class of line split graphs is $\text{FIS}(\mathbb{Z})$, where $\mathbb{Z} = \mathbb{Z}_{\mathbb{BR}} \cup \mathbb{Z}_L$ with $\mathbb{Z}_{\mathbb{BR}} = \{G_1, G_3\}$ (Figure 1) and $\mathbb{Z}_L = \{2K_2, C_4, C_5\}$. We find $\mathbb{Z}' = L^{-1}(\mathbb{Z}_L) = \{2P_3, C_4, C_5\}$ and obtain that a graph $L(G)$ is split if and only if $G \in \text{FS}(2P_3, C_4, C_5)$.

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