# Cycles Containing Many Vertices of Subsets in 1-Tough Graphs with Large Degree Sums[1]

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**Abstract.** Let $G$ be a graph of order $n$ and $X$ a given vertex subset of $G$. Define the parameters $\alpha(X) = max\{|S|| \ S$ is an independent set of vertices of the subgraph $G[X]$ in $G$ induced by $X\}$ and and $\sigma_k(X) = min\{\Sigma_{i=1}^{k} d(x_i)| \ \{x_1, x_2, \cdots, x_k\}$ is an independent vertex set in $G[X]$ $\}$. A cycle $C$ of $G$ is called $X-$ *longest* if no cycle of $G$ contains more vertices of $X$ than $C$. A cycle $C$ of $G$ is called $X-$ *dominating* if all neighbors of each vertex of $X \backslash V(C)$ are on $C$. In particular, $G$ is $X-$ *cyclable* if $G$ has an $X-$cycle, i.e., a cycle containing all vertices of $X$. Our main result is as follows: If $G$ is 1-tough and $\sigma_3(X) \geq n$, then $G$ has an $X-$longest cycle $C$ such that $C$ is an $X-$dominating cycle and $|V(C) \cap X| \geq min\{|X|, |X| + \frac{1}{3}\sigma_3(X) - \alpha(X)\}$, which extends the well known results of D. Bauer et al [2] in terms of $X-$cyclability. Finally, if $G$ is 2-tough and $\sigma_3(X) \geq n$, then $G$ is $X-$cyclable.

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## §1. Results

We use [4] for terminology and notations not defined here and consider simple graph only.

Let $G$ be a graph of order $n$. $X \subseteq V(G)$ and $\omega(G)$ denote the number of components of graph $G$. As introduced by Chvátal [6], a graph $G$ is $t-tough$ if $|S| \geq t \cdot \omega(G\backslash S)$ for any $S \subset V(G)$ with $\omega(G\backslash S) > 1$. The toughness of $G$, denoted by $\tau(G)$, is the maximum value of $t$ for which $G$ is t-tough($\tau(K_n) = \infty$ for all $n \geq 1$). A cycle $C$ of $G$ is called $X-longest$ if no cycle of $G$ contains more vertices of $X$ than $C$, and by $c(X)$ we denote the number of vertices of $X$ in $X-$longest cycle. A cycle $C$ of $G$ is called $X - dominating$ if all neighbors of each vertex of $X\backslash V(C)$ are on $C$. We say that $G$ is $X - cyclable$ if $G$ has an $X-$cycle, i.e., a cycle containing all vertices of $X$. If $X = V(G)$, instead of V(G)-longest cycle and V(G)-cyclability, we use the common terms hamilton cycle and Hamiltonian, and $c(G)$ instead of $c(V(G))$, respectively. We denote by $\alpha(X) = max\{|S|| S$ is an independent set of vertices of the subgraph $G[X]$ in $G$ induced by $X \}$. More generally, for $k \leq \alpha(X)$ we denote by $\sigma_k(X)$ the maximum value of the degree sums (in $G$) of any k pairwise nonadjacent vertices of $X$; for $k > \alpha(X)$, we set $\sigma_k(X) = k(n - \alpha(X))$. We write $\alpha$ and $\sigma_k(G)$ instead of $\alpha(V(G))$ and $\sigma_k(V(G))$, respectively.

Two classical results in hamiltonian graph theory are the following.

**Theorem 1.** [10] Let $G$ be a graph of order $n \geq 3$ with $\sigma_2(G) \geq n$, then $G$ is Hamiltonian.

**Theorem 2.** [7] Let $G$ be a 1-tough graph of order $n \geq 3$ with $\sigma_2(G) \geq n - 4$, then $G$ is Hamiltonian.

In [2], D.Bauer et al obtained the further extension of Theorem 2.

**Theorem 3.** [2] Let $G$ be a 1-tough graph of order $n \geq 3$ with $\sigma_3(G) \geq n$, then $c(G) \geq min\{n, n + \frac{1}{3}\sigma_3(G) - \alpha(G)\}$.

Theorem 3 implies several known results. For details, readers are referred to the surveys of D. Bauer et al in [1] and [2].

We note that, the proof of Theorem 3 relies on the following fact: if $G$ is 1-tough and $\sigma_3(G) \geq n$, then every longest cycle is dominating; Moreover, if $G$ is nonhamiltonian, $G$ contains a longest cycle $C$ with $\mu(C) = max\{d(v)| v \in V(G)\backslash V(C)\} \geq \frac{1}{3}\sigma_3(G)$.

Recently, H.J. Broersma, H. Li, J.P. Li, F. Tian and H.J. Veldman [5] obtained some results involving given subsets.

**Theorem 4.** [5] Let $G$ be a 2-connected graph of order $n \geq 3$ and let $X \subseteq V(G)$ with $\sigma_3(X) \geq n + 2$, then $c(X) \geq min\{|X|, |X| + \frac{1}{3}\sigma_3(X) - \alpha(X)\}$.

Motivated by Theorem 4, we can obtain the following main result, which extends the some results of D. Bauer et al [2] in terms of X-cyclability.

**Theorem 5.** Let $G$ be a 1-tough graph of order $n \geq 3$ and $X \subseteq V(G)$ with $\sigma_3(X) \geq n$, then $G$ contains an X-longest cycle $C$ which is X-dominating.

The proof of Theorem 5 is postponed to section 2. For a note, under the conditions of Theorem 5, we easily obtain that: $X \backslash V(C)$ is an independent set for every X-longest cycle $C$.

Theorem 5 admits the following corollary, which extends the theorem of Bigalke and Jung [3] in terms of X-cyclability.

**Corollary 6 .** Let $G$ be a 1-tough graph of order $n \geq 3$ and $X \subseteq V(G)$ with $\delta(X) \geq \frac{1}{3}n$, then $G$ contains an X-longest cycle $C$ which is X-dominating.

Next key Lemma is the basis for many of the results that follows.

**Lemma 7.** Let $G$ be a graph of order $n$ with $\delta(X) \geq 2$ and $X \subseteq V(G)$ with $\sigma_3(X) \geq n$. Let $G$ contain an $X$-longest cycle $C$ which is $X$-dominating. If $x_0 \in X \backslash V(C)$ and $A = N(x_0)$, then $(X \backslash V(C)) \cup A^X$ is an independent set of vertices, where $A^X$ contains, for each $v \in A$, the first vertex of $X \cap V(C)$ succeeding $v$ on $C$ (in a fixed orientation of $C$).

