THE SPECTRUM OF d–CYCLIC ORIENTED TRIPLE SYSTEMS

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Abstract. An oriented triple systems of order v, denoted $OTS(v)$, is said to be *d–cyclic* if it admits an automorphism consisting of a single cycle of length d and $v-d$ fixed points, $d \geq 2$. In this paper we give necessary and sufficient conditions for the existence of d–cyclic $OTS(v)$. We solve the analogous problem for directed triple systems.

1. Introduction

A *cyclic triple*, denoted $C(a,b,c)$, is the digraph on the vertex set $\{a,b,c\}$ with the arc set $\{(a,b),(b,c),(c,a)\}$. Notice that $C(a,b,c)=C(b,c,a)=C(c,a,b)$. A *transitive triple*, denoted $T(a,b,c)$, is the digraph on the vertex set $\{a,b,c\}$ with the arc set $\{(a,b),(b,c),(a,c)\}$. An *oriented triple* means either a cyclic or a transitive triple. Let D_v denote the complete symmetric digraph on v vertices.

An *oriented triple system* (also called an *ordered triple system*) of order v, denoted $OTS(v)$, is an arc–disjoint partition of D_v into oriented triples. An $OTS(v)$ exists if and only if $v \equiv 0$ or 1 (mod 3) [9]. An $OTS(v)$ in which the oriented triples are all cyclic triples is a *Mendelsohn triple system* of order v, denoted $MTS(v)$. A $MTS(v)$ exists if and only if $v \equiv 0$ or 1 (mod 3), $v \neq 6$ [10]. An $OTS(v)$ in which the oriented triples are all transitive triples is a *directed triple system* of order v, denoted $DTS(v)$. A $DTS(v)$ exists if and only if $v \equiv 0$ or 1 (mod 3) [8].

An automorphism of an $OTS(v)$ based on D_v is a permutation π of the vertex set of D_v which fixes the collection of triples of the $OTS(v)$. The *orbit* of a triple under an automorphism π is the collection of images of the triple under the powers of π.

An $OTS(v)$ admitting an automorphism consisting of a single cycle is said to be *cyclic*. A cyclic $OTS(v)$ exists if and only if $v \equiv 0,1,3,4,7$ or 9 (mod 12),

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$v \neq 9$ [11]. A cyclic $MTS(v)$ exists if and only if $v \equiv 1$ or 3 (mod 6), $v \neq 9$ [4]. A cyclic $DTS(v)$ exists if and only if $v \equiv 1,4$ or 7 (mod 12) [5].

An $OTS(v)$ admitting an automorphism consisting of a fixed point and a cycle of length $v-1$ is said to be *rotational*. A rotational $OTS(v)$ exists if and only if $v \equiv 0$ or 1 (mod 3) [11]. A rotational $MTS(v)$ exists if and only if $v \equiv 1,3$ or 4 (mod 6), $v \neq 10$ [2]. A rotational $DTS(v)$ exists if and only if $v \equiv 0$ (mod 3) [3].

An $OTS(v)$ admitting an automorphism consisting of a single cycle of length d and $v-d$ fixed points, $d \geq 2$, will be called *d-cyclic*. Obviously, a v-cyclic $OTS(v)$ is a cyclic $OTS(v)$ and a $(v-1)$-cyclic $OTS(v)$ is a rotational $OTS(v)$. In [7] Hoffman determined those pairs (v,d) of integers for which there exists a d-cyclic $MTS(v)$ [1].

The purpose of this note is to present necessary and sufficient conditions for the existence of d-cyclic $DTS(v)$s and d-cyclic $OTS(v)$s.

2. Necessary conditions for the existence of d-cyclic $OTS(v)$s

In this section, we give necessary conditions for the existence of d-cyclic $OTS(v)$s and d-cyclic $DTS(v)$s.

