On uniform intersection numbers

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Abstract
In this paper, we consider total clique covers and uniform intersection numbers on multifamilies. We determine the uniform intersection numbers of graphs in term of total clique covers. From this result and some properties of intersection graphs on multifamilies, we determine the uniform intersection numbers of some families of graphs. We also deal with the NP-completeness of uniform intersection numbers.

1 Introduction.

In this paper, we consider finite undirected simple graphs. For a graph G, $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. And p, q denote the cardinalities of $V(G)$ and $E(G)$, respectively. $\Delta(G)$ denotes the maximum degree among the vertices of G. For a vertex v, the *neighborhood* of v is denoted by $N(v)$. For an edge subset $S \subseteq E(G)$, $< S >_E$ denotes the subgraph generated by S and for a vertex subset $S \subseteq V(G)$, $< S >_V$ denotes the subgraph generated by S.

Let $\mathcal{F} = \{S_1, S_2, .., S_p\}$ be a family of distinct nonempty subsets of a set X. Then $S(\mathcal{F})$ denotes the union of the sets in $\mathcal{F}$. The *intersection graph* of $\mathcal{F}$ is denoted by $\Omega(\mathcal{F})$ and defined by $V(\Omega(\mathcal{F})) = \mathcal{F}$, with S_i and S_j adjacent whenever $i \neq j$ and $S_i \cap S_j \neq \emptyset$. Then a graph G is an intersection graph on $\mathcal{F}$ if there exists a family $\mathcal{F}$ such that $G \cong \Omega(\mathcal{F})$. The *intersection number* $\omega(G)$ of a given graph G is the minimum cardinality of a set $S(\mathcal{F})$ such that G is an intersection graph on $\mathcal{F}$. This concept was introduced by P.Erdös et al. ([3]). In this paper, we deal with intersection graphs on uniform families, where $\mathcal{F} = \{S_1, S_2, .., S_p\}$ is a *uniform family* if $\#S_i = \#S_j$ for all $i, j \in \mathcal{F}$. As the case with families, we can define *uniform intersection graphs* and *uniform intersection numbers* $\omega_{ui}(G)$. We already have some results on uniform intersection numbers in [2]. In this paper we consider uniform intersection graphs from the point of view of total clique covers.

In [7] and [8], we deal with *antichains*, that is, *Sperner families*, and *multifamilies*, namely its elements need not be distinct. As is the case with families, we can define *antichain intersection graphs, antichain intersection numbers* $\omega_{ai}(G)$, *multifamily intersection graphs* and *multifamily intersection numbers* $\omega_m(G)$. In [7], we obtained that $\omega_m(G) = min_{\mathcal{Q} \text{ of } tcc}\{\#\mathcal{Q}\}$ for a graph G, where for a graph G, $\mathcal{Q} = \{Q_1, .., Q_n\}$ is a *total clique cover (tcc)* of G if and only if each Q_i is a complete subgraph of G, $\cup_{1 \le i \le n} V(Q_i) = V(G)$ and $\cup_{1 \le i \le n} E(Q_i) = E(G)$. In [8] , we also obtained that $\omega_{ai}(G) = min_{\mathcal{Q} \text{ of } tcc}\{n(\mathcal{Q}) + i(\mathcal{Q})\}$, where $\mathcal{Q} = \{Q_1, .., Q_n\}$ is a *tcc* of a graph G, $n(\mathcal{Q}) = \#\mathcal{Q}$, $i(\mathcal{Q}) = \#\{S_{\mathcal{Q}}(v); (\exists u \neq v)(S_{\mathcal{Q}}(v) \subseteq S_{\mathcal{Q}}(u))\}$, and $S_{\mathcal{Q}}(v) = \{Q_i; v \in V(Q_i)\}$.

