On a conjecture of Mahmoodian and Soltankhah regarding the existence of (v, k, t) trades

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Abstract

In 1992, Mahmoodian and Soltankhah conjectured that, for all $0 \leq i \leq t$, a (v, k, t) trade of volume $2^{t+1} - 2^{t-i}$ exists. In this paper we prove this conjecture and, as a corollary, show that if $s \geq (2t-1)2^t$ then there exists a (v, k, t) trade of volume s.

Introduction

A (v, k, t) **trade** T of **volume** s consists of two disjoint collections T_1 and T_2, each containing s k-subsets (**blocks**) of some set V, such that every t-subset of V is contained in the same number of blocks in T_1 and T_2. Note that not all elements of V need appear in a block of T. The set of elements of V contained in T is called the **foundation** of the trade, denoted by $F(T)$. Let $m(T) = s$ and $f(T) = |F(T)|$; whence $f(T) \leq v \leq |V|$. The trade T is often written as $T_1 - T_2$, where the following example illustrates this notation.

Example 1 $T = T_1 - T_2 = +135 + 146 + 236 + 245 - 136 - 145 - 235 - 246$ is a $(6, 3, 2)$ trade, with $F(T) = \{1, 2, 3, 4, 5, 6\}$, $m(T) = 4$ and $f(T) = 6$.

Trades, which are also known as null t-designs, have many uses in the theory of designs. They can be used to construct t-designs with different support sizes (Hedayat [3]), and are related to the design intersection problem (Billington [1]) and the problem of finding defining sets of designs (Street [7]).

An obvious question to ask is, given parameters k and t, for which volumes does there exist a (v, k, t) trade? Hwang [4] showed that $m(T) \geq 2^t$ and $f(T) \geq k + t + 1$. Trades of volume 2^t are called **basic trades**.

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For $0 \le i \le t$, define $s_i = 2^t + 2^{t-1} + \cdots + 2^{t-i} = 2^{t+1} - 2^{t-i}$. Mahmoodian and Soltankhah [6] constructed (v,k,t) trades of volumes s_1, s_2, s_3, for $t \ge 1$, $t \ge 2$, $t \ge 3$ respectively. It was further shown that trades of volume s, where $s_0 < s < s_1$, do not exist. This led to the following conjecture.

Conjecture 2 ([6]) *For all $0 \le i \le t$, there exists a (v,k,t) trade of volume s_i.*

In this paper we will prove Conjecture 2 by *adding* two basic trades, with foundations chosen so that the resultant trade has volume s_i. All the trades we construct are **simple**; that is, they have no repeated blocks.

Results

Graham, Li and Li [2] determined the structure of basic trades. For the following theorem however, we will use the form given by Khosrovshahi, Majumdar and Widel [5].

Theorem 3 ([5]) *If $v \ge k+t+1$ and $k \ge t+1$, then there exists a (v,k,t) (basic) trade of volume 2^t. Such a trade has the following form:*

$$T = T_1 - T_2 = S_0(S_1 - S_2)(S_3 - S_4)\ldots(S_{2t+1} - S_{2t+2}),$$

with $S_i \subset V$ for $i = 0, \ldots, 2t+2$, $S_i \cap S_j = \emptyset$ for $i \ne j$, $|S_{2i-1}| = |S_{2i}| \ge 1$ for $i = 1, \ldots, t+1$, and $|S_0| + \sum_{i=1}^{t+1} |S_{2i}| = k$. □

Example 4 Let $S_0 = \emptyset$ and $S_i = \{i\}$ for $i = 1, \ldots, 6$. Then the basic trade $T = T_1 - T_2$ constructed as in Theorem 3 is that given in Example 1.