Lemma 2.1. *The fixed points of an automorphism of an $OTS(v)$ form a subsystem.*

Proof. Let α_i, α_j be two fixed points under the automorphism π. The ordered pair (α_i, α_j) occurs in exactly one triple of the $OTS(v)$, say t. So t is fixed under π and the fixed points form a subsystem. ❑

From Lemma 2.1 it follows that if a d-cyclic $OTS(v)$ exists, then $f=v-d \equiv 0$ or 1 (mod 3). Further, since in a d-cyclic $OTS(v)$ the automorphism π has a cycle of length $v-f$, using the standard idea of difference methods we have $v-f-1 \geq f$, therefore $v \geq 2f+1$.

These facts give us:

Lemma 2.2. *If a d-cyclic $OTS(v)$ exists, then $v \equiv 0$ or 1 (mod 3), $f \equiv 0$ or 1 (mod 3) and $v \geq 2f+1$, where $f=v-d$.*

We now consider a d-cyclic $DTS(v)$. In this case we have $w=v-2f-1 \equiv 0$ (mod 3). But if $v \equiv 0$ (mod 3) and $f \equiv 0$ (mod 3), then $w \equiv 2$ (mod 3), and if $v \equiv 1$ (mod 3) and $f \equiv 1$ (mod 3), then $w \equiv 1$ (mod 3). It follows that:

Lemma 2.3. *If a d-cyclic $DTS(v)$ exists, then $v \geq 2f+1$ and, further, $v \equiv 0$ (mod 3) and $f \equiv 1$ (mod 3) or $v \equiv 1$ (mod 3) and $f \equiv 0$ (mod 3), where $f=v-d$.*

3. Constructions of d-cyclic $DTS(v)$s.

In this section, we construct d-cyclic $DTS(v)$s with vertex set $Z_d \cup \{\alpha_1, \alpha_2, \dots, \alpha_f\}$, where $\alpha_1, \alpha_2, \dots, \alpha_f$ are the fixed points of the

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[1] We notice that there does not exist a Steiner triple systems of order v with an automorphism consisting of a simple cycle of length ≥ 2 and more than one fixed point [1].

automorphism π and $f=v-d\geq 2$.

We require the use of two structures. An *(A,n)–system* is a collection of ordered pairs (a_r,b_r) for $r=1,2,\dots,n$ that partition the set $\{1,2,\dots,2n\}$ with the property that $b_r=a_r+r$ for $r=1,2,\dots,n$. It is proved that an (A,n)–system exists if and only if $n\equiv 0$ or 1 (mod 4) [13, 6]. A *(B,n)–system* is a collection of ordered pairs (a_r,b_r) for $r=1,2,\dots,n$ that partition the set $\{1,2,\dots,2n-1,2n+1\}$ with the property that $b_r=a_r+r$ for $r=1,2,\dots,n$. It is proved that a (B,n)–system exists if and only if $n\equiv 2$ or 3 (mod 4) [12, 6].

Theorem 3.1. *A d–cyclic DTS(v), with $f=v-d\geq 2$, exists if and only if $v\geq 2f+1$ and, further, $v\equiv 0$ (mod 3) and $f\equiv 1$ (mod 3) or $v\equiv 1$ (mod 3) and $f\equiv 0$ (mod 3).*

Proof. From Lemma 2.3 the conditions are necessary. We now prove that the conditions are also sufficient. Let $w=v-2f-1$. Since $v\equiv 0$ (mod 3) and $f\equiv 1$ (mod 3) or $v\equiv 1$ (mod 3) and $f\equiv 0$ (mod 3), we have $w\equiv 0$ (mod 3). We consider four cases for w.

Case 1: $w\equiv 0$ (mod 12), say $w=12t$.

Let $\Gamma=\{T(0,r,b_r+4t)$: $r=1,2,\dots,4t$ and b_r are from an $(A,4t)$–system (omit these triples if t=0)$\}\cup\{T(0,\alpha_i,w+i)$: $i=1,2,\dots,f\}$.

Case 2: $w\equiv 3$ (mod 12), say $w=12t+3$.

Let $\Gamma=\{T(0,r,b_r+4t+1)$: $r=1,2,\dots,4t+1$ and b_r are from an $(A,4t+1)$–system$\}\cup$ $\cup\{T(0,\alpha_i,w+i)$: $i=1,2,\dots,f\}$.