We note that, a weak version of Lemma 7 can be found in J.P. Li [8]. For convenience, we give the full argument in section 2.

Lemma 7 has many applications. The next Theorem is obtained by combining Lemma 7 with Theorem 5. An outline proof of Theorem 8 is given in section 2.

**Theorem 8.** Let $G$ be a 1-tough graph of order $n \geq 3$ and let $X \subseteq V(G)$ with $\sigma_3(X) \geq n$, then $c(X) \geq min\{|X|, |X| + \frac{1}{3}\sigma_3(X) - \alpha(X)\}$.

Theorem 8 admits the following result, which extends one result of Bigalke and Jung [3] in terms of X-cyclability.

**Theorem 9.** Let $G$ be a 1-tough graph of order $n \geq 3$ and let $X \subseteq V(G)$ with $\sigma_3(X) \geq max\{n, 3\alpha(X) - 2\}$, then $G$ is $X$-cyclable.

Proof: By assumption, $\sigma_3(X) \geq max\{n, 3\alpha(X) - 2\}$, then $\sigma_3(X) \geq 3\alpha(X) - 2$. Also, by Theorem 8, $c(X) \geq min\{|X|, |X| + \frac{1}{3}\sigma_3(X) - \alpha(X)\} \geq \{|X|, |X| - \frac{2}{3}\}$. Since $c(X)$ is an integer, we have $c(X) = |X|$, i.e., $G$ is X-cyclable.

We now turn our attention to graph with $t(G) = \tau \geq 1$. For $X \subseteq V(G)$, we easily get $\alpha(X) \leq \alpha(G) \leq \frac{n}{\tau+1}$, then Theorem 8 immediately implies our next result.

**Corollary 10.** Let $G$ be a graph of order $n \geq 3$ and let $X \subseteq V(G)$ with $t(G) = \tau \geq 1$. If $\sigma_3(X) \geq n$, then $c(X) \geq min\{|X|, |X| + \frac{1}{3}\sigma_3(X) - \frac{n}{\tau+1}\}$.

A special case of Corollary 10 is the following result.

**Corollary 11.** Let $G$ be a 2-tough graph of order $n \geq 3$ and $X \subseteq V(G)$. If $\sigma_3(X) \geq n$, then $G$ is X-cyclable.

## §2. Proofs

In order to prove our main results, we introduce some additional terminology and notations.

Let $C$ a cycle of graph $G$. We denote by $\overrightarrow{C}$ the cycle with a given orientation, and by $\overleftarrow{C}$ the cycle with the reverse orientation. If $u, v \in V(C)$, then $u\overrightarrow{C}v$ denotes the consecutive vertices of $C$ from $u$ to $v$ in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We will consider $u\overrightarrow{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. We use $u^+$ to denote the successor of u on $\overrightarrow{C}$ and $u^-$ to denote its predecessor. If $S \subseteq V(C)$, then $S^+ = \{v^+ | v \in S\}$ and $S^- = \{v^- | v \in S\}$.

**Proof of Theorem 5.** Let $G$ satisfy the conditions of Theorem 5. Assume that no $X$-longest cycle of $G$ is $X$-dominating. Choose a cycle $C$ and a path $P$ satisfying the following conditions:

(a). $C$ is an $X$-longest cycle;

(b). Subject to (a), $|M(C)|$ is minimal, where $M(C)$ denotes the set of all edges of $G\backslash V(C)$ which are incident with at least one vertex of $X$. (Here, $M(C) \neq \emptyset$ by assumption);

(c). $P$ connects two vertices $v_r$ and $v_s$ of $C$, is internally disjoint from $C$ and contains a vertex $x_0 \in X$ incident with an edge $e_0$ of $M(C)$;

(d). Subject to (a), (b) and (c), let $H$ be the subgraph of $G\backslash V(C)$, which contains $x_0$, $e_0$ and $|V(H)|$ is minimal.

Set $R = V(G)\backslash V(C)$ and $A = N(H)$. We give an orientation of $C$, and let $v_1, v_2, \cdots, v_{|A|}$ be the vertices of $A$, occuring on $\overrightarrow{C}$ in consecutive order. So $v_r, v_s \in A$. Hence, we obtain the assertion:

(1) for any $i \in \{1, 2, \cdots, |A|\}$, either $v_i^+\overrightarrow{C}v_{i+1}^- \cap X \neq \emptyset$ or there exists a vertex $u' \in v_i^+\overrightarrow{C}v_{i+1}^-$ such that $u'$ is adjacent to a vertex $w' \in X\backslash V(C)$.

Assuming the contrary to (1), i.e., $v_i^+\overrightarrow{C}v_{i+1}^- \cap X = \emptyset$ and there exists no vertex $u' \in v_i^+\overrightarrow{C}v_{i+1}^-$ such that $u'$ is adjacent to a vertex $w' \in X\backslash V(C)$, we consider the cycle $C' = v_i H v_{i+1}\overrightarrow{C}v_i$, we easily get either $|V(C')\cap X| > |V(C)\cap X|$ or $|V(C')\cap X| = |V(C)\cap X|$ which contradicts to the condition

(b) or (d). This leads to a contradiction to the choice of $C$. Indeed, the assertion (1) holds.

Let $u_{r1}$ be the first vertex on $v_r^+\overrightarrow{C}v_{r+1}^-$ such that either $u_{r1}\in X$ or $u_{r1}$ is adjacent to a vertex $u_{r2}\in X\backslash V(C)$. Set $x_r = u_{r1}$ if $u_{r1}\in X$ and $x_r = u_{r2}$ otherwise. Define $u_{s1}\in v_s^+\overrightarrow{C}v_{s+1}^-$ and $u_{r2}$, $x_s$ similarly. We have that

(2) $x_i\neq x_j$, $x_ix_j\notin E(G)$ and $N(x_i)\cap N(x_j)\cap R=\emptyset$ $\quad i,j\in\{0,r,s\}$ and $i\neq j$.

otherwise we contradict (a) or (d). A similar argument shows that

(3) $x_iv\notin E(G)$ whenever $v\in v_j^+\overrightarrow{C}u_{j1}\cup\{x_j\}$, $i\in\{0,r,s\}$, $j\in\{r,s\}$ and $i\neq j$.

