Some consequences of a result of Brouwer

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ABSTRACT. We prove two new characterization theorems for finite Moufang polygons, one purely combinatorial, another group-theoretical. Both follow from a result of Andries Brouwer on the connectedness of the geometry opposite a flag in a finite generalized polygon.

1 Introduction

Some time ago in Oberwolfach, I heard Andries Brouwer talk about the geometry far away from a flag in a spherical building. It crossed my mind that in case of generalized polygons (which are after all the rank 2 spherical buildings), this theory could have some implications on characterization results. So I asked Andries if he knew something on the connectedness of these geometries for generalized polygons. The next morning Andries came with the answer, which he published in the meantime, see Brouwer [1]. The result is apparently also useful for other purposes, for instance, Bernhard Mühlherr and Mark Ronan use it to prove the Moufang property for almost all twin buildings without an ∞-stroke in the diagram. The applications I had in mind were not so far-reaching and I have collected them in the present paper.

A *generalized n-gon* (or *polygon*) is a point-line geometry the incidence graph of which has girth $2n$ and diameter n (the girth being the length of the smallest circuit), and we assume in this paper $n \geq 3$. If every element of a generalized polygon Γ is incident with at least three other elements,

*Senior Research Associate of the Belgian National Fund for Scientific Research then Γ is said to be *thick*. For a thick generalized polygon, it is easily seen that the number $s+1$ of points on a line is a constant and the number $t+1$ of lines through a point is constant. The pair (s,t) is sometimes called the *order* or the *parameters* of the polygon. Generalized polygons were introduced by Tits [6], who remarked in [7] that from Theorem D of Fong & Seitz [4] follows that all finite generalized polygons satisfying the so-called *Moufang condition* arise naturally from finite Chevalley groups of (relative) rank 2. These examples are sometimes called *classical* and they belong to the following list: the symplectic $(Sp_4(q))$, orthogonal $(O_4(q), O_5^-(q))$ and Hermitian $(H_3(q), H_4(q))$ quadrangles; the split Cayley $(G_2(q))$ and triality $({}^3D_4(q))$ hexagons and the Ree-Tits octagons $({}^2F_4(q))$ or their duals.

It is still an unsolved question whether there exist thick non-Moufang finite generalized hexagons or octagons. Every characterization theorem of the Moufang ones can therefore be useful in proving or disproving this existence. This is the main motivation for this paper. Another motivation is the fact that we deal with results valid for all polygons and not only quadrangles or hexagons or etc. Common properties, characterizations and proofs must eventually lead to a better understanding of the corresponding exceptional Chevalley groups.

2 Main results

Let Γ be a finite generalized n-gon. Then $n \in \{3,4,6,8\}$ by a result of Feit & Higman [3]. For a positive integer k, a *k-path* is a sequence of $k+1$ consecutively incident elements which are all distinct. A 1-path is also called a *flag* or a *chamber*. A $(2n-1)$-path the extremities of which are incident is an *ordered apartment* (the last two terms are inherited from the *building*-terminology). The elements of an ordered apartment form an ordinary apartment. The polygon Γ *satisfies the k-Moufang condition*, or *is a k-Moufang polygon*, if for any k-path $\gamma = (x_0, x_1, \ldots, x_k)$, the collineation group of Γ fixing every element incident with $x_1, x_2 \ldots$ or x_{k-1} acts transitively on the set of apartments containing all elements of γ. With this definition, a *Moufang* polygon is simply an n-Moufang n-gon, for some n, and a *Tits* polygon (i.e. a polygon arising from an irreducible Tits system, or a group with a BN-pair, of relative rank 2) is nothing other than a 1-Moufang polygon. It is straightforward to see that k-Moufang implies $(k-1)$-Moufang. In Van Maldeghem & Weiss [9], it is shown that 4-Moufang implies n-Moufang, $n \geq 4$, and for finite generalized n-gons, 3-Moufang implies n-Moufang. Our first main result is:

Theorem 2.1 *Every thick finite 2-Moufang polygon is a 3-Moufang polygon, and hence a Moufang polygon.*

All the results mentioned so far can be proved without imposing the classification of the finite simple groups. Assuming that classification, it is possible to show that 1-Moufang implies Moufang, see Buekenhout & Van Maldeghem [2], and that finishes the whole business. A classification free proof of that result seems not to be for the near future and so the present result could be a step in that direction.

For our second main result, we need some further preparations. *Opposite* elements in a generalized n-gon are elements at mutual distance n. If two elements x and y are not opposite, then the projection of x onto y is the unique element incident with y closest to x (with respect to the natural distance function δ in Γ, namely the one inherited from the incidence graph). The distance of an element x to a flag $\{p, L\}$ is the minimum of $\delta(x,p)$ and $\delta(x, L)$.

Now we fix a flag $\mathcal{F} = \{p, L\}$ of Γ (and assume that p is a point and L a line). Given an element x, the projection of x onto $\{p, L\}$ is the unique element incident with p or L, different from both p and L and closest to x. It is equal to the projection of x onto p if the projection of x onto L is p (and dually). A k-path $(x_0, x_1, \dots, x_k)$ such that $\delta(x_k, \mathcal{F}) = \delta(x_0, \mathcal{F})+k = n-1$ will be called a *staircase up*; a *staircase down* is a staircase up in reversed order. A *passage* is a sequence of consecutively incident elements all at distance $n-1$ from $\mathcal{F}$. A *tour* is the juxtaposition of a staircase up, a passage and a staircase down (in that order), with the additional condition that the last element is the same as the first one. Note that all elements of a staircase up (respectively down) have the same projection onto $\mathcal{F}$ and we call that projection the *base step* of the staircase.

