A characterization of halved cubes

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Abstract

The vertex set of a halved cube Q'_d consists of a bipartition vertex set of a cube Q_d and two vertices are adjacent if they have a common neighbour in the cube. Let $d \geq 5$. Then it is proved that Q'_d is the only connected, $\binom{d}{2}$-regular graph on 2^{d-1} vertices in which every edge lies in two d-cliques and two d-cliques do not intersect in a vertex.

1 Introduction

Let G be a bipartite graph with bipartition $V(G) = X \cup Y$. A *halved graph* G' of G is defined as follows. $V(G') = X$ and $uv \in E(G')$ whenever u and v have a common neighbour in G. G has another halved graph with vertex set Y. When we consider the d-cube Q_d both halved graphs are isomorphic and we talk about the *halved d-cube* Q'_d.

Partial Hamming graphs are exactly those graphs which can be isometrically embedded into a Cartesian product of complete graphs, cf. [9]. We refer also to [2, 8] where these graphs are called Hamming graphs. In case every one of the factors is the complete graph K_2 on two vertices one obtains an isometric embedding into a hypercube and speaks of a partial

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binary Hamming graph. By a scale embedding of a graph G into a graph H we mean a mapping

$$\psi : V(G) \to V(H)$$

for which there exists a positive integer λ such that

$$d_H(\psi(u), \psi(v)) = \lambda d_G(u, v)$$

for all $u, v \in V(G)$, where d_H and d_G denote the usual path distance in G and H, respectively. If one relaxes the condition of isometry and considers so-called scale embeddings into hypercubes a class larger than that of partial Hamming graphs arises. It has been characterized by Assouad and Deza [1] as the class of graphs isometrically embeddable into the metric space ℓ_1. These graphs have in turn been characterized by Deza and Grishukhin [3] and Shpectorov [14] as isometric subgraphs of Cartesian products of complete graphs, cocktail party graphs and halved cubes.

It was this recent study of ℓ_1-graphs that motivated us to consider halved cubes. As it is clear from the above, halved cubes play an important role in the characterization of ℓ_1-graphs. In fact, without going into details, by a result of Graham and Winkler [6] about so-called canonical isometric embeddings of graphs into Cartesian products together with an algorithm of Feder [5], a good algorithm for recognizing isometric subgraphs of halved cubes would suffice for a good algorithm for recognizing ℓ_1-graphs. An $O(mn)$ algorithm for recognizing isometric subgraphs of halved cubes and thus of ℓ_1-graphs was recently obtained by Deza and Shpectorov, [4]. Here n denotes the number of vertices and m the number of edges of a given graph. We also wish to recall that Aurenhammer, Formann, Idury, Schäffer and Wagner [2] and Imrich and Klavžar [8] proved that it can be decided in $O(mn)$ time whether a given graph is a partial Hamming graph.

As usual, for a vertex $u \in V(G)$ let $N(u) = \{v;\ uv \in E(G)\}$. A clique is a maximal complete subgraph. If Q is a clique we will also use Q to denote its vertex set. A clique on d vertices will be called a *d-clique*. The cocktail party graph on $2n$ vertices is the complete graph K_{2n} minus a complete matching.

In this note we first study the structure of halved cubes and then give a characterization of these graphs. A halved cube on 2^{d-1} vertices is the only connected, $\binom{d}{2}$-regular graph in which every edge lies in two d-cliques and two d-cliques do not intersect in a single vertex.

2 The characterization

We will first summarize several properties of halved cubes. Then we will prove that some of these properties already imply that a given graph is a halved cube thus obtaining the desired characterization.

The vertex set of the d-cube Q_d may be represented by all sequences of length d over $\{0, 1\}$ where two vertices are adjacent if they differ in exactly one position. We may henceforth consider vertices of the ***halved d-cube*** Q'_d as sequences of length d over $\{0, 1\}$. In the sequel we will, without loss of generality, assume that a vertex of Q'_d is such a sequence with an even number of 1's. In particular, $(0, 0, \ldots, 0) \in Q'_d$. Then two vertices of Q'_d are adjacent if and only if they differ in two positions.

Clearly, Q'_d has 2^{d-1} vertices. In addition, from the coordinate representation of Q'_d it follows immediately that Q'_d is a $\binom{d}{2}$-regular graph. (We also recall that halved cubes are distance-regular graphs, cf. [7].)