Case 3: $w\equiv 6$ (mod 12), say $w=12t+6$.

Let $\Gamma=\{T(0,r,b_r+4t+2)$: $r=1,2,\dots,4t+2$ and b_r are from a $(B,4t+2)$–system$\}\cup$ $\cup\{T(0,\alpha_1,w)\}\cup\{T(0,\alpha_i,w+i)$: $i=2,3,\dots,f\}$.

Case 4: $w\equiv 9$ (mod 12), say $w=12t+9$.

Let $\Gamma=\{T(0,r,b_r+4t+3)$: $r=1,2,\dots,4t+3$ and b_r are from a $(B,4t+3)$–system$\}\cup$ $\cup\{T(0,\alpha_1,w)\}\cup\{T(0,\alpha_i,w+i)$: $i=2,3,\dots,f\}$.

In all cases, the union of orbits of all triples of Γ and of the set of triples of a $DTS(f)$ on $\{\alpha_1,\alpha_2,\dots,\alpha_f\}$ form a d–cyclic $DTS(v)$. ❑

4. Constructions of *d*–cyclic *OTS*(*v*)s.

In this section, we determine the spectrum of d–cyclic $OTS(v)$s.

A d–cyclic $MTS(v)$, when $f=v-d\geq 2$, exists if and only if $v\geq 2f+1$, $v\equiv 0$ or 1 (mod 3), $f\equiv 0$ or 1 (mod 3), $v\neq 6$, $f\neq 6$ and $(v,f)\neq(12,3)$ [7].

From this and Theorem 3.1 it follows that:

Lemma 4.1. A *d–cyclic OTS(v), with $f=v-d\geq 2$, exists if $v\geq 2f+1$, $v\equiv 0$ or 1 (mod 3), $f\equiv 0$ or 1 (mod 3), $(v,f)\neq(12,3)$ and, for any integer $k\geq 5$, $(v,f)\neq(3k,6)$.*

In the following lemmas we construct d–cyclic $OTS(v)$s with vertex set $Z_d\cup\{\alpha_1,\alpha_2,\dots,\alpha_f\}$, where $\alpha_1,\alpha_2,\dots,\alpha_f$ are the fixed points of the automorphism π and $f=v-d\geq 2$.

Lemma 4.2. *A 9–cyclic OTS(12) exists.*

Proof. Let $\Gamma=\{C(0,3,6), C(0,6,3), T(0,1,5), C(\alpha_1,0,2), C(\alpha_2,0,7), C(\alpha_3,0,8)\}$. The union of orbits of all triples of Γ and the set of triples of an $OTS(3)$ on $\{\alpha_1,\alpha_2,\alpha_3\}$ form a 9–cyclic $OTS(12)$. ❑

Lemma 4.3. *For every $v\equiv0$ (mod 3), $v\geq15$, a $(v-6)$–cyclic $OTS(v)$ exists.*

Proof. Let $v\equiv0$ (mod 3), $v\geq15$, and $w=v-7$; we have $w\equiv2$ (mod 3), $w\geq8$. We consider four cases for w.

Case 1: $w\equiv2$ (mod 12), say $w=12t+2$.

Let $\Gamma=\{T(0,r,b_r+4t)$: $r=1,2,\dots,4t$, and b_r are from an $(A,4t)$–system, with $r\neq4t-3$, $4t-4$, and therefore $b_r\neq4t-2$, $8t-2$ respectively$\}$ $\cup$ $\{C(0,4t+1,8t+2)$, $C(0,8t+2,4t+1)$, $C(\alpha_1,0,4t-3)$, $C(\alpha_2,0,8t-2)$, $C(\alpha_3,0,4t-4)$, $C(\alpha_4,0,12t-2)$, $C(\alpha_5,0,12t+1)$, $C(\alpha_6,0,12t+2)\}$ if $t>1$, and $\Gamma=\{C(0,5,10)$, $C(0,10,5)$, $C(\alpha_1,0,1)$, $C(\alpha_2,0,6)$, $C(\alpha_3,0,3)$, $C(\alpha_4,0,7)$, $C(\alpha_5,0,13)$, $C(\alpha_6,0,14)$, $T(0,2,11)$, $T(0,4,12)\}$ if $t=1$.