First we consider *uniform multifamilies* of nonempty subsets of a set X, namely its elements have same cardinalities and need not to be distinct. We can also define *intersection numbers with respect to uniform multifamilies* $\omega_{mui}(G)$, as is the case with families. In Section 2 we determine the uniform multifamily intersection number of a graph in terms of total clique covers. Then for the complete graph K_p, $\omega_{ui}(K_p) = l$ if $\binom{l-1}{[\frac{l-1}{2}]+1} < p \le \binom{l}{[\frac{l}{2}]+1}$ and $\omega_{mui}(K_p) = 1$. But for the complete bipartite graph $K(p,p) \cong G$, $\omega_{ui}(G) = \omega_{mui}(G) = p^2$. In general $\omega_{mui}(G) \le \omega_{ui}(G) \le \omega_{mui}(G) + \#V(G)$. We also consider the differences between $\omega_{ui}(G)$ and $\omega_{mui}(G)$ and structures of graphs which lead to those differences. By these structures, in Section 2 and 3 we consider the uniform intersection numbers of some graphs. In Section 4 we consider the complexity of several intersection numbers. Terminology and notation of combinatorics and graph theory can be found in [1] and [4].

2 Uniform intersection numbers.

A *tcc* $\mathcal{Q}_0$ of G is *minimal* if there exists no *tcc* $\mathcal{Q}$ of G such that $\mathcal{Q} \subseteq \mathcal{Q}_0$. For a vertex subset S of G , $\mathcal{P} = \{P_1, P_2, ..., P_k\}$ is a *clique packing* of $< S >_V$ if each P_i is a clique of $< S >_V$ whose cardinality is more than or equal to 2 and $\cup_{1 \le i \le k} V(P_i) \subseteq S$, and $\mathcal{Q} = \{Q_1, Q_2, ..., Q_k\}$ is a *clique cover* of $< S >_V$ if each Q_i is a clique of $< S >_V$ and $\cup_{1 \le i \le k} V(Q_i) = S$.

For a family $\mathcal{F} = \{S_1, S_2, .., S_p\}$ such that $\cup_{1 \le i \le p} S_i = \{a_1, a_2, .., a_n\}$, we denotes $A_{\mathcal{F}}(a_i) = \{S_j; a_i \in S_j \in \mathcal{F}\}$ (or $A(a_i)$). Similarly, for a total clique cover $\mathcal{Q} = \{Q_1, .., Q_n\}$ of a graph G, put $S_{\mathcal{Q}}(v) = \{Q_i \in \mathcal{Q}; v \in V(Q_i)\}$ (or $S(v)$). If $G \cong \Omega(\mathcal{F})$, then every $Q(a_i) = \Omega(A_{\mathcal{F}}(a_i))$ is a clique of G

and $Q(\mathcal{F}) = \{Q(a_1), .., Q(a_n)\}$ is a total clique cover of G . Conversely, let $\mathcal{Q} = \{Q_1, .., Q_n\}$ be a total clique cover of G, then $\mathcal{F}(\mathcal{Q}) = \{S_\mathcal{Q}(v); v \in V(G)\}$ satisfies $\Omega(\mathcal{F}(\mathcal{Q})) \cong G$. Therefore we can identify an element a_i with a clique Q_i, and $\{S_\mathcal{Q}(v); v \in V(G)\}$ with $\mathcal{F}$ according to correspondences showing above. In the follow we often abuse these identification without any mention. By these facts, we obtain that $\omega_m(G) = min_{\mathcal{Q}:tcc \text{ of } G}\{\#\mathcal{Q}\}$ for a graph G and the following result.

Theorem 1 *Let G be a connected graph with $p \geq 2$ vertices. Then $\omega_{mui}(G) = min_{\mathcal{Q}_0}\{\#\mathcal{Q}_0 + (\#\mathcal{P}_1 + \#\mathcal{I}_1) + (\#\mathcal{P}_2 + \#\mathcal{I}_2) + ... + (\#\mathcal{P}_l + \#\mathcal{I}_l)\}$, where $\mathcal{Q}_0$ is a minimal tcc of G, $\mathcal{P}_i$ is a clique packing of $< V_i >_V$, $\mathcal{I}_i$ is the set of all isolated vertices of $< V_i >_V$, $V_i = \{v \in V_{i-1}; \sum_{j=0}^{i-1} \#S_{\mathcal{P}_j \cup \mathcal{I}_j}(v) < max_{v \in V(G)}\{\#S_{\mathcal{Q}_0}(v)\}\}$, $S_{\mathcal{P}_j \cup \mathcal{I}_j}(v) = \{Q; v \in Q \in \mathcal{P}_j \cap \mathcal{I}_j\}$ and $V_1 = \{v \in V(G); \#S_{\mathcal{Q}_0}(v) < max_{v \in V(G)}\{\#S_{\mathcal{Q}_0}(v)\}\}$.*