Note that Q'_3 is isomorphic to the complete graph K_4 on four vertices and that Q'_4 is isomorphic to the cocktail party graph on 8 vertices. To simplify the presentation we may henceforth assume that $d \geq 5$.

Proposition 1 *(i) There are only two types of cliques of Q'_d, namely 4-cliques and d-cliques.*

(ii) Every vertex of Q'_d lies in d d-cliques.

(iii) Q'_d has 2^{d-1} d-cliques.

Proof. (i) We include the proof of (i) for the sake of completeness although it can be found in [7].

Let u, v and w be distinct vertices of a clique Q of Q'_d. We may, without loss of generality, assume that $u = (0, 0, 0, 0, \ldots)$, $v = (1, 1, 0, 0, \ldots)$, and $w = (1, 0, 1, 0, \ldots)$, where all three vertices agree in the remaining coordinates.

Let z be another vertex of Q. It must have exactly one 1 in its first two coordinates for otherwise it would not be adjacent to at least one of u and v.

If $z = (0, 1, \ldots)$, it must agree with w in coordinates $3, 4, \ldots, d$ and there is only one such vertex. Clearly the vertices u, v, w and z induce a clique.

If $z = (1, 0, \ldots)$ it must be of the form $(1, 0, 0, \ldots, 1, \ldots)$. Clearly these $d-3$ vertices, together with u, v and w form a d-clique.

(ii) By the argument from (i), the d-cliques of Q'_d are induced by the neighborhoods of vertices of Q_d with an odd number of 1's. Now, since every vertex of Q'_d is in d such neighborhoods, it is contained in precisely d such cliques.

(iii) This follows by the same argument as (ii). $\square$

We next give properties of halved cubes with respect to a given edge.

Proposition 2 *Let uv be an edge of Q'_d. Then*

(i) $|N(u) \cap N(v)| = 2(d-2)$.

(ii) uv belongs to precisely two d-cliques of Q'_d, say Q and Q'.
(iii) $Q \cap Q' = \{u, v\}$.
(iv) $Q - \{u, v\}$ and $Q' - \{u, v\}$ are joined by a matching.

Proof. We may without loss of generality assume $u = (0, 0, 0, 0, \ldots, 0)$ and $v = (1, 1, 0, 0, \ldots, 0)$. Let w be a vertex adjacent to both u and v. Then w starts out $(1, 0, \ldots)$ or $(0, 1, \ldots)$ and it has exactly one 1 in the remaining $d-2$ coordinates. Thus there are $2(d-2)$ vertices in $N(u) \cap N(v)$. Furthermore, the vertex sets

$$\{u, v, (1,0,1,0,\ldots,0), (1,0,0,1,\ldots,0), \ldots, (1,0,0,0,\ldots,1)\}$$

and

$$\{u, v, (0,1,1,0,\ldots,0), (0,1,0,1,\ldots,0), \ldots, (0,1,0,0,\ldots,1)\}$$

induce the two cliques containing uv. All the rest now easily follows. □

A connected graph G is a $(0,2)$-*graph* if any two distinct vertices in G have exactly two common neighbors or none at all, cf. [12, 13]. Note that in bipartite graphs this condition applies only to pairs of vertices at distance two.

We will need the following result due to Mulder [13, page 55], cf. also [11].

Theorem 3 *Let G be a d-regular $(0,2)$-graph. Then $|V(G)| = 2^d$ if and only if G is Q_d.*

We are ready now to characterize halved cubes.

Theorem 4 *Let $d \geq 5$. Let G be a connected, $\binom{d}{2}$-regular graph on 2^{d-1} vertices. Then G is the halved cube Q'_d if and only if*

(i) every edge of G is contained in exactly two d-cliques,
(ii) for any d-cliques Q and Q', $|Q \cap Q'| \neq 1$.

Proof. If G is a halved cube then Proposition 2 yields (i) and (ii). Conversely, suppose that (i) and (ii) hold. Since G is a $\binom{d}{2}$-regular graph on 2^{d-1} vertices, $|E(G)| = d(d-1)2^{d-3}$. Thus, because of (i), there are $\dfrac{2|E(G)|}{\binom{d}{2}} = 2^{d-1}$ d-cliques of G. In addition, since G is $\binom{d}{2}$-regular and every edge is in two d-cliques, every vertex of G belongs to $\dfrac{2\binom{d}{2}}{d-1} = d$ d-cliques.