Suppose (x, y) is a passage (respectively two consecutive element of a staircase down) and z is an element having the same projection onto (respectively the same projection onto and the same distance from) $\mathcal{F}$ as x does. Then there exists a unique element u incident with z such that u has the same projection onto (respectively distance from) $\mathcal{F}$ as y does. We denote u by $z_{(x,y)}$. Now consider a tour $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, where $\gamma_1 = (x_0, x_1, \dots, x_k)$ is a staircase up, $\gamma_2 = (y_0, y_1, \dots, y_\ell)$ is a passage (with the technical convention that $x_k = y_0$) and $\gamma_3 = (y_{\ell+1}, \dots, y_m = x_0)$ is a staircase down. For a given staircase up γ_1' with the same base step as γ_1, one can uniquely define a path $\gamma_{2,3}' = (z_0, z_1, \dots, z_m)$ inductively by $z_i = (z_{i-1})_{(y_{i-1}, y_i)}$ and z_0 is the last element of Γ_1'. We call $\gamma_{2,3}'$ *parallel* to $\gamma_2\gamma_3$. If for every choice of γ and γ_1', the juxtaposition of γ_1' and $\gamma_{2,3}'$ is a tour, then we call the chamber $\mathcal{F}$ a *master chamber*. Our second main result reads:

Theorem 2.2 *If in a thick finite generalized polygon γ all chambers in one apartment are master chambers, then Γ is a Moufang polygon, except possibly in the cases where Γ is a hexagon of order* $(3,3)$ *or an octagon of order* $(2,4)$ *or* $(4,2)$.

Note that both theorems are well-known in the case of projective planes, where they even hold in the infinite case. So we may as well assume that $n \geq 4$. The exclusion of hexagons of order $(3,3)$ and octagons with $st = 8$ is not a serious problem and could with some effort be solved, but it is believed that they are unique and so the effort is not worth the goal.

Also remark that the converse of both theorems is true. These are easy consequences of the 3-Moufang property of Moufang polygons.

3 Proof of Theorem 2.1

Suppose Γ is a thick finite 2-Moufang n-gon, $n \geq 4$. Fix a chamber $\{p, L\}$, with p a point and L a line. Suppose first that $s = 2$. Every collineation fixing all lines through p and fixing one further point on L fixes all points on L. Therefore Γ is 3-Moufang and hence Moufang by Van Maldeghem & Weiss [9]. Now suppose that $(s,t) = (3,3)$ and $n = 6$. Then the result follows directly from Yanushka [10]. So by Brouwer [1], we may assume that the geometry induced by Γ on the points and lines at distance $n - 1$ from $\{p, L\}$ is connected. Let x and y be two points opposite p (note that n is even) with common projection p' onto L. We show that there exists a collineation fixing all points on L and all lines through p and mapping x to y. Let $(x, M_1, x_1, M_2, x_2, \ldots, y)$ be a shortest path from x to y all elements of which are at distance $n-1$ from $\{p, L\}$. Let u be the projection of p' onto M_2. By an inductory argument, we may assume $y = u$. By the 1-Moufang assumption, there exists a collineation θ of Γ fixing all lines through p and mapping x to x_1. Dually, there exists a collineation σ fixing all points on L and mapping M_1 to M_2 (and thus fixing x_1 and mapping x to u). The collineation $\theta\sigma^{-1}\theta^{-1}\sigma$ fixes all points on L and all lines through p and maps x to u. Hence our claim. But this means that Γ is 3-Moufang. The theorem is proved.

A nice corollary is the following:

Corollary 3.1 *A finite generalized quadrangle which contains an apartment all points and lines of which are elation points and elation lines has the Moufang property.*

Indeed, an elation point p is exactly a point such that there exists a collineation group fixing all lines through p and acting regularly on the points opposite p, see e.g. Payne & Thas [5]. Dually, on finds the definition of an elation line. But the assumptions of Corollary 3.1 readily imply that every point is an elation point and every line is an elation line and so the quadrangle is 2-Moufang.

An interesting question is whether all finite generalized quadrangles all points of which are elation points can be classified without the classification of the finite simple groups.

4 Proof of Theorem 2.2

Suppose every chamber in a certain apartment Σ of a finite thick generalized n-gon Γ is a master chamber. With the assumptions of Theorem 2.2, it follows from Brouwer [1] that the geometry at distance $n-1$ from such a chamber $\mathcal{F} = \{p, L\}$ is connected. Let x and y be two points opposite p having the same projection onto L. We construct a collineation θ fixing all points on L and all lines through p and mapping x to y in the classical way as follows. If a is any element, then there exist a passage and a staircase down the juxtaposition of which is a path from x to a. The last element of the parallel to that path starting in y is by definition the image of a by θ. This definition does not depend on the choice of the passage or the staircase down, because another choice gives rise to a tour and so the parallels must also form a tour by the master chamber condition. Incidence is automatically preserved since every flag can be put into a passage or a staircase down. This completes the proof of the theorem.

An interesting problem is to find weaker conditions by putting certain conditions on the tours, e.g. Van Maldeghem, Payne & Thas [8] prove that if $n = 4$, then it suffices to require the property only for apartments (viewed as tours). A similar result for hexagons or octagons is not known and seems hard to prove.

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