Proof. Let $\mathcal{F} = \{S(v); v \in V(G)\}$ be a uniform multifamily of nonempty subsets of a set X such that $\Omega(\mathcal{F}) \cong G$, $S(\mathcal{F}) = \{a_1, .., a_n\}$ and $n = \omega_{mui}(G)$. Then $Q(a_i) = \{S(v); a_i \in S(v) \in \mathcal{F}\}$ is a clique and $\mathcal{Q}(\mathcal{F}) = \{Q(a_i); a_i \in S(\mathcal{F})\}$ is a *tcc* of G. We assume that $\mathcal{Q}(\mathcal{F})$ is not minimal. Let $\mathcal{Q}_0 \subseteq \mathcal{Q}(\mathcal{F})$ be a minimal *tcc* of G, $V_1 = \{v \in V(G); \#S_{\mathcal{Q}_0}(v) < max_{u \in V(G)}\{\#S_{\mathcal{Q}_0}(u)\}\}$ and $\mathcal{I}_1$ be the set of all isolated vertices on $< V_1 >_V$. Since $\mathcal{F}$ is uniform, $\mathcal{Q}(\mathcal{F}) - \mathcal{Q}_0$ is a clique cover of $< V_1 >_V$. Thus there exists a clique packing $\mathcal{P}_1 \subseteq \mathcal{Q}(\mathcal{F}) - \mathcal{Q}_0$. Next we consider $\mathcal{Q}_1 = \mathcal{P}_1 \cup \mathcal{I}_1$. In the following we repeat the similarly step. If $\mathcal{Q}(\mathcal{F}) - \mathcal{Q}_0 - \mathcal{Q}_1 - ... - \mathcal{Q}_{i-1} \neq \emptyset$, let $V_i = \{v \in V(G); \sum_{j=0}^{i-1} \#S_{\mathcal{Q}_j}(v) < max_{v \in V(G)}\{\#S_{\mathcal{Q}_0}(v)\}\}$ and $\mathcal{I}_i$ be the set of all isolated vertices of $< V_i >_V$. Then there exists a clique packing $\mathcal{P}_i \subseteq \mathcal{Q}(\mathcal{F}) - \mathcal{Q}_0 - \mathcal{Q}_1 - ... - \mathcal{Q}_{i-1}$ of $< S_i >_V$. So we next consider on $\mathcal{Q}_i = \mathcal{P}_i \cup \mathcal{I}_i$. We finally get $\mathcal{Q}(\mathcal{F}) = \mathcal{Q}_0 \cup \mathcal{Q}_1 \cup ... \cup \mathcal{Q}_l$ and $\omega_{mui}(G) = \sum_{j=0}^{l} \#\mathcal{Q}_j \geq min\{\sum_{j=0}^{l} \#\mathcal{Q}_j\}$

Conversely let $\mathcal{Q}_0$ be a minimal *tcc* of G, $\mathcal{P}_i$ be a clique packing of $< V_i >_V$, $\mathcal{I}_i$ be the set of all isolated vertices of $< V_i >_i$, $V_i = \{v \in V(G); \sum_{j=1}^{i-1} \#S_{\mathcal{Q}_i}(v) < max_{v \in V(G)}\{\#S_{\mathcal{Q}_0}(v)\}\}$, $V_1 = \{v \in V(G); \#S_{\mathcal{Q}_0}(v) < max_{v \in V(G)}\{\#S_{\mathcal{Q}_0}\}$ and $S(v) = \{Q; v \in Q \in \cup_{i=0}\mathcal{Q}_i\}$. Then $\#S(v) = \#S(u) = max_{w \in V(G)}\{\#S_{\mathcal{Q}_0}(w)\}$ for $\forall u, v \in V(G)$ and $\mathcal{F} = \{S(v); v \in V(G)\}$ is a uniform multifamily and $\Omega(\mathcal{F}) \cong G$. Thus $\omega_{mui}(G) \leq \sum_{i=0}^{l} \#\mathcal{Q}_i$ and $\omega_{mui}(G) \leq min\{\sum_{i=1}^{l} \#\mathcal{Q}_i\}$ □

By the Theorem 1, we are now in a position to give numerous examples of uniform intersection numbers. An *independent set* in a graph G is a set of vertices of G, no two of which are adjacent.