Lemma 5 ([3]) *Suppose $T = T_1 - T_2$ and $R = R_1 - R_2$ are (v,k,t) and (v^*,k,t) trades respectively. Then $T + R = (T_1 + R_1) - (T_2 + R_2)$ is a (v^{**},k,t) trade.* □

Note that, if in Lemma 5 $F(T) \cap F(R) = \emptyset$, then $m(T+R) = m(T) + m(R)$. The following theorem uses non-disjoint foundations for basic trades T and R to construct a trade of volume s_i.

Theorem 6 *For all $0 \le i \le t$, there exists a (v,k,t) trade of volume s_i.*

Proof Let T and R be basic (v,k,t) trades of the form

$$\begin{aligned} T &= T_1 - T_2 = S_0(S_1 - S_2)(S_3 - S_4)\ldots(S_{2t+1} - S_{2t+2}), \\ R &= R_1 - R_2 = -S_0(S_1 - S_2)\ldots \\ &\quad \ldots(S_{2t-2i-1} - S_{2t-2i})(\tilde{S}_{2t-2i+1} - S_{2t-2i+2})\ldots(\tilde{S}_{2t+1} - S_{2t+2}), \end{aligned}$$

where $0 \leq i \leq t$, and $\tilde{S}_j$ is chosen so that $\tilde{S}_j \cap S_l = \emptyset$ for $j = 2t-2i+1, 2t-2i+3, \ldots, 2t+1$, $l = 0, 1, \ldots, 2t+2$.

That $T+R$ is a (v^*, k, t) trade follows from Lemma 5. It remains to find $m(T+R)$. There are two cases to consider, $i = t$ and $i < t$, depending on whether 2^{t-i} is odd or even. When $i = t$, the only block common to T and R is $S_0 S_2 \ldots S_{2t+2}$. This block is in $T_1 \cap R_2$ or $T_2 \cap R_1$, hence $m(T+R) = m(T) + m(R) - 1 = 2^{t+1} - 1$. When $0 \leq i \leq t-1$, $|T_1 \cap R_2| = |T_2 \cap R_1| = 2^{t-i-1}$. Therefore,

$$m(T+R) = m(T) + m(R) - 2.2^{t-i-1} = 2^{t+1} - 2^{t-i},$$

as required. □

Example 7 Let $T = (1-2)(3-4)(5-6)$, as in Example 4, and choose $R = -(1-2)(\tilde{3}-4)(\tilde{5}-6)$. Then

$$\begin{aligned} T+R &= +135 + 236 + 245 + 1\tilde{3}6 + 14\tilde{5} + 2\tilde{3}\tilde{5} \\ &\quad -136 - 145 - 235 - 1\tilde{3}\tilde{5} - 2\tilde{3}6 - 24\tilde{5} \end{aligned}$$

is an $(8,3,2)$ trade with $m(T+R) = 6$ and $f(T+R) = 8$.

Corollary 8 *For $s \geq (2t-1)2^t$ there exists a (v,k,t) trade of volume s.*

Proof Let $0 \leq a \leq 2^t - 1$. Then a has a unique binary representation $\sum_{i=1}^{t} a_i 2^{t-i}$, where $a_i \in \{0,1\}$. Therefore, if $s^* = t.2^{t+1} - a$, where $0 \leq a \leq 2^t - 1$, then

$$s^* = \sum_{i=1}^{t} a_i s_i + \sum_{i=1}^{t} (1-a_i) 2^{t+1}.$$

Any integer $s > (2t-1)2^t$ can be written in the form $s = b.2^t + s^*$, where $s^* = t.2^{t+1} - a$, with $0 \leq a \leq 2^t - 1$, and $b \geq 0$. Using disjoint foundations, choose b basic trades, for each $a_i = 1$ a trade of volume s_i and for each $a_i = 0$ two trades of volume 2^t (or a trade of volume 2^{t+1} as in [6], if a smaller foundation size is required). Now use Lemma 5 to construct a trade of volume s as required.

For $s = (2t-1)2^t$, choose $(2t-1)$ basic trades on disjoint foundations, and use Lemma 5 to construct the required trade. □

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