For any $v_i\in A\backslash\{v_r,v_s\}$, set $u_{i1}=v_i^+$. Let $u_{i2}=u_{i1}^+$ if $N(u_{i1})\cap R=\emptyset$, otherwise let $u_{i2}$ be an arbitrary vertex in $N(u_{i1})\cap R$. Then we obtain

(4) $x_r(x_s)\neq u_{i2}$, $u_{i2}\neq u_{j2}$ $\qquad (i,j\in\{1,2,\cdots,|A|\}\backslash\{r,s\}$ and $i\neq j)$.

otherwise contradicting (a) or (d). Furthermore, we get that

(5) $x_ku_{im}\notin E(G)$ $\qquad (i\in\{1,2,\cdots,|A|\}\backslash\{r,s\}$; k=r, s; m= 1, 2.)

We can also obtain the following observations (otherwise contradicting the choice of $C$):

(6) if $v\in u_{r1}^+\overrightarrow{C}v_s^-$ and $x_sv\in E(G)$, then $x_rv^+\notin E(G)$;

(7) if $v\in u_{s1}^+\overrightarrow{C}v_r^-$ and $x_rv\in E(G)$, then $x_sv^+\notin E(G)$.

Let $U=V(C)\cup\{x_r,x_s\}\cup\{u_{i2}|i\in\{1,2,\cdots,|A|\}\backslash\{r,s\}\}$. We define a bijection $f:U\longrightarrow U$ as follows:

(8) if $x_i\neq u_{i1}$, then $f(u_{i1})=x_i$ and $f(x_i)=u_{i1}^+$ $(i=r,s)$;

(9) if $u_{i2}\notin V(C)$, then $f(u_{i1})=u_{i2}$ and $f(u_{i2})=u_{i1}^+$ $(i\in\{1,2,\cdots,|A|\backslash\{r,s\})$;

(10) if $f(v)$ is not yet defined as above, then $f(v)=v^+$.

We consider the following sets:

$$
\begin{aligned}
A(x_r) &= \{v\in(u_{r1}\overrightarrow{C}u_{s1}^-\cup\{x_r\}\cup\{u_{i2}|\ i\in\{r+1,r+2,\cdots,s-1\}\}) \\
&\qquad |\ x_rf(v)\in E(G)\} \\
A(x_s) &= \{v\in(u_{r1}\overrightarrow{C}u_{s1}^-\cup\{x_r\}\cup\{u_{i2}|\ i\in\{r+1,r+2,\cdots,s-1\}\}) \\
&\qquad |\ x_sv\in E(G)\} \\
B(x_r) &= \{v\in(u_{s1}\overrightarrow{C}u_{r1}^-\cup\{x_s\}\cup\{u_{j2}|\ j\in\{s+1,s+2,\cdots,r-1\}\}) \\
&\qquad |\ x_rv\in E(G)\} \\
B(x_s) &= \{v\in(u_{s1}\overrightarrow{C}u_{r1}^-\cup\{x_s\}\cup\{u_{j2}|\ j\in\{s+1,s+2,\cdots,r-1\}\}) \\
&\qquad |\ x_sf(v)\in E(G)\} \\
D(x_i) &= \{v\in V(G)\backslash U|x_iv\in E(G)\} \qquad i\in\{0,r,s\}.
\end{aligned}
$$

Set $AB(x_r,x_s)=A(x_r)\cup A(x_s)\cup B(x_r)\cup B(x_s)$ and $D(x_r,x_s)=D(x_r)\cup D(x_s)$. Noting that $f$ is bijection, we obtain

$$d(x_i)=|A_i|+|B_i|+|D_i| \qquad (i=r,s)$$

and

$$d(x_0)=|A\cap N(x_0)|+|D(x_0)|.$$

Observing (2) to (7), we have that the sets $A(x_r)$, $A(x_s)$, $B(x_r)$, $B(x_s)$, $D(x_0)$, $D(x_r)$ and $D(x_s)$ are pairwise disjoint, and the $x_0$, $u_{i1}$ ($i \in \{1, 2, \cdots, |A|\} \setminus \{r, s\}$) are in none of these sets. Noting that $x_0$, $x_r$, $x_s \in X$, we conclude that

$$\begin{aligned}
\sigma_3(X) &\leq d(x_0) + d(x_r) + d(x_s) \\
&\leq |A \cap N(x_0)| + |D(x_0)| + |A(x_r)| + |B(x_r)| + |D(x_r)| \\
&\qquad + |A(x_s)| + |B(x_s)| + |D(x_s)| \\
&= |A \cap N(x_0)| + |D(x_0)| + |AB(x_r, x_s)| + |D(x_r, x_s)| \qquad (11) \\
&\leq |A| + (|H| - 1) + (|V(C)| - (|A| - 2)) \\
&\qquad + (|V(G)| - |V(C)| - |V(H)|) \\
&= n + 1
\end{aligned}$$

On the other hand, since $x_0$, $x_r$ and $x_s$ are three pairwise nonadjacent vertices of $X$, we get

(12) $\qquad d(x_0) + d(x_r) + d(x_s) \geq \sigma_3(X) \geq n$

It follows that $x_0$, and hence every vertex of $X \cap V(H)$, is adjacent to all but at most one vertex in $A$. This implies the existence of a $(v_i, v_j)$-path $P_{i,j}$ with all internal vertices in $H$ for all $i, j \in \{1, 2, \cdots, |A|\}$ with $i \neq j$, such that $P_{i,j}$ contains at least one vertex in $X \cap V(H)$ and either one edge in $M(C)$ or is adjacent to at least one edge in $M(C)$. Again using (11), we obtain that at most one of the following assertions holds:

(13) (i). at most one vertex $u_{i1}$ of $\{u_{11}, u_{21}, \cdots, u_{|A|1}\} \setminus \{u_{r1}, u_{s1}\}$ satisfying $u_{i1} \notin X$;

(ii). at most one vertex of $V(G) \setminus (V(C) \cup V(H))$ is not in $D(x_r, x_s)$.

Without loss of generality, we may assume that $u_{i1} \in X$ for any $i \in \{1, 2, \cdots, |A|\} \setminus \{r, s\}$ (otherwise we easily obtain a contradiction in the similar argument below).