Example 2 *Let G be a connected graph with $p \geq 3$ vertices and q edges, where its maximum degree is $\Delta(G)$. Then $E(G)$ is a total clique cover of G*

and $max_{v \in V(G)}\{\#S_{E(G)}(v)\} = \Delta(G)$. *So each vertex needs more* $\Delta(G) - deg_G(v)$ *elements. Thus we get the following upper bound:* $\omega_{mui}(G) \leq \omega_{ui}(G) \leq \#E(G) + \sum_{v \in V(G)}\{\Delta(G) - deg_G(v)\} = \Delta(G) \times p - q$ *([2]).* □

Example 3 *Let* G *be a triangle-free graph and the vertex set* $W = \{v \in V(G);\ deg_G(v) \neq \Delta(G)\}$ *is an independent set. Then each maximal complete subgraph of* G *is* K_2*, so a minimal tcc is* $E(G)$ *and* $< V_1 >_V$ *is a union of isolated vertices. Therefore* $\omega_{mui}(G) = \omega_{ui}(G) = \Delta(G) \times p - q$ *if and only if* G *is* K_3 *or a triangle-free graph and the vertex set* $W = \{v \in V(G); deg_G(v) \neq \Delta(G)\}$ *is an independent set. We also get the following results. If* G *is a connected triangle-free graph with* $p \geq 3$ *vertices, then* $\omega_{mui}(G) = \omega_{ui}(G) = \#E(G)$ *if and only if* G *is regular. If* G *is a connected regular graph with* $p \geq 3$ *vertices, then* $\omega_{mui}(G) = \omega_{ui}(G) = \#E(G)$ *if and only if* G *is triangle-free ([2]).* □

3 General results.

We next consider some structures of graphs which lead differences between $\omega_{ui}(G)$ and $\omega_{mui}(G)$. We obtain the next result.

Proposition 4 *Let* G *be a connected graph and* $\mathcal{F} = \{S(v); v \in V(G)\}$ *be a uniform multifamily such that* $\Omega(\mathcal{F}) \cong G$ *and* $\#S(\mathcal{F}) = \omega_{mui}(G)$*. For vertices* u *and* v *with* $S(u) = S(v)$*,* u *and* v *are adjacent and* $N(u) - \{v\} = N(v) - \{u\}$*.*

Proof. Let $S(u) = \{a_1, .., a_l\} = S(v)$. Since each a_i is a clique containing u and v, there exists an edge joining u and v. If $N(v) - \{u\} \neq N(u) - \{v\}$, then there exists a vertex $w \in (N(v) - \{u\}) - (N(u) - \{v\})$ (or $(N(u) - \{v\}) - (N(v) - \{u\})$) and a clique $a_i \in S(u)$ (or $S(v)$) containing w . Since a_i belong to $S(v)$, $v, w \in a_i$ and $w \in N(v) - \{u\}$, giving the required contradiction. □

Proposition 4 is also valid for the non-uniform cases. In [7] and [8], we obtained the same results of Proposition 4 for the ordinary family case and the antichain case.

The graph G of Figure 1 shows that the converse of Proposition 4 does not hold. For vertices u and v, $N(u) - \{v\} = N(v) - \{u\}$, but $S(u) \neq S(v)$ in the minimal uniform multifamily $\mathcal{F} = \{\ S(u) = \{\ Q_1 = \{w_1, w_2, u, v\}, Q_2 = \{w_3, w_4, u\},\ Q_3 = \{w_4, w_5, u, v\}\ \},\ S(v) = \{Q_1, Q_3, Q_4 = \{w_2, w_3, v\}\},\ S(w_1) = \{\ Q_1,\ Q_5 = \{w_1, w_2\},\ Q_6 = \{w_1\}\ \},\ S(w_2) = \{Q_1, Q_4, Q_5\},$