Case 2: $w\equiv5$ (mod 12), say $w=12t+5$.

Let $\Gamma=\{T(0,r,b_r+4t+1)$: $r=1,2,\dots,4t+1$ and b_r are from an $(A,4t+1)$–system, with $r\neq4t-1$, $4t-2$, and therefore $b_r\neq4t$, $8t+1$ respectively$\}$ $\cup$ $\{C(0,4t+2,8t+4)$, $C(0,8t+4,4t+2)$, $C(\alpha_1,0,4t-1)$, $C(\alpha_2,0,8t+1)$, $C(\alpha_3,0,4t-2)$, $C(\alpha_4,0,12t+2)$, $C(\alpha_5,0,12t+4)$, $C(\alpha_6,0,12t+5)\}$.

Case 3: $w\equiv8$ (mod 12), say $w=12t+8$.

Let $\Gamma=\{T(0,r,b_r+4t+2)$: $r=1,2,\dots,4t+2$, and b_r are from a $(B,4t+2)$–system, with $r\neq4t$, $4t-1$, and therefore $b_r\neq4t+1$, $8t+3$ respectively$\}$ $\cup$ $\{C(0,4t+3,8t+6)$, $C(0,8t+6,4t+3)$, $C(\alpha_1,0,4t)$, $C(\alpha_2,0,8t+3)$, $C(\alpha_3,0,4t-1)$, $C(\alpha_4,0,12t+5)$, $C(\alpha_5,0,12t+6)$, $C(\alpha_6,0,12t+8)\}$ if $t>1$; $\Gamma=\{T(0,1,9)$, $T(0,4,15)$, $T(0,5,17)$, $T(0,6,19)$, $C(0,7,14)$, $C(0,14,7)$, $C(\alpha_1,0,2)$, $C(\alpha_2,0,3)$, $C(\alpha_3,0,10)$, $C(\alpha_4,0,16)$, $C(\alpha_5,0,18)$, $C(\alpha_6,0,20)\}$ if $t=1$; $\Gamma=\{C(0,3,6)$, $C(0,6,3)$, $C(\alpha_1,0,1)$, $C(\alpha_2,0,4)$, $C(\alpha_3,0,2)$, $C(\alpha_4,0,7)$, $C(\alpha_5,0,5)$, $C(\alpha_6,0,8)\}$ if $t=0$.

Case 4: $w\equiv11$ (mod 12), say $w=12t-1$.

Let $\Gamma=\{T(0,r,b_r+4t-1)$: $r=1,2,\dots,4t-1$, and b_r are from a $(B,4t-1)$–system, with $r\neq4t-3$, $4t-4$, and therefore $b_r\neq4t-2$, $8t-3$ respectively$\}$ $\cup$ $\{C(0,4t,8t)$, $C(0,8t,4t)$, $C(\alpha_1,0,4t-3)$, $C(\alpha_2,0,8t-3)$, $C(\alpha_3,0,4t-4)$, $C(\alpha_4,0,12t-4)$, $C(\alpha_5,0,12t-3)$, $C(\alpha_6,0,12t-1)\}$ if $t>1$, and $\Gamma=\{C(0,4,8)$, $C(0,8,4)$, $C(\alpha_1,0,1)$, $C(\alpha_2,0,5)$, $C(\alpha_3,0,2)$, $C(\alpha_4,0,6)$, $C(\alpha_5,0,9)$, $C(\alpha_6,0,11)$, $T(0,3,10)\}$ if $t=1$.

In all cases, the union of orbits of all triples of Γ and the set of triples of an $OTS(6)$ on $\{\alpha_1,\alpha_2,\dots,\alpha_6\}$ form a $(v-6)$–cyclic $OTS(v)$. ❑

The results of this section and Lemma 2.2 combine to give us:

Theorem 4.1. *A d–cyclic $OTS(v)$ exists if and only if $v\geq2f+1$, $v\equiv0$ or 1 (mod 3) and $f\equiv0$ or 1 (mod 3), where $f=v-d\geq2$.*

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