So we set $x_i = u_{i1}$ for any $i \in \{1, 2, \cdots, |A|\}$. For each $i \in \{1, 2, \cdots, |A|\}$, if $v \in x_i \overrightarrow{C} v_{i+1}^-$ such that $x_i v \in E(G)$, let $u_{i1}'$ be the first vertex on $v^- \overleftarrow{C} x_i$ such that $u_{i1}' \in X$ or $u_{i1}'$ is adjacent to a vertex $u_{i2}' \in X \setminus V(C)$. Set $x_i' = u_{i1}'$ if $u_{i1}' \in X$ and $x_i' = u_{i2}'$ otherwise. We call that $x_i'$ is an $i-vertex$ respect to $v$, maybe $x_i = x_i'$. In particular, $x_i$ is an i-vertex.

If $x_r'$ is an r-vertex and $x_s'$ is an s-vertex, substitute $x_r'$ and $x_s'$ for $x_r$ and $x_s$, the observations (2) through (12) still hold. Moreover, observations (2) through (12) actually hold for arbitrary $r$ and $s$ with $1 \leq r < s \leq |A|$. From (11) and (12), we also deduce the follows ($1 \leq r < s \leq |A|$):

**(14)** if $x_r'$ is an r-vertex and $x_s'$ is an s-vertex, then at most one of the following assertions holds:

(i). at most one vertex of $V(C)\backslash\{u_{i1}|i \in \{1,2,\cdots,|A|, i \neq j\}$ is not in $AB(x'_r, x'_s)$;

(ii). at most one vertex of $V(G)\backslash(V(C) \cup V(H))$ is not in $D(x'_r, x'_s)$.

Without loss of generality, we may assume that $x'_r$, $x'_s \in V(C) \cap X$.

Now, we give some notations. For any $i \in \{1,2,\cdots,|A|\}$, let $w_{i1}$ be the last vertex on $v_i^+ \overrightarrow{C} v_{i+1}^-$ such that either $w_{i1} \in X$ or $w_{i1}$ is adjacent to a vertex $w_{i2} \in X\backslash V(C)$. Set $y_i = w_{i1}$ if $w_{i1} \in X$ and $y_i = w_{i2}$ otherwise. With the similar arguments as above (in the given reverse orientation of $C$), we have that $y_i \in V(C) \cap X$ for any $i \in \{1,2,\cdots,|A|\}$.

For $s \in \{1,2,\cdots,|A|\}$, we can obtain the next three observations.

(15) if $v \in v_{s+1}^+ \overrightarrow{C} v_s$ and $x_s v \in E(G)$, then $y_s v^- \notin E(G)$, otherwise we easily construct an X-longer cycle than $C$.

(16) if $v \in v_{s+1}^+ \overrightarrow{C} v_s$ and $x_s v \in E(G)$, then

(i). if $N(v_{s+1}^-) \cap R = \emptyset$ or $v_{s+1}^- \notin X$, then $v_{s+1}^{--} v^- \notin E(G)$;

(ii). if $N(v^-) \cap R = \emptyset$ or $v^- \notin X$, then $y_s v^{--} \notin E(G)$.

otherwise contradicting assumptions (a) to (d).

(17) if $v \in v_{s+1} \overrightarrow{C} v_s^-$ and $x_s v \in E(G)$, then

(i). $y_s v^+ \notin E(G)$;

(ii). if $N(v_{s+1}^-) \cap R = \emptyset$ or $v_{s+1}^- \notin X$, then $v_{s+1}^{--} v^+ \notin E(G)$;

(iii). if $N(v^+) \cap R = \emptyset$ or $v^+ \notin X$, then $y_s v^{++} \notin E(G)$.

otherwise contradicting assumptions (a) to (d).

Using observations as above, we now derive contradictions in all cases distinguished below. If $v \in V(G)$, then by $N'(v)$ we denote the set of vertex $x$ such that there is a $(v, x)$-path of length at least 1 with all internal vertices in $V(G)\backslash V(C)$. In particular, $N(v) \subseteq N'(v)$. For $S \subseteq V(G)$, set $N'(S) = \cup_{v \in S} N'(v)\backslash S$. (Noting $x_i \in X \cap V(C)$ for any $i \in \{1,2,\cdots,|A|\}$.

**Case 1.** For all $i \in \{1,2,\cdots,|A|\}$, $N'(v_i^+ \overrightarrow{C} x_i) \cap V(C) \subseteq v_i \overrightarrow{C} v_{i+1} \cup A$ and $N'(y_i \overrightarrow{C} v_{i+1}^-) \cap V(C) \subseteq v_i \overrightarrow{C} v_{i+1} \cup A$.

Suppose there exist integer $r$, $s$ and vertices $x$, $y$ such that $1 \leq r < s \leq |A|$, $x \in x_r^+ \overrightarrow{C} v_{r+1}^-$, $y \in x_s^+ \overrightarrow{C} v_{s+1}^-$ and $xy \in E(G)$. Since by the hypotheses of Case 1 $x_s x$, $x_r y \notin E(G)$, we get either $x_r x^+$ or $x_s y^+$ is in $E(G)$, otherwise $x$, $y \notin AB(x_r, x_s)$, contradicting (14). Without loss of generality, we assume $x_r x^+ \in E(G)$. So we get $x_s y^+ \notin E(G)$, otherwise we get an $X$-longer cycle than $C$. Hence we obtain

(i) . If exists $y' \in N(y^+) \cap R$, then $y' \notin D(x_s)$ (otherwise we get an X-longer cycle then $C$). So $y, y' \notin AB(x_r, x_s)$, contradicting (14).

(ii) . If $N(y^+) \cap R = \emptyset$, then $x_s y^{++} \notin E(G)$ (otherwise contradicting assumptions (a) to (d)). So $y, y^+ \notin AB(x_r, x_s)$, contradicting (14).

This contradiction with (14) shows that in the case no edge, and similarly no path with all internal vertices in $V(G)\backslash V(C)$, joins two vertices in different sets of the collection $\{v_i^+ \overrightarrow{C} v_{i+1}^- | 1 \le i \le |A|\}$, then $\omega(G\backslash A) \ge |A| + 1$, contradicting the fact that $G$ is 1-tough.

**Case 2.** For some $i \in \{1, 2, \cdots, |A|\}$, $N'(v_i^+ \overrightarrow{C} x_i) \cap V(C) \not\subseteq v_i \overrightarrow{C} v_{i+1} \cup A$ or $N'(y_i \overrightarrow{C} v_{i+1}^-) \cap V(C) \not\subseteq v_i \overrightarrow{C} v_{i+1} \cup A$.