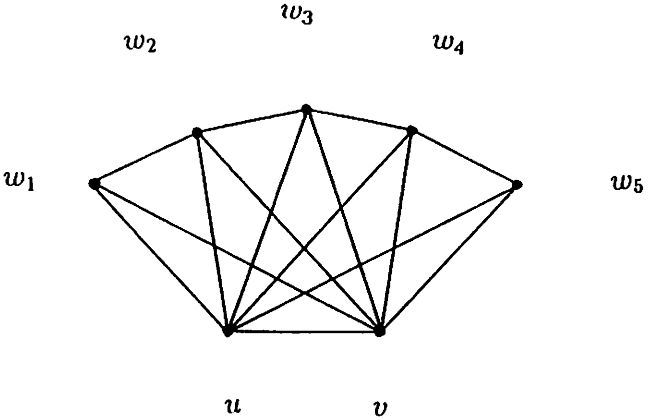


Figure 1:

$S(w_3) = \{ Q_2, Q_4, Q_7 = \{w_3, w_4\} \}$, $S(w_4) = \{Q_2, Q_3, Q_7\}$, $S(w_5) = \{ Q_3, Q_8 = \{w_5\}, Q_9 = \{w_5\} \} \}$.

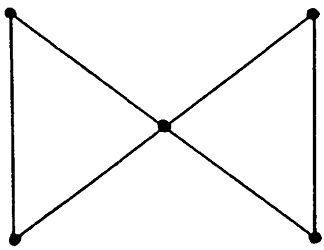
However by restricting the choice of the pair of vertices u, v of Proposition 4, its converse becomes true. The next result gives a class of graphs satisfying the condition.

Proposition 5 *Let G be a connected graph and $\mathcal{F} = \{S(v); v \in V(G)\}$ be a uniform multifamily such that $\Omega(\mathcal{F}) \cong G$, $\#S(\mathcal{F}) = \#\{a_1, .., a_n\} = \omega_{mui}(G)$ and $max_{v \in V(G)}\{\#S(v)\} \leq \Delta(G) - 1$. For vertices u and v such that $N(u) - \{v\}$ and $N(v) - \{u\}$ are independent sets, $deg(u) = deg(v) = \Delta(G)$ and for each vertex $w \in N(u) - \{v\}$ whose degree is $\Delta(G)$, $N(w) - \{u, v\}$ is independent, $S(u) = S(v)$ if and only if u and v are adjacent and $N(u) - \{v\} = N(v) - \{u\}$.*

Proof. Since the necessity is true by Proposition 4 , we show the sufficiency. Since $N(u) - \{v\}$ is an independent set, there are at least $\Delta(G) - 1$ cliques in a clique cover of $N(u) - \{v\}$. So $\#S(u) = \Delta(G) - 1 = \#S(v)$. If $S(u) \neq S(v)$, then there exists a clique Q which belongs to $S(u) - S(v)$. Since $N(u) - \{v\}$ is an independent set, $Q = \{u, w\}$ and $w \in N(u) - \{v\} = N(v) - \{u\}$. By $\#S(u) = \#S(v) = \Delta(G) - 1$, there exists no $\{u, v, w\}$ in $S(v)$ and $S(u)$. Since $N(w) - \{u, v\}$ is independent, $\#S(w) > \Delta(G) - 1$, giving the required contradiction. □

By Proposition 5 , we obtain some results on uniform intersection numbers. The *8-graph* is a graph which is obtained from two triangles T_1 and T_2 by identifying one vertex on T_1 with one vertex on T_2 (see Figure 2).

A graph G is called *8-free* if G has no induced subgraphs isomorphic to the 8-graph.



8-graph

Figure 2:

Corollary 6 *Let G be K_4-free and 8-free graph. If there exists a uniform multifamily $\mathcal{F}$ such that $\Omega(\mathcal{F}) \cong G$, $\#S(\mathcal{F}) = \omega_{mui}(G)$ and $max_{v \in V(G)}\{S(v)\} \leq \Delta(G) - 1$, then $\omega_{ui}(G) \leq \omega_{mui}(G) + min_Q\{\#Q\}$, where Q is a clique cover of G such that $S_Q(u) \neq S_Q(v)$ for all u, v such that u and v are adjacent, $deg(u) = deg(v) = \Delta(G)$*

Corollary 7 *Let G be K_4-free and 8-free graph which has a $K_{1.\Delta(G)}$ as an induced graph. Then $\omega_{ui}(G) = \omega_{mui}(G) + min_{\mathcal{F}}\{v \in V(G); (\exists u \in V(G))(S_{\mathcal{F}}(v) = S_{\mathcal{F}}(u))\}$, where $\mathcal{F}$ is a uniform multifamily such that $\Omega(\mathcal{F}) \cong G$ and $S(\mathcal{F}) = \omega_{mui}(G)$.*