Let Q and Q' be d-cliques of G with $|Q \cap Q'| = s$ for $s \geq 1$. Then by (ii), $s \geq 2$. Let $u \in Q \cap Q'$ and let $Q, Q', Q_1, Q_2, \ldots, Q_{d-2}$ be the d-cliques containing u. Note first that for any i, $Q_i \cap (Q \cap Q') = \{u\}$, for otherwise an edge of this intersection would belong to at least three d-cliques. Thus by (ii), Q_i must intersect $Q \setminus Q'$ for $i = 1, 2, \ldots, d-2$. Furthermore, if for $w \in Q \setminus Q'$ we have $w \in Q_i \cap Q_j$, $i \neq j$, then the edge uw would not satisfy (i). If follows that $d - s \geq d - 2$, thus $s = 2$. Hence if $Q \cap Q' \neq \emptyset$ then $|Q \cap Q'| = 2$.

Let $n = 2^{d-1}$ and denote the vertices of G by $V(G) = \{u_1, u_2, \ldots, u_n\}$. Let H be a graph which we get from G in the following way. To every d-clique Q of G we add a new vertex and join it to every vertex of Q. These are the newly defined edges of H. The original edges of G are all removed. Note that H is bipartite. Since G contains n d-cliques we may write $V(H) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$. By construction, $d_H(v_i) = d$, for every $i = 1, 2, \ldots, n$, and since every u_i is in d d-cliques, we conclude that H is d-regular.

We claim that H is a $(0,2)$-graph. H is connected because G is connected and every edge of G lies in a d-clique. Let $d_H(u_i, u_j) = 2$ and let v_k be a common neighbor of u_i and u_j. Then $u_i u_j$ must be an edge of G and since it is contained in two d-cliques, there is another common neighbor of u_i and u_j, say v_ℓ. Furthermore, v_k and v_ℓ are their only common neighbors for otherwise $u_i u_j$ would lie in more that two d-cliques of G. Now, let u_k be a common neighbor of vertices v_i and v_j and let Q_i and Q_j be the cliques of G corresponding to v_i and v_j. Since $u_k \in Q_i \cap Q_j$ we have $|Q_i \cap Q_j| = 2$. But this means that v_i and v_j have precisely two common neighbors and the claim is proved.

We have seen that H is a d-regular $(0,2)$-graph on 2^d vertices. Thus H is Q_d by Theorem 3. To complete the proof we are going to show that G is the halved graph of H. More precisely, we need to show that $u_i u_j \in E(G)$ if and only if $d_H(u_i, u_j) = 2$. Let $u_i u_j \in E(G)$. Then $u_i u_j$ belongs to a d-clique Q and by construction there is a vertex of H adjacent to every vertex of Q. In particular, $d_H(u_i, u_j) = 2$. Conversely, let $d_H(u_i, u_j) = 2$. Because in H all the edges of G are removed there is a vertex v_k (not in G) such that $u_i v_k \in E(H)$ and $v_k u_j \in E(H)$. But this implies that u_i and u_j belong to a common clique of G, hence $u_i u_j \in E(G)$. □

We note that condition (ii) of Theorem 4 can be replaced by the following equivalent condition:

(ii') for any d-cliques Q and Q', $|Q \cap Q'| \leq 2$.

In the proof of Theorem 4 we have shown that (ii) implies (ii'). Suppose now that (ii') holds and assume that $|Q \cap Q'| = 1$ for d-cliques Q and Q'. Let $u \in Q \cap Q'$. Let $V(Q) = \{u, w_1, w_2, \ldots, w_{d-1}\}$. Clearly, $uw_i \in Q$ for $i = 1, 2, \ldots, d-1$. Let $Q_i \neq Q$ be the second d-clique containing uw_i, $i = 1, 2, \ldots, d-1$. Then $Q_i \neq Q'$. Furthermore, if $i \neq j$ then $Q_i \neq Q_j$, for otherwise $|Q_i \cap Q| \geq 3$. It follows that u is contained in at least $d+1$ d-cliques, a contradiction.

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