Assume, e.g. $z_r \in N'(v_s^+ \overrightarrow{C} x_s)$, where $z_r \in v_r^+ \overrightarrow{C} v_{r+1}^-$, $r < s$ and $|v_r^+ \overrightarrow{C} z_r|$ is minimum. Moreover, we have $z_r \in x_r^+ \overrightarrow{C} v_{r+1}^-$, otherwise we get an X-longer cycle than $C$, a contradiction. For convenience, we may assume $x_s z_r \in E(G)$, in the other cases we easily get a contradiction in the similar manner.

By (3), $z_r \neq x_r$. Let $z$ be a vertex in $x_r^+ \overrightarrow{C} z_r^-$ with $x_r z \in E(G)$ such that $|z \overrightarrow{C} z_r|$ is minimum. Let $x_z$ be the $r$-vertex respect to $z$. For convenience we suppose $x_z \in X \cap V(C)$, maybe $x_z = x_r$. Either $z = z_r^-$ or $z = z_r$, otherwise $z, z^+ \notin AB(x_r, x_s)$, contradicting (14). So we distinguish two subcases.

**Case 2.1.** $z = z_r^-$

In the case, $x_r z_r^+ \notin E(G)$. Moreover, we have

(18). $\quad x_r z_r^{++} \notin E(G)$.

In the fact, suppose $x_r z_r^{++} \in E(G)$. If $N(z_r^+) \cap R = \emptyset$, we get a cycle $C' = v_r P_{r,s} v_s \overleftarrow{C} z_r^{++} x_r \overleftarrow{C} z_r x_s \overrightarrow{C} v_r$, which satisfies either $|V(C') \cap X| > |V(C) \cap X|$ or $|V(C') \cap X| = |V(C) \cap X|$ with $|M(C')| < |M(C)|$, contradicting the choice of $C$. If $z^* \in N(z_r^+) \cap R$, then $x_r z^*, x_s z^* \notin E(G)$, otherwise we get an X-longer cycle $C'$ than $C$. Hence $z, z^* \notin AB(x_r, x_s) \cup D(x_r, x_s)$, contradicting (14). Indeed, $x_r z_r^{++} \notin E(G)$.

By (15) and (17), $y_s z_r^+, y_s z_r^- \notin E(G)$. Hence $x_s y_s' \notin E(G)$ for any $y_s' \in y_s^+ \overrightarrow{C} v_{s+1}$, i.e., $y_s$ is not an s-vertex, otherwise $z_r^-, z_r^+ \notin AB(x_r, y_s)$, contradicting (14). Thus $x_r y_s \in E(G)$, otherwise $z, y_s \notin AB(x_r, x_s)$, contradicting (14).

Note that $x_r z_r^{++} \notin E(G)$, we have $x_s z_r^+ \in E(G)$, otherwise $z, z_r^+ \notin AB(x_r, x_s)$, contradicting (14). With the similar argument as (18), we have $x_r z_r^{+++} \notin E(G)$. Using inductive method, we obtain $z_r \overrightarrow{C} v_{r+1} \subseteq N(x_s)$ and $z_r \overrightarrow{C} v_{r+1} \cap N(x_r) = \emptyset$, otherwise contradicting (14).

Also note that $y_s v_{r+1}^- \notin E(G)$ and $y_s \in X$ ($x_s \neq y_s$, otherwise we get an X-longer cycle than $C$), there exists $x_s^* \in X \cap x_s^+ \overrightarrow{C} y_s$ satisfying $z_r \overrightarrow{C} v_{r+1} \not\subseteq N(x_s^*)$. Choose that $x_s'$ is the first vertex in $X \cap x_s^+ \overrightarrow{C} y_s$ satisfying such condition. Moreover, we may assume that no vertex on $x_s^+ \overrightarrow{C} (x_s')^-$ is not adjacent to any vertex $x \in X \backslash V(C)$, otherwise we replace $x$ by $x'_s$ and also get a contradiction with the similar manner. For convenience, we also assume $x_s^+ \overrightarrow{C}(x'_s)^- \cap X = \emptyset$. Note that, in this case, if we replace $x_s$ by $x'_s$, then observations (2)-(18) still hold. Hence there exists exactly one vertex on $z_r \overrightarrow{C} v_{r+1}$ which is not a neighbor of $x'_s$, otherwise there exist $w, w' \in z_r \overrightarrow{C} v_{r+1}$ which are not neighbors of $x'_s$, then $w, w' \notin AB(x_r, x'_s)$, contradicting (14). So we may assume that $w$ is only one vertex on $z_r \overrightarrow{C} v_{r+1}$ which is not a neighbor of $x'_s$.

Suppose $zx'_s \in E(G)$, then $x_z y_s^+ \notin E(G)$, otherwise we get an X-longer cycle $C' = x_z y_s^+ \overrightarrow{C} v_r P_{r,r+1} \overrightarrow{C} x_s v_{r+1}^- \overleftarrow{C} z x'_s \overrightarrow{C} y_s x_r \overrightarrow{C} x_z$, a contradiction. If $y_s^+ x'_s \in E(G)$, then $z_r^-, z_r^+ \notin AB(x_r, y_s)$, contradicting (14). If $y_s^+ x'_s \notin E(G)$, then $y_s, w \notin AB(x_z, x'_s)$ (note $y_s x_z \notin E(G)$ in the case, otherwise the cycle $x'_s z \overrightarrow{C} v_{r+1}^- x_s \overleftarrow{C} v_{r+1} P_{r+1,s+1} v_{s+1} \overrightarrow{C} x_z y_s \overleftarrow{C} x'_s$ contradicts the choice of $C$), contradicting (14). Indeed $zx'_s \notin E(G)$. Hence $w, z \notin AB(x_r, x'_s)$, a contradiction.

**Case 2.2.** $z = z_r$

Case 2.2.1 $x_s v \notin E(G)$ for any $v \in y_s^+ \overrightarrow{C} v_{s+1}$

For any vertex $v \in y_s \overrightarrow{C} v_{s+1}^-$, we have $x_z v \notin E(G)$, otherwise we get an X-longer cycle than $C$. In particular $x_z y_s, x_z v_{s+1}^- \notin E(G)$. Hence, by (14), $y_s = v_{s+1}^-$. Moreover, we get

(19). $x_z y_s^- \notin E(G)$.