Proof. Since G has a $K_{1.\Delta(G)}$ as an induced subgraph, $max_{v \in V(G)}\{\#S(v)\} \geq \Delta(G)$. If $S(u) = S(v)$, then there exist $Q = \{u, v\} \in S(u) = S(v)$, or $R_1 = \{u, v, w\}$, $R_2 = \{u, v, w\} \in S(u) = S(v)$. In the first case, we replace Q with $Q'_1 = \{u\}$, $Q'_2 = \{v\}$. In the second case we replace R_1, R_2 with $R'_1 = \{u, v\}$, $R'_2 = \{v, w\}$, $R'_3 = \{u, w\}$. In the both case we obtain the requesting uniform family, and $\omega_{ui}(G) = \omega_{mui}(G) + min_{\mathcal{F}}\{v \in V(G); (\exists u \in V(G))(S_{\mathcal{F}}(v) = S_{\mathcal{F}}(u))\}$. □

4 Complexity of intersection numbers.

In this section we consider the complexity of intersection numbers. In [6], we dealt with complexity on several kinds of intersection numbers. We also know that the problem of determining of $\omega_m(G)$ is NP-complete ([5]). From a graph G with $V(G) = \{v_1, v_2, ..., v_p\}$, wa can construct the graph H as follows: To the graph G, add new vertices u_1, u_2, ..., u_p and new edges $\{u_1, v_1\}$, $\{u_2, v_2\}$, ..., $\{u_p, v_p\}$. Clearly H has no vertices pair x, y such that x and y are adjacent and $N(x) - \{y\} = N(y) - \{x\}$. Thus $\omega(H) = \omega_m(H) = \omega_m(G) + p$ and $\omega_{ai}(H) = \omega_m(G) + 2p$. So we obtain the following fact.

Fact 8 *For a given graph G and an integer k it is NP-complete to decide whether $\omega(G) \leq k$ and it is also NP-complete to decide whether $\omega_{ai}(G) \leq k$.*

We also have the results which concern with the NP-completeness on uniform intersection numbers. This result is based on the NP-completeness of $\omega_m(G)$.

Theorem 9 *For a given graph G and an integer k it is NP-complete to decide whether $\omega_{mui}(G) \leq k$ and it is also NP-complete to decide whether $\omega_{ui}(G) \leq k$.*

Proof. We reduce to it the problem of determining $\omega_m(G)$, which is NP-complete (see [5]). Let a graph G be given and $V(G) = \{v_1, v_2, ..., v_p\}$. The graph H is constructed as follows: To the graph G, add new vertices x, u_1, u_2, ..., u_p, w_1, w_2, ..., w_p, new edges $\{u_1, v_1\}$, $\{u_2, v_2\}$, ..., $\{u_p, v_p\}$ and join the vertex x to all vertices of $\{ u_1, u_2, ..., u_p, w_1, w_2, ..., w_p \}$. Then $deg(x) = 2p$. Thus $\omega_{mui}(H) = \omega_m(G) + min_{\mathcal{F}:tcc,\Omega(\mathcal{F})\cong G,\#S(\mathcal{F})=\omega_m(G)}$ $\{\sum_{1\leq i\leq p}\{(2p - \#S_{\mathcal{F}}(v_i)) + (2p - 1 - (2p - \#S_{\mathcal{F}}(v_i)))\} + 2p + p(2p-1)\} =$ $\omega_m(G) + min_{\mathcal{F}:tcc,\Omega(\mathcal{F})\cong G,\#S(\mathcal{F})=\omega_m(G)}$ $\{\sum_{1\leq i\leq p}\{2p-1\} + 2p + p(2p-1)\}$ $= \omega_m(G) + 2p(2p-1) + 2p = \omega_m(G) + 4p^2$

Clearly H has no vertices pair x, y such that x and y are adjacent and $N(x) - \{y\} = N(y) - \{x\}$. Hence $\omega_{mui}(H) = \omega_{ui}(H) = \omega_m(G) + 4p^2$ □

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