In the fact, suppose $x_z y_s^- \in E(G)$. If there exists $y' \in N(y_s) \backslash V(C)$, then $y' x_s \notin E(G)$, otherwise we get an X-longer cycle than $C$. Hence $y_s, y' \notin AB(x_z, x_s) \cup D(x_z, x_s)$, contradicting (14). This shows that $N(y_s) \subseteq V(C)$. So we get the cycle $C' = x_z y_s^- \overleftarrow{C} x_s z_r \overrightarrow{C} v_s P_{s,s+1} v_{s+1} \overrightarrow{C} x_z$ such that $|V(C') \cap X| > |V(C) \cap X|$ or $|V(C') \cap X| = |V(C) \cap X|$ with $|M(C')| < |M(C)|$, a contradiction. Indeed $x_z y_s^- \notin E(G)$.

Thus $y_s x_s \in E(G)$, otherwise $y_s, y_s^- \notin A(x_z, x_s)$. Without loss of generality, we may assume that $y_s^- \in X$ (otherwise we find an s-vertex respect to $y_s$, say $y'_s$, and substitute $y'_s$ for $y_s^-$), i.e., $y_s^-$ is an s-vertex respect to $y_s$.

It is clear to see $x_z z_r^+ \notin E(G)$. Moreover, we get

(20). $x_z z_r^{++} \notin E(G)$.

In the fact, suppose $x_z z_r^{++} \in E(G)$. If $N(z_r^+) \subseteq V(C)$, we get an cycle $C' = v_s \overleftarrow{C} z_r^{++} x_z \overleftarrow{C} x_r z_r x_s \overrightarrow{C} v_r P_{r,s} v_s$ such that $|V(C') \cap X| > |V(C) \cap X|$ or $|V(C') \cap X| = |V(C) \cap X|$ with $|M(C')| < |M(C)|$, a contradiction. If there exists $y \in N(z_r^+) \backslash V(C)$, then $yx_r, yx_s \notin E(G)$, otherwise we easily get an X-longer cycle than $C$. Then $y_s, y \notin AB(x_z, x_s)$, contradicting (14). Indeed, $x_z z_r^{++} \notin E(G)$.

Recalling that $x_z y_s \notin E(G)$, we conclude that $x_s z_r^+ \in E(G)$, since otherwise $z_r^+, y_s \notin A(x_z, x_s)$. Noting that $N(y_s) \subseteq V(C)$ (otherwise if there exists $y \in N(y_s) \backslash V(C)$, then $y, y_s \notin AB(x_z, x_r) \cup D(x_z, x_r)$), we get

$y_s^- z_r, y_s^- z_r^+ \notin E(G)$. It follows that $z_r, z_r^+ \notin AB(x_z, y_s^-)$, a contradiction.
Case 2.2.2 $x_s v \in E(G)$ for some $v \in y_s^+ \overrightarrow{C} v_{s+1}$

In the case, without loss of generality, we may assume $x_s v_{s+1} \in E(G)$, i.e., $y_s$ is an s-vertex respect to $v_{s+1}$. It is clear to see $x_z z_r^+ \notin E(G)$. Moreover, we get

(21). $y_s z_r \in E(G)$.

In the fact, suppose $y_s z_r \notin E(G)$. If there exists $z \in N(z_r^+) \cap R$, then we get $z \notin D(x_z, y_s)$, otherwise, we easily get an X-longer cycle than $C$, so $z, z_r \notin AB(x_z, y_s)$, contradicting (14). If $N(z_r^+) \cap R = \emptyset$, we get $z_r^+ \notin AB(x_z, y_s)$, also $x_z z_r^{++} \notin E(G)$, otherwise we get a cycle $C' = v_s \overleftarrow{C} z_r^{++} x_z \overleftarrow{C} x_r x_s \overrightarrow{C} v_r P_{r,s} v_s$ with $|V(C') \cap X| \geq |V(C) \cap X|$ and $|M(C')| < |M(C)|$, contradicting the choice of $C$. So $z_r, z_r^+ \notin AB(x_z, y_s)$, contradicting (14). Indeed $y_s z_r \in E(G)$.

Now, for any $v \in y_r \overrightarrow{C} v_{r+1}^-$, we have $x_z v \notin E(G)$, otherwise the cycle $x_z v \overleftarrow{C} z_r y_s \overleftarrow{C} v_{r+1} P_{r+1,s+1} v_{s+1} \overleftarrow{C} x_z$ is an X-longer than $C$. It is clear that $y_s v \notin E(G)$ for any $v \in y_r \overrightarrow{C} v_{r+1}^-$. Below, we will show

(22). $y_r^- = z_r^+$.

If there exists $z \in N(z_r^+) \cap R$, we have $y_r = v_{r+1}^-$, otherwise $z, y_r \notin AB(x_z, y_s) \cup D(x_z, y_z)$. So $N(y_r) \cap R = \emptyset$, otherwise if there exists $y' \in N(y_r) \cap R$, then $z, y' \notin D(x_z, y_s)$ (if $z = y'$, we have $y_r^- y_s \notin E(G)$, then $z, y_r^- \notin AB(x_z, y_s) \cup D(x_z, y_s)$), contradicting (14). Moreover, $y_r^- y_s \notin E(G)$, otherwise the cycle $y_s y_r^- \overleftarrow{C} v_{s+1} P_{s+1,r+1} \overleftarrow{C} y_s$ contradicting to the choice of $C$. Hence $z, y_r^- \notin AB(x_z, y_s)$, a contradiction. If $N(z_r^+) \cap R = \emptyset$, we have $x_z z_r^{++} \notin E(G)$. As the similar argument above, we get $y_r^- y_s \notin E(G)$. It follows that $z_r^+, y_r^- \notin AB(x_z, y_s)$. Hence, by (14), $y_r^- = z_r^+$.

We now show that

(23). $x_s$ connects to all vertices in $x_s^+ \overrightarrow{C} v_{s+1}$ by some paths whose internal vertices in $V(G) \backslash V(C)$.

Assuming the contrary, let $v$ be the vertex in $x_s^+ \overrightarrow{C} v_{s+1}$ such that $x_s$ is not connected to $v$ by a path whose internal vertices in $V(G) \backslash V(C)$ and $|v \overrightarrow{C} v_{s+1}|$ is minimum. Then $v \in x_s^+ \overrightarrow{C} v_{s+1}^-$ and $x_s v^+ \in E(G)$. We let $v^* = v^-$ if $N(v) \cap R = \emptyset$ and $v^*$ be an arbitrary vertex in $N(v) \backslash V(C)$, let $z = z_r^+$ if $N(z_r^+) \cap R = \emptyset$ and $z$ be an arbitrary vertex in $N(z_r^+) \backslash V(C)$. Hence, we easily obtain $v^*, z \notin AB(x_r, x_s)$, otherwise we get a cycle contradicting the choice of $C$ (note $y_s z_r \in E(G)$ by (21)). This contradiction proves (23).

Similarly we have

(24). $x_r$ connects to all vertices in $x_r^+ \overrightarrow{C} v_r$ by some paths whose internal vertices in $V(G) \backslash V(C)$.

By (21), we have $y_s z_r \in E(G)$, hence $x_z z_r^{++} \notin E(G)$, otherwise we get an X-longer cycle than $C$. If there exists $y \in N(y_r^-) \backslash V(C)$, then

$yy_s, y_r^- y_s \notin E(G)$, hence $y, y_r^- \notin AB(x_z, y_s) \cup A(x_z, y_s)$. This shows that $N(y_r^-) \subseteq V(C)$. Moreover, $x_s z_r^{++}(u_s y_r) \notin E(G)$by (16). Recalling that $x_s z_r^{++} \notin E(G)$, we now note that for all $i \in \{1, 2, \cdots, |A|\} \backslash \{r\}$ the assumption $x_i z_r^+ \in E(G)$ or $x_i z_r^{++} \in E(G)$ leads to a contradiction by applying the above arguments substitute $s$ for $i$. Thus $x_i z_r^+, x_i z_r^{++} \notin E(G)$ for all $i \in \{1, 2, \cdots, |A|\} \backslash \{r\}$. By (20), noting $y_s z_r \in E(G)$ and $x_r$ is r-vertex, we have $x_r z_r^+$, $x_r z_r^{++} \notin E(G)$. Hence $x_i z_r \in E(G)$ $(i \in \{1, 2, \cdots, |A|\} \backslash \{r\})$, for otherwise $z_r, z_r^+ \notin AB(x_r, x_i)$, contradicting (14). It now follows that (23) remains true if $s$ is replaced by $i$ $(i \in \{1, 2, \cdots, |A|\} \backslash \{r\})$. By (14), $AB(x_r, x_s) = V(C) \backslash (\{z_r\} \cup \{x_j | j \in \{1, 2, \cdots, |A|\} \backslash \{r, i\}$, implying that

$$N'(x_r) \cap V(C) \subseteq v_r^+ \overrightarrow{C} z_r \cup A$$

and

$$N'(x_i) \cap V(C) \subseteq v_i^+ \overrightarrow{C} v_{i+1} \cup A \cup \{z_r\} \quad (i \in \{1, 2, \cdots, |A|\} \backslash \{r\}.$$

Using the properties as above, we conclude that no edge, and similarly no path with all internal vertices in $V(G) \backslash V(C)$, joins two vertices in different sets of the collection $\{v_i^+ \overrightarrow{C} v_{i+1}^- | 1 \leq i \leq |A|, i \neq r\} \cup \{v_r^+ \overrightarrow{C} z_r^-\} \cup \{z_r^+ \overrightarrow{C} v_{r+1}^-\}$. But then $\omega(G \backslash (A \cup \{z_r\})) \geq |A \cup \{z_r\}| + 1$, our final contradiction.

□

**Proof of Lemma 7.** By assumption, $X \backslash V(C)$ is an independent set in $G[X]$ and a standard argument shows that $A^X$ is an independent set in $G[X]$ since $C$ is a $X$-longest cycle which is $X$-dominating. Hence it suffices to show that no vertex in $X \backslash V(C))$ is adjacent to any vertex in $A^X$.

Let $x_1, x_2, \cdots, x_{|A|}$ be the vertices of $A^X$, occuring on $\overrightarrow{C}$. Suppose that exists $x' \in X \backslash V(C)$ such that $x' x_i \in E(G)$ for some $i \in \{1, 2, \cdots, |A|\}$. Clearly $x' \neq x_0$ since $C$ is a $X$-longest cycle. Without loss of generality, we assume that $i = 1$, i.e. $x' x_1 \in E(G)$. We consider following sets of vertices.

$$\begin{aligned}
A_1 &= \{v \in x_1 \overrightarrow{C} x_{|A|}^- \mid x' v^+ \in E(G)\} \\
A_2 &= \{v \in x_1 \overrightarrow{C} x_{|A|}^- \mid x_{|A|} v \in E(G)\} \\
B_1 &= \{v \in x_{|A|} \overrightarrow{C} x_1^- \mid x' v \in E(G)\} \\
B_2 &= \{v \in x_{|A|} \overrightarrow{C} x_1^- \mid x_{|A|} v^+ \in E(G)\} \\
D &= \{v \in V(G) \backslash V(C) \mid x_{|A|} v \in E(G)\}
\end{aligned}$$

Then we get $A_1 \cap A_2 = \emptyset$, otherwise, if $v \in A_1 \cap A_2$, we can obtain a $X$-longer cycle $C'$ than $C$ as follows:

$$v_{|A|} x_0 v_1 \overleftarrow{C} x_{|A|} v \overleftarrow{C} v^+ x' x_1 \overrightarrow{C} v_{|A|} \qquad (\text{if} \quad v \in v_{|A|} \overrightarrow{C} x_{|A|}^-)$$

or

$$v_{|A|} x_0 v_1 \overleftarrow{C} x_{|A|} v \overleftarrow{C} x_1 x' v^+ \overrightarrow{C} v_{|A|} \qquad (\text{if} \quad v \in x_1 \overrightarrow{C} v_{|A|}^-)$$

and we get $B_1 \cap B_2 = \emptyset$, otherwise, if $v \in B_1 \cap B_2$, we can obtain a $X$-longer cycle $C'$ than $C$ as follows:

$$v_{|A|}x_0v_1\overleftarrow{C}vx'x_1\overrightarrow{C}v_{|A|} \qquad (\text{if } v = x_{|A|})$$

or

$$v_{|A|}x_0v_1\overleftarrow{C}v^+x_{|A|}\overrightarrow{C}vx'x_1\overrightarrow{C}v_{|A|} \qquad (\text{if } v \in x_{|A|}^+\overrightarrow{C}v_1^-)$$

or

$$v_{|A|}x_0v_1\overleftarrow{C}x_{|A|}v^+\overrightarrow{C}v_{|A|} \qquad (\text{if } v \in v_1\overrightarrow{C}x_1^-)$$

Since $N(x') = A_1^+ \cup B_1 \cup \{x_1\}$, then $d(x') = |A_1| + |B_1| + 1$. Similarly, we have $d(x_{|A|}) = |D| + |A_2| + |B_2|$. Moreover, we obtain that

$$\begin{aligned} d(x_0) + d(x') + d(x_{|A|}) &= |A| + |A_1| + |B_1| + 1 + |D| + |A_2| + |B_2| \\ &= |A| + |A_1 \cup A_2| + |B_1 \cup B_2| + |D| + 1 \end{aligned}$$

Clearly, we can obtain the fact that $x_i \notin A_1$ for each $i \in \{1, 2, \cdots, |A|\}$, otherwise we can find a $X$-longer cycle $C''$ than $C$. Since $x_1, x_2, \cdots, x_{|A|}$ are independent, then $x_i \notin A_2$ for each $i \in \{1, 2, \cdots, |A| - 1\}$. It is easy to see that $v_1 \notin B_1 \cup B_2$, otherwise we can find a $X$-longer cycle $C''$ than $C$. Thus, noting that $x, x', x_{|A|} \in X$, we obtain the following

$$\begin{aligned} n \le \sigma_3(X) &\le d(x_0) + d(x') + d(x_{|A|}) \\ &\le |A| + (|V(C)| - (|A| - 1) - 1) + (n - (|V(C)| - 2) + 1) \\ &\le n - 1 \end{aligned}$$

a contradiction. Thus we prove the lemma.

□

**Outline Proof of Theorem 8.** Let $C$ be an $X-$longest cycle of $G$ which is X-dominating. Assume $C$ is chosen such that $max\{d(v)|v \in X\backslash V(C)\}$ is maximum. If $X\backslash V(C) = \emptyset$, there is nothing to prove. Thus we assume $X\backslash V(C) = \{u_0, u_1, \cdots, u_t\}$, such that $d(u_0) \ge d(u_1) \ge \cdots \ge d(u_t)$. Let $A = N(u_0)$ and $v_1, v_2, \cdots, v_{|A|}$ be the vertices of $A$ and $x_1, x_2, \cdots, x_{|A|}$ be the vertices of $A^X$, occuring on $\overrightarrow{C}$ and, without loss of generality, we assume that $d(x_1) = min\{d(x_i)|1 \le i \le |A|\}$. From Lemma 7, we have $|X\backslash V(C)| + |A^X| \le \alpha(X)$. Hence $|X \cap V(C)| \ge |X| + |A^X| - \alpha(X) = |X| + d(u_0) - \alpha(X)$. Thus it suffices to show that $d(u_0) \ge \frac{1}{3}\sigma_3(X)$. It is clearly true if $t \ge 2$.

Suppose $t = 1$, $d(u_0) < \frac{1}{3}\sigma_3(X)$ and consider $x_1$. Set $R = V(G)\backslash V(C)$. For $j \in \{2, 3, \cdots, |A|\}$, set $u_{j1} = v_j^+$, and let $u_{j2} = u_{j1}^+$ if $N(u_{j1}) \cap R = \emptyset$, otherwise let $u_{j2}$ be the an arbitrary vertex in $N(u_{j1}) \cap R$.

Suppose $x_1u_{j2} \in E(G)$ for some $j \in \{2, 3, \cdots, |A|\}$, then the cycle $C' = u_0v_j\overleftarrow{C}x_1u_{j2}\overrightarrow{C}v_1u_0$ is $X-$longer than $C$, unless $u_{j1} \in X$ and $N(u_{j1}) \cap R = \emptyset$. Moreover, if $u_{j1} \in X$ and $N(u_{j1}) \cap R = \emptyset$, the cycle $C'$ is $X-$dominating and has $|V(C'') \cap X| = |V(C) \cap X|$ but includes $u_0$ and omits $u_{j1}$. However, $u_0, u_1, u_{j1}$ are independent and belongs to $X$, thus $d(u_0) + d(u_1) + d(u_{j1}) \ge$

$\sigma_3(X)$. This implies $d(u_{j1}) > \frac{1}{3}\sigma_3(X) > d(u_0)$, contradicting the choice of $C$. Indeed, $x_1u_{j2} \notin E(G)$ for any $j \in \{2,3,\cdots,|A|\}$.

Clearly, we have that $x_1u_{j1} \notin E(G)$ for any $j \in \{2,3,\cdots,|A|\}$. We denote $s = |\{j|u_{j2} \notin V(C), j \in \{2,3,\cdots,|A|\}|$. We note that, any distinct two vertices of $\{x_1, u_{21}, u_{22}, \cdots, u_{|A|1}\}$ have no neighbors in $V(G)\backslash V(C)$, $u_0$, $u_1 \notin N(x_1)$, $u_{j2} \neq u_0, u_1$ ($j \in \{2,3,\cdots,|A|\}$) and $u_{j2} \notin N(x_1)$. Since $N(x_1) \cap V(C) \subseteq V(C)\backslash(\{x_1\} \cup \{u_{j1}, u_{j2}|j \in \{2,3,\cdots,|A|\}\})$, hence we obtain th follows

$$d(x_1) \leq |V(C)| - (d(u_0) + d(u_0) - 1 - s) + |N(x_1)\backslash V(C)|$$

But then $n \leq d(x_1) + d(u_0) + d(u_1) \leq d(x_1) + 2d(u_0) \leq |V(C)| + s + |N(x_1)\backslash V(C)| + 1 \leq n - 2 + 1 = n - 1$, a contradiction.

Finally, the proof for $t = 0$ is modeled along the lines of the proof of Theorem 5. Whenever a contradiction is obtained in the proof of Theorem 5 by finding an $X$−longest cycle, we now find a contradiction either in the same way, or by finding an X-dominating cycle $C'$ such that $|V(C') \cap X| = |X| - 1$ and $v_0 \in X\backslash V(C')$ has $d(v_0) > d(u_0)$. The argument, although quite lengthy is involved, is tedious and is thus omitted here. The full proof can be found in the appendix of [9].

Thus we prove the theorem.

□


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